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The Property of a Special Type of Exponential Spline Function

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Abstract

Approximation theory experienced a long term history. Since 50' last century, the rise of spline function as well as the advance of calculation promotes the growth of classical approximation theory and makes them develop a profound theory in maths, and application values have shown among the field of scientific calculation and engineering technology and etc. At present, the study of spline function had made a great progress and had a lot of fruits, as for that, the reader could look up the book [1] or [2]. Nevertheless, the research staff pays less attention to exponential spline function, since polynomial spline function is a special case of that, so it is much essential and meaningful for one to explore the nature of exponential spline function.

Keywords

Exponential Spline Function, Interpolation, Error Estimation

1. Introduction

At the beginning, we introduce the definition of exponential spline function. From literature [3], we could learn the definition: if function $S(t)$ satisfies equation $L\left[S(t)\right]=\sum_k c_k\delta(t-t_k)$, we describe it as exponential spline function, where L is a differential operator $Lf(t)=D^{n+1}f+a_nD^nf+\cdots+a_0D^0f$. Here, $a_i\in R$ $(0\le i\le n)$ are constant coefficient and D^k represent kth-order derivative. By this definition, we learn that $S(t)$ exists continuous derivative $n-1$ and in each interval $S(t)$ is linear combination of $\left\{t^{k-1}e^{\alpha_{(m)}t}\right\}_{m=1,\cdots,N_d;k=1,\cdots,k_{(m)}}$ $\left(\sum_{m=1}^{N_d}k_{(m)}=n+1\right)$, where the $\alpha_{(m)}$'s are the N_d distinct roots of characteristic polynomial and $\alpha_{(m)}$ is of order $k_{(m)}$. As exists a single root 0 for characteristic polynomial, $S(t)$ is polynomial

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spline function. Next we will deal with the case of there being unique real root.

2. Main Result

Theorem 1:

If the differential operator's characteristic polynomial is $L(s)=(s-\alpha)^{n+1}$ $(\alpha\in R)$, where α is a root of multiplicity $n+1$. Then the expression for exponential spline function of this special case is

$$S(x)=S_0(x)+\sum_{j=1}^{N}c_j\mathrm{e}^{\alpha x}(x-x_j)_+^n \quad x\in[a,b].$$

Proof:

Let $S(x)$ be on interval $[x_i,x_{i+1}](i=0,1,\cdots,N)$, $S(x)=S_i(x)\in span\{\mathrm{e}^{\alpha x},\mathrm{e}^{\alpha x}x,\cdots,\mathrm{e}^{\alpha x}x^n\}$ Suppose $\eta(x)=S_1(x)-S_0(x)$ And we have $\mathrm{e}^{-\alpha x}\eta(x)=\mathrm{e}^{-\alpha x}(S_1(x)-S_0(x))$

$$\left[\mathrm{e}^{-\alpha x}\eta(x)\right]^{(i)}=\sum_{k=0}^{i}C_i^k\left(\mathrm{e}^{-\alpha x}\right)^{(k)}\eta^{(i-k)}(x) \quad (i\le n-1)$$

Since there exists order $n-1$ continuous derivatives for $S(x)$,

Hence

$$\eta^{(i)}(x_1)=0 \quad (i=0,1,\cdots,n-1)$$

So that $\left[\mathrm{e}^{-\alpha x}\eta(x)\right]^{(i)}\Big|_{x=x_1}=0\,(0\le i\le n-1)$

Furthermore, $\mathrm{e}^{-\alpha x}\eta(x)$ is polynomial of *n*th degrees.

Therefore $\mathrm{e}^{-\alpha x}\eta(x)=c_1(x-x_1)^n$.

We get $S_1(x)=S_0(x)+c_1\mathrm{e}^{\alpha x}(x-x_1)^n$,

put $x_+=\begin{cases}x & x\ge 0\\ 0 & x<0\end{cases}$.

In terms of this idea, we obtain $S(x)=S_0(x)+\sum_{j=1}^{N}c_j\mathrm{e}^{\alpha x}(x-x_j)_+^n \quad x\in[a,b]$.

Theorem 2: The dimension of the exponential spline function space is $n+N+1$.

Proof:

Suppose $S(x)=p(x)+\sum_{j=1}^{N}c_j\mathrm{e}^{\alpha x}(x-x_j)_+^n$, $p(x)\in span\{\mathrm{e}^{\alpha x},\mathrm{e}^{\alpha x}x,\cdots,\mathrm{e}^{\alpha x}x^n\}$

We have $\left(S_{i+1}(x)-S_i(x)\right)^{(m)}=c_i\sum_{k=0}^{m}C_m^k\left(\mathrm{e}^{\alpha x}\right)^{(k)}\left[(x-x_i)^n\right]^{(m-k)} \quad (0\le m\le n-1)$

Since $\left[(x-x_i)^n\right]^{(m-k)}\Big|_{x=x_i}=0$

So that $S^{(m)}(x)$ is continuous at the knot $x_i\,(m=0,\cdots,n-1)$, hence $S(x)$ has order $n-1$ continuous derivatives on interval $[a,b]$.

When characteristic polynomial has single real root, the linear space can be written as

$$span\left\{\mathrm{e}^{\alpha x},\mathrm{e}^{\alpha x}x,\cdots,\mathrm{e}^{\alpha x}x^n,\mathrm{e}^{\alpha x}(x-x_1)_+^n,\cdots,\mathrm{e}^{\alpha x}(x-x_N)_+^n\right\}$$

Next we prove that $\mathrm{e}^{\alpha x},\mathrm{e}^{\alpha x}x,\cdots,\mathrm{e}^{\alpha x}x^n,\mathrm{e}^{\alpha x}(x-x_1)_+^n,\cdots,\mathrm{e}^{\alpha x}(x-x_N)_+^n$ is linearly independent

Set $\sum_{i=0}^{n}c_i\mathrm{e}^{\alpha x}x^i+\sum_{i=1}^{N}\alpha_i\mathrm{e}^{\alpha x}(x-x_i)_+^n=0.$ On the interval $[x_0,x_1]$, above equation become $\sum_{i=0}^{n}c_i\mathrm{e}^{\alpha x}x^i=0$, we have $c_i = 0\left(i = 0,1,\cdots,n\right)$ On the interval $\left[x_1, x_2\right]$, we can get $\alpha_1 \mathrm{e}^{\alpha x}\left(x - x_1\right)_+^n = 0$, so that $\alpha_1 = 0$, For the interval $\left[x_i, x_{i+1}\right]$, By means of the same technique, we can obtain $\alpha_i = 0$, hence $\mathrm{e}^{\alpha x}, \mathrm{e}^{\alpha x}x, \cdots, \mathrm{e}^{\alpha x}x^n, \mathrm{e}^{\alpha x}\left(x - x_1\right)_+^n, \cdots, \mathrm{e}^{\alpha x}\left(x - x_N\right)_+^n$ is linearly independent. So that we conclude $\dim S = n + N + 1$.

According to theorem 1. 4. 23 of the book [4], we can prove next conclusion is true.

Corollary: There exists the $S\left(x\right)$ for every *f* belonging to $L^p\left[a,b\right]$, such that

$$\left\| f\left(x\right) - S\left(x\right) \right\|_p = \min_{s(x)\in S} \left\| f\left(x\right) - s\left(x\right) \right\|_p$$

Theorem 3: If condition of interpolation and boundary satisfy:

$$\begin{cases} S\left(x_i\right) = f\left(x_i\right) & i = 0,1,\cdots,N+1 \\ S'\left(a\right) = f'\left(a\right) & S'\left(b\right) = f'\left(b\right) \end{cases} \tag{1}$$

then there exist the 3rd degree exponential spline function satisfied with condition. And we have formula of error evaluation

$$\left\| f\left(x\right) - S\left(x\right) \right\|_\infty \le c_0 \mathrm{e}^{|\alpha|(b-a+4)} M h^4 \quad \left(\text{where } c_0 = \frac{5}{384}, M = \max_{0\le i\le 4} \left\| f^{(i)} \right\|_\infty\right)$$

Proof:

Suppose $p\left(x\right)$ is 3rd degree polynomial spline function, let $S\left(x\right) = \mathrm{e}^{\alpha x} p\left(x\right)$

Hence $S'\left(x\right) = \alpha \mathrm{e}^{\alpha x} p\left(x\right) + \mathrm{e}^{\alpha x} p'\left(x\right)$

Both of them can be denoted by: $\begin{pmatrix} S\left(x\right) \\ S'\left(x\right) \end{pmatrix} = \mathrm{e}^{\alpha x} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} p\left(x\right) \\ p'\left(x\right) \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, $|A| \ne 0$, so that A is invertible matrix. $A^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$

$$\text{This lead to } \begin{pmatrix} p\left(x\right) \\ p'\left(x\right) \end{pmatrix} = \mathrm{e}^{-\alpha x} \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} S\left(x\right) \\ S'\left(x\right) \end{pmatrix} \tag{2}$$

Since $p\left(x\right) \in C^2\left[a,b\right]$, hence $S\left(x\right) \in C^2\left[a,b\right]$, we can get $S\left(x\right)$ is exponential spline function.

If boundary condition is $S'\left(a\right) = f'\left(a\right)$, $S'\left(b\right) = f'\left(b\right)$, by matrix relation (2), let $p'\left(a\right) = \mathrm{e}^{-\alpha a}\left(-\alpha f\left(a\right) + f'\left(a\right)\right)$ and $p'\left(b\right) = \mathrm{e}^{-\alpha b}\left(-\alpha f\left(b\right) + f'\left(b\right)\right)$

Since one of 3rd degree polynomial spline function meet the constraint of interpolation $p\left(x_i\right) = \mathrm{e}^{-\alpha x_i} f\left(x_i\right)$, boundary condition is $p'\left(a\right)$ and $p'\left(b\right)$.

So that exponential spline function satisfied with condition (1) exists. That is $\mathrm{e}^{\alpha x} p\left(x\right)$.

Next we prove formula of error evaluation. Suppose $f\left(x\right) \in C^4\left[a,b\right]$, $S\left(x\right)$ is 3rd degree exponential spline function satisfied with condition (1).

Let $S\left(x\right) = \mathrm{e}^{\alpha x} p\left(x\right)$ (where $p\left(x\right)$ is 3rd degree polynomial spline function)

$$\begin{aligned} \left\| f\left(x\right) - S\left(x\right) \right\|_\infty &= \left\| f\left(x\right) - \mathrm{e}^{\alpha x} p\left(x\right) \right\|_\infty = \left\| \mathrm{e}^{\alpha x}\left(\mathrm{e}^{-\alpha x} f\left(x\right) - p\left(x\right)\right) \right\|_\infty \\ &\le \left\| \mathrm{e}^{\alpha x} \right\|_\infty \left\| \mathrm{e}^{-\alpha x} f\left(x\right) - p\left(x\right) \right\|_\infty \end{aligned}$$

Since $p\left(x_i\right) = \mathrm{e}^{-\alpha x_i} f\left(x_i\right)$

$$p'\left(a\right) = -\alpha \mathrm{e}^{-\alpha a} S\left(a\right) + \mathrm{e}^{-\alpha a} S'\left(a\right) = -\alpha \mathrm{e}^{-\alpha a} f\left(a\right) + \mathrm{e}^{-\alpha a} f'\left(a\right) = \left(\mathrm{e}^{-\alpha x} f\left(x\right)\right)' \Big|_{x=a}$$

$$p'\left(b\right) = \left(\mathrm{e}^{-\alpha x} f\left(x\right)\right)' \Big|_{x=b}$$

By formula of error evaluation for 3rd degree polynomial spline function, we can have

$$\left\| \mathrm{e}^{-\alpha x} f\left(x\right) - p\left(x\right) \right\|_\infty \le c_0 \left\| \left(\mathrm{e}^{-\alpha x} f\left(x\right)\right)^{(4)} \right\|_\infty h^4$$

$$\left\|\left(e^{-\alpha x} f(x)\right)^{(4)}\right\|_{\infty} = \left\|\sum_{k=0}^{4} C_4^k \left(e^{-\alpha x}\right)^{(4-k)} f^{(k)}\right\|_{\infty} = \left\|\sum_{k=0}^{4} C_4^k (-\alpha)^{4-k} e^{-\alpha x} f^{(k)}\right\|_{\infty}$$

In terms of book [5], we have

$$\left\|\sum_{k=0}^{4} C_4^k (-\alpha)^{4-k} e^{-\alpha x} f^{(k)}\right\|_{\infty} \le \left\|e^{-\alpha x}\right\|_{\infty} M \sum_{k=0}^{4} C_4^k |\alpha|^{4-k} = \left\|e^{-\alpha x}\right\|_{\infty} M \left(1+|\alpha|\right)^4$$

$$\text{Since } \ 1+x \le e^x \ (x \ge 0)$$

$$\text{Hence } \ \left(1+|\alpha|\right)^4 \le e^{4|\alpha|}$$

$$\text{Furthermore } \ \left\|e^{\alpha x}\right\|_{\infty} \left\|e^{-\alpha x}\right\|_{\infty} = e^{|\alpha|(b-a)}$$

By above expressions, we can conclude that

$$\left\|f(x) - S(x)\right\|_{\infty} \le c_0 e^{|\alpha|(b-a+4)} M h^4 .$$

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