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Riesz Means of Dirichlet Eigenvalues for the Sub-Laplace Operator on the Engel Group

Jingjing Xue
Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, China
Email: xuejingjingsx@163.com

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ABSTRACT

In this paper, we are concerned with the Riesz means of Dirichlet eigenvalues for the sub-Laplace operator on the Engel group and deriver different inequalities for Riesz means. The Weyl-type estimates for means of eigenvalues are given.

Keywords: Engel Group; Sub-Laplace Operator; Eigenvalues; Riesz Mean

1. Introduction

The Engel group G is a Carnot group of step $r=3$ (see [1]), its Lie algebra is generated by the left-invariant vector fields

$$X_1=\frac{\partial}{\partial x_1}-\frac{x_2}{2}\frac{\partial}{\partial x_3}+\left(\frac{-x_1x_2}{12}-\frac{x_3}{2}\right)\frac{\partial}{\partial w},$$
$$X_2=\frac{\partial}{\partial x_2}+\frac{x_1}{2}\frac{\partial}{\partial x_3}+\frac{x_1^2}{12}\frac{\partial}{\partial w},$$
$$X_3=\frac{\partial}{\partial x_3}+\frac{x_1}{2}\frac{\partial}{\partial w},$$
$$X_4=\frac{\partial}{\partial w},$$

where $P=(x_1,x_2,x_3,w)$ is a point of G. It is easy to see that

$$[X_1,X_2]=X_3,[X_1,X_3]=X_4,[X_2,X_3]=0,$$
$$[X_1,X_4]=[X_2,X_4]=0,$$

and $[X_3,X_4]=0$. So the Lie algebra of G is

$$g=V_1\oplus V_2\oplus V_3,$$

where $V_1=span\{X_1,X_2\}$, $V_2=span\{X_3\}$ and $V_3=span\{X_4\}$. The sub-Laplace operator on G is of the form $\Delta_E=X_1^2+X_2^2$.

In the paper, we investigate the Riesz means of the Dirichlet problem

$$\begin{cases}-\Delta_E u=\lambda u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases} \tag{1.1}$$

in the Engel group G. Here Ω is a bounded and noncharacteristics domain in G, with smooth boundary $\partial\Omega$. The existence of eigenvalues for (1.1) is from [2]. Let us by $R_\sigma(z)$ denote the Riesz means of order σ of the sequence $\{\lambda_k\}$ of eigenvalues of (1.1).

The Riesz means of Dirichlet eigenvalues for the Laplace operator in the Euclidean space have been extensively studied(see [3-5]). In recent years, E. M. Harrell II and L. Hermi in [6] treated the Riesz means $R_\sigma(z)$ of order σ of $\{\lambda_k\}$ on the bounded domain $\Omega\subset R^d$ and pointed out that: for $0<\sigma\le 2$ and $z\ge\lambda_1$,

$$R_{\sigma-1}(z)\ge\left(1+\frac{d}{4}\right)\frac{1}{z}R_\sigma(z)$$
$$\text{and } R'_\sigma(z)\ge\left(1+\frac{d}{4}\right)\frac{\sigma}{z}R_\sigma(z), \tag{1.2}$$

and $\frac{R_\sigma(z)}{z^{\sigma+\frac{d\sigma}{4}}}$ is a nondecreasing function of z; for $2<\sigma<+\infty$ and $z\ge\lambda_1$,

$$R_{\sigma-1}(z)\ge\left(1+\frac{d}{2\sigma}\right)\frac{1}{z}R_\sigma(z)$$
$$\text{and } R'_\sigma(z)\ge\left(\sigma+\frac{d}{2}\right)\frac{1}{z}R_\sigma(z), \tag{1.3}$$

and $\frac{R_\sigma(z)}{z^{\sigma+\frac{d}{2}}}$ is a nondecreasing function of *z*, and then the Weyl-type estimates of means of eigenvalues is derived.

Jia *et al.* in [7] extended (1.2), (1.3) to the Heisenberg group.

The main results of this paper are the following.

Theorem 1.1 *For* $0<\sigma\le 2$ *and* $z\ge\lambda_1$, *we have*

$$R_{\sigma-1}(z)\ge\frac{3}{2z}R_\sigma(z), \tag{1.4}$$

$$R'_\sigma(z)\ge\frac{3}{2}\frac{\sigma}{z}R_\sigma(z), \tag{1.5}$$

and $\frac{R_\sigma(z)}{z^{\frac{3\sigma}{2}}}$ is a nondecreasing function of z; for $2<\sigma<+\infty$ and $z\ge\lambda_1$, we have

$$R_{\sigma-1}(z)\ge\left(1+\frac{1}{\sigma}\right)\frac{1}{z}R_\sigma(z), \tag{1.6}$$

$$R'_\sigma(z)\ge(\sigma+1)\frac{1}{z}R_\sigma(z), \tag{1.7}$$

and $\frac{R_\sigma(z)}{z^{\sigma+1}}$ is a nondecreasing function of *z*.

Theorem 1.2 *Suppose that* $z\ge 3\overline{\lambda_j}$, *then*

$$R_2(z)\ge\frac{4jz^3}{27\overline{\lambda_j}}, \tag{1.8}$$

and therefore

$$R_1(z)\ge\frac{2jz^2}{9\overline{\lambda_j}}, \tag{1.9}$$

$$N(z)=R_0(z)\ge\frac{jz}{3\overline{\lambda_j}}, \tag{1.10}$$

Moreover, for all $k\ge j\ge 1$, we have the upper bound

$$\lambda_{k+1}\le\frac{3k}{j}\overline{\lambda_j}. \tag{1.11}$$

Theorem 1.3 *For* $k>\frac{4j}{3}$, *we have*

$$\frac{\overline{\lambda_k}}{\overline{\lambda_j}}\le\frac{9k}{8j}. \tag{1.12}$$

Authors in [6] combined the Weyl-type estimates of means of eigenvalues established in [6] and the result in [8] to obtain the Weyl-type estimates of eigenvalues. But it is not easy to extend the result in [8] to the Engel group. The Weyl-type estimates of eigenvalues for (1.1) still are open questions.

This paper is arranged as follows. In Section 2 the definition of Riesz means and Lemmas are described; Section 3 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 is appeared in Section 4. In Section 5 the proof of Theorem 1.3 is given.

2. Preliminaries

Definition 2.1 *For an increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of real numbers and* $z\ge 0$, *the Riesz means* $R_\sigma(z)$ *of order* $\sigma>0$ *of* $\{\lambda_k\}$ *is defined by*

$$R_\sigma(z)=\sum_{k=1}^{\infty}(z-\lambda_k)_+^\sigma,$$

where $(z-\lambda_k)_+=\max\{0,z-\lambda_k\}$ is the ramp function.

Clearly,

$$R'_\sigma(z)=\sigma R_{\sigma-1}(z). \tag{2.1}$$

Similarly to Theorem 1 of [9], we immediately have

Lemma 2.2 *Denoting the* L^2*-normalized eigenfunctions of* (1.1) *by* $\{u_j\}$, *let*

$$T_{\alpha\, jm}=\left|\left(X_\alpha u_j,u_m\right)\right|^2$$

for $\alpha=1,2; j,m=1,2,\cdots$. Then for each fixed α, we have

$$R_\sigma(z)=$$

$$2\sum_{j,m:\lambda_j\ne\lambda_m}\frac{(z-\lambda_j)_+^\sigma-(z-\lambda_m)_+^\sigma}{\lambda_m-\lambda_j}T_{\alpha\, jm} \tag{2.2}$$

$$+4\sum_{j,q:\lambda_j\le z<\lambda_q}\frac{(z-\lambda_j)^\sigma}{\lambda_q-\lambda_j}T_{\alpha\, jq}.$$

Lemma 2.3 ([10]) *Let* $0<x<y$ *and* $\sigma\ge 0$, *then*

$$\frac{y^\sigma-x^\sigma}{y-x}\le C_\sigma\left(y^{\sigma-1}+x^{\sigma-1}\right),$$

where

$$C_\sigma=\begin{cases}\frac{\sigma}{2}, & 0\le\sigma<1,\\ 1, & 1\le\sigma\le 2,\\ \frac{\sigma}{2}, & 2\le\sigma<+\infty.\end{cases}$$

3. The Proof of Theorem 1.1

In this section, we prove Theorem 1.1 and two corollaries.

Proof. Let us use (2.2) and denote the first term on the right-hand side of (2.2) by $G(\sigma,z,\alpha)$. Applying Lemma 2.3 it follows

$$G(\sigma,z,\alpha)=2\sum_{j,m:\lambda_j\neq\lambda_m}\frac{(z-\lambda_j)_+^\sigma-(z-\lambda_m)_+^\sigma}{\lambda_m-\lambda_j}T_{\alpha jm}$$
$$=2\sum_{j,m:\lambda_j\le z,\lambda_m\le z,\lambda_j\neq\lambda_m}\frac{(z-\lambda_j)^\sigma-(z-\lambda_m)^\sigma}{\lambda_m-\lambda_j}T_{\alpha jm}$$
$$\le 2C_\sigma\sum_{j,m:\lambda_j\le z,\lambda_m\le z,\lambda_j\neq\lambda_m}\left[(z-\lambda_j)^{\sigma-1}+(z-\lambda_m)^{\sigma-1}\right]T_{\alpha jm}$$
$$=4C_\sigma\sum_{j,m:\lambda_j\le z,\lambda_m\le z}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jm}$$
$$=4C_\sigma\sum_{j:\lambda_j\le z,\text{all } m}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jm}$$
$$-4C_\sigma\sum_{j,q:\lambda_j\le z<\lambda_q}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jq},$$

here we used the symmetry on j and m in the last step.

Putting the above estimate into (2.2), we have

$$\begin{aligned}R_\sigma(z)&\le 4C_\sigma\sum_{j:\lambda_j\le z,\text{all } m}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jm}\\&-4C_\sigma\sum_{j,q:\lambda_j\le z<\lambda_q}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jq}\\&+4\sum_{j,q:\lambda_j\le z<\lambda_q}\frac{(z-\lambda_j)^\sigma}{\lambda_q-\lambda_j}T_{\alpha jq}\\&=4C_\sigma\sum_{j:\lambda_j\le z,\text{all } m}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jm}\\&+4\sum_{j,q:\lambda_j\le z<\lambda_q}T_{\alpha jq}(z-\lambda_j)^{\sigma-1}\left(\frac{z-\lambda_j-C_\sigma(\lambda_q-\lambda_j)}{\lambda_q-\lambda_j}\right)\\&=4C_\sigma\sum_{j:\lambda_j\le z,\text{all } m}(z-\lambda_j)_+^{\sigma-1}T_{\alpha jm}+4H(\sigma,z,\alpha),\end{aligned}\quad(3.1)$$

where we denote

$$H(\sigma,z,\alpha)=\sum_{j,q:\lambda_j\le z<\lambda_q}T_{\alpha\, jq}(z-\lambda_j)^{\sigma-1}\left(\frac{z-\lambda_j-C_\sigma(\lambda_q-\lambda_j)}{\lambda_q-\lambda_j}\right).\quad(3.2)$$

Since $\{u_m\}$ is a complete orthonormal set, it follows

$$\sum_{m=1}^{\infty}T_{\alpha\, jm}=\|X_\alpha u_j\|^2$$

and

$$\begin{aligned}\sum_{\alpha=1}^{2}\sum_{m=1}^{\infty}T_{\alpha jm}&=\|X_1u_j\|^2+\|X_2u_j\|^2=\int|\nabla_E u_j|^2\\&=\int(\nabla_E u_j)\cdot(\nabla_E u_j)=-\int u_j\cdot\Delta_E u_j\\&=\int\lambda_j u_j^{\,2}=\lambda_j.\end{aligned}$$

Returning to (3.1) with them, it yields

$$2R_\sigma(z)\le 4C_\sigma\sum_j(z-\lambda_j)_+^{\sigma-1}\lambda_j+4\sum_{\alpha=1}^{2}H(\sigma,z,\alpha).\quad(3.3)$$

Since

$$\sum_j(z-\lambda_j)_+^{\sigma-1}\lambda_j=zR_{\sigma-1}(z)-R_\sigma(z),$$

we have

$$2R_\sigma(z)\le 4C_\sigma\left(zR_{\sigma-1}(z)-R_\sigma(z)\right)+4\sum_{\alpha=1}^{2}H(\sigma,z,\alpha),$$

namely,

$$(1+2C_\sigma)R_\sigma(z)-2zC_\sigma R_{\sigma-1}(z)\le 2\sum_{\alpha=1}^{2}H(\sigma,z,\alpha).\quad(3.4)$$

We consider three cases: 1) $1\le\sigma\le 2$; 2) $0<\sigma<1$ and 3) $\sigma>2$.

1) $1\le\sigma\le 2$. In this case, it sees $C_\sigma=1$ and

$$\frac{z-\lambda_j-C_\sigma(\lambda_q-\lambda_j)}{\lambda_q-\lambda_j}=\frac{z-\lambda_q}{\lambda_q-\lambda_j}.$$

Since $\lambda_q>z$, it follows

$$\frac{z-\lambda_j-C_\sigma(\lambda_q-\lambda_j)}{\lambda_q-\lambda_j}<0,$$

and therefore

$$H(\sigma,z,\alpha)<0.$$

Substituting this into (3.4), we obtain

$$(1+2C_\sigma)R_\sigma(z)-2zC_\sigma R_{\sigma-1}(z)\le 0$$

and

$$R_{\sigma-1}(z)\ge\frac{3}{2z}R_\sigma(z).$$

Now (1.4) is proved.

Using (2.1), we have

$$\frac{1}{\sigma}R'_\sigma(z)\ge\frac{3}{2z}R_\sigma(z),$$

and (1.5) is proved.

Since

$$\left(\frac{R_\sigma(z)}{z^{\frac{3\sigma}{2}}}\right)'=\frac{R'_\sigma(z)z^{\frac{3\sigma}{2}}-R_\sigma(z)\frac{3\sigma}{2}z^{\frac{3\sigma}{2}-1}}{z^{3\sigma}}=\frac{z^{\frac{3\sigma}{2}-1}\left[zR'_\sigma(z)-\frac{3\sigma}{2}R_\sigma(z)\right]}{z^{3\sigma}}\ge 0,$$

it follows that $\frac{R_\sigma(z)}{z^{\frac{3\sigma}{2}}}$ is a nondecreasing function of z.

2) $0<\sigma<1$. Now $C_\sigma=\frac{\sigma}{2}\in\left(0,\frac{1}{2}\right)$, so $1-C_\sigma>0$ and

$$\frac{z-\lambda_j-C_\sigma\left(\lambda_q-\lambda_j\right)}{\lambda_q-\lambda_j}<\frac{\lambda_q-\lambda_j-C_\sigma\left(\lambda_q-\lambda_j\right)}{\lambda_q-\lambda_j}=1-C_\sigma. \tag{3.5}$$

Then

$$H(\sigma,z,\alpha)\le\left(1-C_\sigma\right)\sum_{j,q:\lambda_q>z}T_{\alpha\,jq}\left(z-\lambda_j\right)_+^{\sigma-1}\le\left(1-C_\sigma\right)\sum_{j,q}T_{\alpha\,jq}\left(z-\lambda_j\right)_+^{\sigma-1}$$

and

$$\sum_{\alpha=1}^{2}H(\sigma,z,\alpha)\le\left(1-C_\sigma\right)\sum_j\left(z-\lambda_j\right)_+^{\sigma-1}\lambda_j=\left(1-C_\sigma\right)\left(zR_{\sigma-1}(z)-R_\sigma(z)\right).$$

Substituting this into (3.4), we obtain

$$\left(1+2C_\sigma\right)R_\sigma(z)-2zC_\sigma R_{\sigma-1}(z)\le\left(2-2C_\sigma\right)\left[zR_{\sigma-1}(z)-R_\sigma(z)\right],$$

namely,

$$3R_\sigma(z)\le 2zR_{\sigma-1}(z),$$

and (1.4) is proved.

The remainders are discussed similarly to 1).

3) $\sigma>2$. In this case $C_\sigma=\frac{\sigma}{2}>1$, so $1-C_\sigma<0$ and

$$H(\sigma,z,\alpha)\le\left(1-C_\sigma\right)\sum_{j,q:\lambda_q>z}T_{\alpha\,jq}\left(z-\lambda_j\right)_+^{\sigma-1}<0.$$

Substituting this into (3.4), we have

$$\left(1+2C_\sigma\right)R_\sigma(z)\le 2zC_\sigma R_{\sigma-1}(z)$$

and (1.6) is proved.

Noting (2.1), it implies

$$\frac{1}{\sigma}R_\sigma'(z)\ge\left(1+\frac{1}{\sigma}\right)\frac{1}{z}R_\sigma(z)$$

and (1.7) is proved.

Similarly,

$$\left(\frac{R_\sigma(z)}{z^{\sigma+1}}\right)'=\frac{R_\sigma'(z)z^{\sigma+1}-R_\sigma(z)(\sigma+1)z^\sigma}{z^{2(\sigma+1)}}=\frac{z^\sigma\left[zR_\sigma'(z)-(\sigma+1)R_\sigma(z)\right]}{z^{2(\sigma+1)}}\ge 0,$$

thus $\frac{R_\sigma(z)}{z^{\sigma+1}}$ is a nondecreasing function of z.

Corollary 3.1 *For all* $\sigma\ge 2$ *and* $z\ge(1+\sigma)\lambda_1$,

$$\sigma^\sigma\lambda_1^{-1}\left(\frac{z}{1+\sigma}\right)^{1+\sigma}\le R_\sigma(z)\le L_{\sigma,2}^{cl}|\Omega|z^{\sigma+1}, \tag{3.6}$$

where $L_{\sigma,2}^{cl}=\frac{\Gamma(\sigma+1)}{4\pi\Gamma(\sigma+2)}$.

Proof. 1) Noting $R_\sigma(z_0)=\sum_k\left(z_0-\lambda_k\right)_+^\sigma\ge\left(z_0-\lambda_1\right)_+^\sigma$, for any $z_0>\lambda_1$, it follows from Theorem 1.1 that for all $z\ge z_0$,

$$\frac{R_\sigma(z)}{z^{\sigma+1}}\ge\frac{R_\sigma(z_0)}{z_0^{\sigma+1}}\ge\frac{\left(z_0-\lambda_1\right)_+^\sigma}{z_0^{\sigma+1}}.$$

So

$$R_\sigma(z)\ge\left(z_0-\lambda_1\right)_+^\sigma\left(\frac{z}{z_0}\right)^{\sigma+1}. \tag{3.7}$$

Since (3.7) holds for arbitrary $z_0>\lambda_1$, it yields

$$R_\sigma(z)\ge\max_{z_0>\lambda_1}\left[\left(z_0-\lambda_1\right)_+^\sigma\left(\frac{z}{z_0}\right)^{\sigma+1}\right].$$

Due to

$$\left[\left(z_0-\lambda_1\right)_+^\sigma\left(\frac{1}{z_0}\right)^{\sigma+1}\right]'=\frac{\sigma\left(z_0-\lambda_1\right)_+^{\sigma-1}z_0^{(\sigma+1)}-(\sigma+1)\left(z_0-\lambda_1\right)_+^\sigma z_0^\sigma}{z_0^{2(\sigma+1)}}=\frac{\left(z_0-\lambda_1\right)_+^{\sigma-1}\left[\sigma z_0-(\sigma+1)\left(z_0-\lambda_1\right)_+\right]}{z_0^{\sigma+2}},$$

we see that when $z_0=(\sigma+1)\lambda_1$, it gets

$$\max_{z_0>\lambda_1}\left[\left(z_0-\lambda_1\right)_+^\sigma\left(\frac{z}{z_0}\right)^{\sigma+1}\right]=\sigma^\sigma\lambda_1^{-1}\left(\frac{z}{1+\sigma}\right)^{1+\sigma}.$$

For $z\ge z_0=(\sigma+1)\lambda_1$, we have

$$R_\sigma(z)\ge\sigma^\sigma\lambda_1^{-1}\left(\frac{z}{1+\sigma}\right)^{1+\sigma}$$

and the inequality in the left-hand side of (3.6) is valid.

2) By the Berezin-Lieb inequality (see [11]), we have

$$\frac{R_\sigma(z)}{z^{\sigma+1}}\to L_{\sigma,2}^{cl}|\Omega|,\ z\to\infty.$$

Notice that $\frac{R_\sigma(z)}{z^{\sigma+1}}$ is nondecreasing to z, it follows

$$\frac{R_\sigma(z)}{z^{\sigma+1}}\le L_{\sigma,2}^{cl}|\Omega|$$

and the inequality in the right-hand side of (3.6) is proved.

Corollary 3.2 1) *For* $1 \le \sigma \le 2$ *and* $z \ge (\sigma+2)\lambda_1$,

$$R_\sigma(z) \ge \frac{(\sigma+1)^\sigma}{(\sigma+2)^{\sigma+1}} \lambda_1^{-1} z^{\sigma+1}. \quad (3.8)$$

2) For $0 \le \sigma < 1$ and $z \ge (\sigma+3)\lambda_1$,

$$R_\sigma(z) \ge \frac{3(\sigma+2)^{\sigma+1}}{2(\sigma+3)^{\sigma+2}} \lambda_1^{-1} z^{\sigma+1}. \quad (3.9)$$

Proof. 1) By Corollary 3.1 we know that for $1 \le \sigma \le 2$ and $z \ge (\sigma+2)\lambda_1$, it holds

$$R_{\sigma+1}(z) \ge (\sigma+1)^{\sigma+1} \lambda_1^{-1} \left(\frac{z}{\sigma+2}\right)^{\sigma+2}. \quad (3.10)$$

Using Theorem 1.1, we have

$$R_\sigma(z) \ge \left(1+\frac{1}{\sigma+1}\right)\frac{1}{z} R_{\sigma+1}(z), \text{ for } 1 \le \sigma \le 2. \quad (3.11)$$

Combining (3.10) and (3.11), it follows

$$R_\sigma(z) \ge \left(1+\frac{1}{\sigma+1}\right)\frac{1}{z}(\sigma+1)^{\sigma+1} \lambda_1^{-1}\left(\frac{z}{\sigma+2}\right)^{\sigma+2} = \frac{(\sigma+1)^\sigma}{(\sigma+2)^{\sigma+1}} \lambda_1^{-1} z^{\sigma+1}$$

and (3.8) is proved.

2) By Corollary 3.1, it shows that for $0 \le \sigma < 1$ and $z \ge (\sigma+3)\lambda_1$, it holds

$$R_{\sigma+2}(z) \ge (\sigma+2)^{\sigma+2} \lambda_1^{-1} \left(\frac{z}{\sigma+3}\right)^{\sigma+3}. \quad (3.12)$$

From Theorem 1.1, we see that for $0 \le \sigma < 1$,

$$R_\sigma(z) \ge \frac{3}{2z} R_{\sigma+1}(z) \ge \frac{9}{4z^2} R_{\sigma+2}(z). \quad (3.13)$$

In the light of (3.12) and (3.13), it obtains

$$R_\sigma(z) \ge \frac{9}{4z^2} R_{\sigma+2}(z) \ge \frac{9}{4z^2}(\sigma+2)^{\sigma+2} \lambda_1^{-1}\left(\frac{z}{\sigma+3}\right)^{\sigma+3} = \frac{3(\sigma+2)}{2(\sigma+3)} \cdot \frac{3(\sigma+2)^{\sigma+1}}{2(\sigma+3)^{\sigma+2}} \lambda_1^{-1} z^{\sigma+1}.$$

Noting that $\frac{3(\sigma+2)}{2(\sigma+3)} = \frac{3}{2}\left(1-\frac{1}{\sigma+3}\right) \ge 1$, for $0 \le \sigma < 1$, we have

$$R_\sigma(z) \ge \frac{3(\sigma+2)^{\sigma+1}}{2(\sigma+3)^{\sigma+2}} \lambda_1^{-1} z^{\sigma+1}$$

and (3.9) is proved.

Remark 3.3 *Specially*, *we have*

$$R_1(z) \ge \frac{3}{2z} R_2(z) \ge \frac{2}{9} \lambda_1^{-1} z^2, \quad (3.14)$$

$$N(z) = R_0(z) \ge \frac{3}{2z} R_1(z) \ge \frac{9}{4z^2} R_2(z) \ge \frac{z}{3\lambda_1}. \quad (3.15)$$

4. Proof of Theorem 1.2

Denote

$$\overline{\lambda_j} = \frac{1}{j}\sum_{l \le j} \lambda_l \quad \text{and} \quad \overline{\lambda_j^2} = \frac{1}{j}\sum_{l \le j} \lambda_l^2,$$

and let $ind(z)$ be the greatest integer i such that $\lambda_i \le z$.

Let $ind(z) = i$, it implies that $\lambda_i \le z$ and $\lambda_{i+1} > z$, so

$$\begin{aligned} R_2(z) &= \sum_k (z-\lambda_k)_+^2 \\ &= (z-\lambda_1)^2 + (z-\lambda_2)^2 + \cdots + (z-\lambda_i)^2 \\ &= iz^2 - 2z(\lambda_1+\lambda_2+\cdots+\lambda_i) + (\lambda_1^2+\lambda_2^2+\cdots+\lambda_i^2) \\ &= iz^2 - 2iz\frac{\lambda_1+\lambda_2+\cdots+\lambda_i}{i} + i\frac{\lambda_1^2+\lambda_2^2+\cdots+\lambda_i^2}{i} \\ &= i\left(z^2 - 2z\overline{\lambda_i} + \overline{\lambda_i^2}\right) \\ &= ind(z)\left(z^2 - 2z\overline{\lambda_{ind(z)}} + \overline{\lambda_{ind(z)}^2}\right). \end{aligned} \quad (4.1)$$

For any integer j and $z \ge \lambda_j$, it implies $ind(z) \ge j$, and

$$R_2(z) \ge Q(z, j) := j\left(z^2 - 2z\overline{\lambda_j} + \overline{\lambda_j^2}\right)$$

Using Theorem 1.1, we have that for $z \ge z_j \ge \lambda_j$,

$$\frac{R_2(z)}{z^3} \ge \frac{Q(z_j, j)}{z_j^3}$$

or

$$R_2(z) \ge Q(z_j, j)\left(\frac{z}{z_j}\right)^3. \quad (4.2)$$

By the Cauchy-Schwarz inequality, it follows

$$\overline{\lambda_j}^2 \le \overline{\lambda_j^2}$$

and

$$\begin{aligned} Q(z_j, j) &= j\left(z^2 - 2z\overline{\lambda_j} + \overline{\lambda_j^2}\right) \\ &= j\left(z^2 - 2z\overline{\lambda_j} + \overline{\lambda_j}^2 + \overline{\lambda_j^2} - \overline{\lambda_j}^2\right) \\ &= j\left[\left(z-\overline{\lambda_j}\right)^2 + \left(\overline{\lambda_j^2} - \overline{\lambda_j}^2\right)\right] \\ &\ge j\left(z-\overline{\lambda_j}\right)^2. \end{aligned} \quad (4.3)$$

Proof of Theorem 1.2 1) Substituting $z_j = 3\overline{\lambda_j}$ into (4.2) and noticing (4.3), we have

$$R_2(z) \geq j\left(z_j - \overline{\lambda_j}\right)^2 \frac{z^3}{z_j^3} = \frac{4jz^3}{27\overline{\lambda_j}}$$

and (1.8) is proved.

2) We take (1.8) into (3.14) to obtain

$$R_1(z) \geq \frac{3}{2z} \cdot \frac{4jz^3}{27\overline{\lambda_j}} = \frac{2jz^2}{9\overline{\lambda_j}}$$

and (1.9) is proved.

3) Combining (1.8) and (3.15), it implies

$$N(z) = R_0(z) \geq \frac{9}{4z^2} \cdot \frac{4jz^3}{27\overline{\lambda_j}} = \frac{jz}{3\overline{\lambda_j}}$$

and (1.10) is proved.

4) If $\lambda_{k+1} \leq 3\overline{\lambda_j}$, then (1.11) is clearly valid; if $\lambda_{k+1} > 3\overline{\lambda_j}$, then (1.10) shows by letting $z \to \lambda_{k+1}$ that

$$\frac{\lambda_{k+1}}{\overline{\lambda_j}} \leq \frac{3k}{j}.$$

So (1.11) is proved and Theorem 1.2 is proved. □

Corollary 4.1 *We have*

$$\lambda_{k+1} \leq 3\overline{\lambda_k}$$

and

$$\lambda_{k+1} \leq 3k\lambda_1. \tag{4.4}$$

5. Proof of Theorem 1.3

We first recall the following definition before proving Theorem 1.3.

Definition 5.1 *If* $f(z)$ *is superlinear in z as* $z \to \infty$, *then its Legendre transform is defined by*

$$L[f](w) = \sup_z \{wz - f(z)\}. \tag{5.1}$$

Remark 5.2 *If* $f(z) \geq g(z)$ *for all* z, *then* $L[f](w) \leq L[g](w)$ *for all* w; *Since the maximizing value of* z *in* (5.1) *is a nondecreasing function of* w, *it follows that for* w *sufficiently large, the maximizing* z *exceeds* $z_j = 3\overline{\lambda_j}$.

Proof of Theorem 1.3 From (1.9), we have

$$L[R_1](w) \leq L\left[\frac{2jz^2}{9\overline{\lambda_j}}\right](w). \tag{5.2}$$

Now let us calculate $L[R_1](w)$. Since

$$R_1(z) = \sum_k (z - \lambda_k)_+$$

is piecewise linear function of z, it implies that the maximizing value of z in the Legendre transform of R_1 is attained at one of the critical values.

In fact if $\lambda_k < z \leq \lambda_{k+1}$, then

$$\begin{aligned} L[R_1](w) &= \sup_z \{wz - R_1(z)\} \\ &= \sup_z \left\{wz - \sum_k (z-\lambda_k)_+\right\} \\ &= \sup_z \{wz - (z-\lambda_1) - (z-\lambda_2) - \cdots - (z-\lambda_k)\} \\ &= \sup_z \{(w-k)z + \lambda_1 + \lambda_2 + \cdots + \lambda_k\}. \end{aligned}$$

Noting that the maximizing value of z is a non-decreasing function of w, we see $w - k \geq 0$, therefore the critical value $z_{cr} = \lambda_{k+1}$.

It is easy to check $k = [w]$ and

$$\begin{aligned} L[R_1](w) &= \sup_z \{(w-k)z + \lambda_1 + \lambda_2 + \cdots + \lambda_k\} \\ &= (w - [w])\lambda_{[w]+1} + [w] \cdot \frac{\lambda_1 + \lambda_2 + + \lambda_{[w]}}{[w]} \\ &= (w - [w])\lambda_{[w]+1} + [w]\overline{\lambda_{[w]}}, \end{aligned} \tag{5.3}$$

Next we calculate $L\left[\frac{2jz^2}{9\overline{\lambda_j}}\right](w)$. Noting

$$L\left[\frac{2jz^2}{9\overline{\lambda_j}}\right](w) = \sup_z \left\{wz - \frac{2jz^2}{9\overline{\lambda_j}}\right\}$$

and letting

$$f(z) = wz - \frac{2jz^2}{9\overline{\lambda_j}},$$

we know $f'(z) = w - \frac{4jz}{9\overline{\lambda_j}}$. By $f'(z) = 0$, it solves

$$z_* = \frac{9w\overline{\lambda_j}}{4j}. \tag{5.4}$$

Therefore

$$\begin{aligned} L\left[\frac{2jz^2}{9\overline{\lambda_j}}\right](w) &= \sup_z \left\{wz - \frac{2jz^2}{9\overline{\lambda_j}}\right\} \\ &= w \cdot \frac{9w\overline{\lambda_j}}{4j} - \frac{2j}{9\overline{\lambda_j}} \cdot \left(\frac{9w\overline{\lambda_j}}{4j}\right)^2 \\ &= \frac{9\overline{\lambda_j}w^2}{8j}. \end{aligned} \tag{5.5}$$

Taking (5.3) and (5.5) into (5.2), we have

$$(w - [w])\lambda_{[w]+1} + [w]\overline{\lambda_{[w]}} \leq \frac{9\overline{\lambda_j}w^2}{8j}, \tag{5.6}$$

By (5.4), it has

$$w = \frac{4j}{9\overline{\overline{\lambda_j}}} z_*.$$

From Theorem 1.2, $z_* \geq 3\overline{\lambda_j}$, so $w \geq \frac{4j}{9\overline{\overline{\lambda_j}}} \cdot 3\overline{\lambda_j} = \frac{4j}{3}$.

Then it follows that if w is restricted to the value $w \geq \frac{4j}{3}$, then (5.6) is valid.

Meanwhile, for any w, we can always find an integer k such that $k-1 \leq w < k$ and

$$[w] = k - 1.$$

If $k > \frac{4j}{3}$ and w approaches to k from below, then we obtain from (5.5) that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = \lambda_k + (k-1)\overline{\lambda_{k-1}} \leq \frac{9\overline{\lambda_j}}{8j} k^2.$$

Therefore

$$\frac{\overline{\lambda_k}}{\overline{\overline{\lambda_j}}} \leq \frac{9k}{8j}.$$

and Theorem 1.3 is proved. □

Remark 5.3 *If we let* $j = 1$, *then*

$$\frac{\overline{\lambda_k}}{\lambda_1} \leq \frac{9}{8} k. \quad (5.7)$$

We point out that (5.7) is sharper than (4.4). In fact, we get from (4.4) that

$$\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{\lambda_1} \leq 3 \sum_{j=0}^{k-1} j = \frac{3k(k-1)}{2} \leq \frac{3}{2} k^2$$

and

$$\frac{\overline{\lambda_k}}{\lambda_1} \leq \frac{3}{2} k.$$

But $\frac{9k}{8} < \frac{3k}{2}$ is always valid, so (5.7) is sharper than (4.4).

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