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The Harmonic Functions on a Complete Asymptotic Flat Riemannian Manifold*

Huashui Zhan
School of Sciences, Jimei University, Xiamen, China
E-mail: hszhan@jmu.edu.cn
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Abstract

Let M be a simply connected complete Riemannian manifold with dimension $n \geq 3$. Suppose that the sectional curvature satisfies $-b^2 \leq K_M(\rho) \leq -\frac{a^2}{1+\rho}$, where ρ is distance function from a base point of M, a, b are constants and $ab \neq 0$. Then there exist harmonic functions on M.

Keywords: Harmonic Function, Riemannian Manifold, Negative Sectional Curvature

1. Introduction

The existence of the harmonic functions on a complete Riemannian manifold is a well known problem. In what follows, we consider the harmonic function f is not a constant function, that is, $f \neq c$, $c =$ constant. If there is no restrictions imposed on the curvature, then it was proved [1] that there does not exist a harmonic function of the form $L^p(M)$, $1 < p < \infty$, on the manifold. If $p = \infty$, then it was proved [1] that there dose not exist any bounded harmonic function on a complete manifold with nonnegative Ricci curvature. On the other hand, by introducing the sphere at infinity $S(\infty)$, Anderson-Scheon [2] and Sullivan [3] succeeded to prove the existence of the bounded harmonic functions on a complete simply-connected manifold with

$$-b^2 \leq K_M \leq -a^2 < 0,$$

where K_M represents the sectional curvature and $a \neq 0$, $b \neq 0$ are constants. It is naturally to consider whether the same conclusion holds only on the manifold with negative sectional curvature, *i.e.* $-b^2 \leq K_M < 0$? However, this is still an open problem.

Let M be a complete manifold and $o \in M$ be fixed. Then we write

$$K_o^{\min}(\rho) \geq c(\rho), \tag{1}$$

if for any minimal geodesic γ issuing from o, the sectional curvature of the plane which is tangent to γ is greater than or equal to $c(\rho)$, where $c(\rho)$ is a monotone increasing function and ρ is the distance function from the base point o in the manifold. This notion was first introduced by Klingenberg [4]. By using the Toponogov-type comparison theorem with $K_o^{\min} \geq c$ in [5,6], and using the approach of Anderson and Scheon [2], we are able to prove the following result:

Theorem 1. *Let M be a complete simply-connected Riemannian manifold with dimension $n \geq 3$. If*

$$-b^2 \leq K_M^{\min}(\rho) \leq K_M^{\max}(\rho) \leq -\frac{a^2}{1+\rho}, \tag{2}$$

with

$$1 + b < 2a,\ 2(n-1) + (2a - 1 - b) > 7, \tag{3}$$

then there exist bounded harmonic functions on the manifold M, where ρ is distance function from a given base point o in M, $ab \neq 0$.

A special case of the manifolds satisfying the theorem 1 is with the following sectional curvature condition

$$-\frac{a^2}{1+\rho} \leq K_M^{\min}(\rho) \leq K_M^{\max}(\rho) \leq -\frac{a^2}{1+\rho}. \tag{4}$$

In general, since ρ is large enough, the curvature in (4) is close to 0, one would conjecture that the behavior of this manifold would be much closer to the Euclidean spaces and hence there may not exist any bounded harmonic functions. Our theorem states that this conclusion is not true.

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2. The proof of Theorem 1

Let $M^2(c)$ be the complete simply connected surface of constant curvature c. We also assume that all geodesics have unit speed.

Lemma 2. ([5-7]). *Let M be a complete Riemannian manifold and o be a point of M with $K_o^{\min} \geq c$.*

1) Let $\gamma_i:[o,l_i] \to M$, $i=0,1,2$ be minimal geodesics with $\gamma_1(0)=\gamma_2(l_2)=o$, $\gamma_0(0)=\gamma_1(l_1)$ and $\gamma_2(0)=\gamma_0(l_0)$. Then, there exist minimal geodesics $\tilde{\gamma}_i:[o,l_i] \to M^2(c)$, $i=0,1,2$ with $\tilde{\gamma}_1(0)=\tilde{\gamma}_2(l_2)$, $\tilde{\gamma}_0(0)=\tilde{\gamma}_1(l_1)$ and $\tilde{\gamma}_2(0)=\tilde{\gamma}_0(l_0)$ such that

$$L(\gamma_i)=L(\tilde{\gamma}_i), \quad i=0,1,2$$

and

$$\angle(-\gamma_1'(l_1),\gamma_0'(0)) \geq \angle(-\tilde{\gamma}_1'(l_1),\tilde{\gamma}_0'(0)),$$

$$\angle(-\gamma_0'(l_0),\gamma_2'(0)) \geq \angle(-\tilde{\gamma}_0'(l_0),\tilde{\gamma}_2'(0)).$$

2) Let $\gamma_i:[o,l_i] \to M$, $i=1,2$ be two minimizing geodesics starting from p. Let $\tilde{\gamma}_i:[o,l_i] \to M^2(c)$, $i=0,1,2$ be minimizing geodesics starting from same point such that $\angle(-\gamma_1'(0),\gamma_2'(0)) \geq \angle(-\tilde{\gamma}_1'(0),\tilde{\gamma}_2'(0))$. Then

$$d(\gamma_1(l_1),\gamma_2(l_2)) \leq d_c(\tilde{\gamma}_1(l_1),\tilde{\gamma}_2(l_2)),$$

where d_c denotes the distance function in $M^2(c)$.

If $K_o^{\max}(\rho) \leq c$, then we have the parallel result as Lemma 2.

Let M be a complete Riemannian manifold, o a point of M with $K_o^{\max}(\rho) \leq c(\rho)$, and $c(\rho)$ a monotone increasing function. For any given $\rho_0 > 0$, it is obvious that $K_o^{\max}(\rho) \leq c(\rho_0)$, $0 \leq \rho \leq \rho_0$. By Lemma 1 and the hyperbolic cosine theorem in $M^2(c(\rho_0))$, we can easily prove the following lemma

Lemma 3. *Let M be a complete Riemannian manifold and o a point of M. For any given $r>0$, let o_1, x_1, x_2 be three points in M such that $\rho = d(o_1,x_1)=d(o_1,x_2)$. Suppose that* (2) *is true. Denote the ray from o to x_1 by γ_1, the ray from o to x_2 by γ_2 and the angle of γ_1 and γ_2 at o by θ;*

$$\begin{aligned} &2\rho + \frac{2\sqrt{1+\rho+d(o,o_1)}}{a}(\ln\theta+1) \\ &\leq d(x_1,x_2) \leq 2\rho + \frac{2}{b}(\ln\theta+1), \end{aligned} \tag{3}$$

where ρ is large enough and θ is small enough.

Now we consider a simply connected Riemannian manifold M with negative sectional curvature. As usual, two rays γ_1 and γ_2 on M are equivalent, that is, $\gamma_1 \sim \gamma_2$ if and only if $d(\gamma_1(t)\ \gamma_2(t)) \leq c$, for all $t \geq 0$. If we denote the set of all rays in M by Γ, then the Matrin boundary at infinity is defined as $\Gamma/\sim$.

If γ_1 and γ_2 are emanating from the same point o of M, $\gamma_1(0)=\gamma_2(0)$, from (2),

$$2\rho + \frac{2\sqrt{1+\rho+d(o,o_1)}}{a}(\ln\theta+1) \leq c,$$

as $t \to 0$, $\theta = 0$. This means that $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 = \gamma_2$. Then, $\Gamma/\sim = S(\infty) = \{\gamma : \text{remanating from a fixed point } o\}$ and it is equivalent to the unite sphere S_o in T_oM.

Moreover, by Lemma 3, we can also construct a C^α topological structure on $\bar{M} = M \cup S(\infty)$ as [1]. By using this fact, we can prove the following Theorem 4, and Theorem 1 as its Corollary.

Theorem 4. *Let M be a simply connected Riemannian manifold with* (2) *and* (3). *For any $\varphi \in C^0(S(\infty))$, there is a unique harmonic function $u \in C^\infty(M) \cap C^0(\bar{M})$ such that $u|_{S(\infty)} = \varphi$.*

Proof: We first fix the base point o. Let $S(\infty)$ be equivalent to the unit tangent sphere $S_o(1)=S^{n-1}$. From [1], without loss of generality, we may assume that $\varphi \in C^\infty(S_o(1))$. Since $K_M \leq 0$, M is diffeomorphic to R^n. Denote $\{(r,\theta) | \theta \in S_o(1) = S^{n-1}\}$ as the normal coordinate around o. Then, $\varphi = \varphi(\theta)$, $\theta \in S_o(1)$. Now, we define an extension of φ and still denote it by φ, so that

$$\varphi(r,\theta) = \varphi(\theta), \text{for all } r > 0.$$

Then φ is a differential function on $M \setminus \{0\}$. Write

$$\operatorname{osc}_{B_x(1)} \varphi = \sup_{y \in B_x(1)} |\varphi(y) - \varphi(x)|,$$

Now, we proceed to prove Theorem 4 via the following steps:

1) $\operatorname{osc}_{B_x(1)} \varphi = O\left(e^{-\frac{1}{\sqrt{1+\rho}}\rho}\right)$. According to the definition of φ, if $y \in B_x(1)$, then

$$|\varphi(y)-\varphi(x)| = |\varphi(\theta)-\varphi(\theta')| \leq c|\theta-\theta'|,$$

where θ, θ' is the geodesic sphere coordinate of y, x, respectively. By Lemma 3, we have

$$2\rho(x) + 2\sqrt{1+\rho}(\ln(\theta'-\theta)-1) \leq \rho_x(y) \leq 1,$$

$$|\theta'-\theta| \leq c e^{-\frac{1}{\sqrt{1+\rho}}\rho}.$$

2) Consider $\varphi \to \tilde{\varphi}$ such that $\Delta\tilde{\varphi} = O\left(e^{-\frac{1}{\sqrt{1+\rho}}\rho}\right)$.

Let $\chi \in C_0^\infty(R)$, $1 \geq \chi \geq 0$, $\sup p\ \chi \subset [-1,1]$. Set

$$\tilde{\varphi} = \frac{\int_M \chi\left(\rho_x^2(y)\right)\varphi(y)\,dy}{\int_M \chi\left(\rho_x^2(y)\right)dy}.$$

Then

$$|\varphi - \tilde{\varphi}| = \frac{\left|\int_{B_x(1)} \chi\left(\rho_x^2(y)\right)\left(\varphi(y) - \varphi(x)\right)dy\right|}{\int_{B_x(1)} \chi\left(\rho_x^2(y)\right)dy} \le \sup_{B_x(1)} |\varphi(y) - \varphi(x)| = \operatorname{osc}_{B_x(1)}\varphi.$$

At the same time, we have

$$\Delta\tilde{\varphi}(x_0) = \Delta\left(\tilde{\varphi}(x) - \varphi(x_0)\right)\Big|_{x=x_0} = \int_M \Delta\left(\frac{\chi\left(\rho_y^2(x)\right)}{\int_M \chi\left(\rho_y^2(x)\right)dx}\right)\left(\tilde{\varphi}(x) - \varphi(x_0)\right)dy\Big|_{x=x_0}.$$

Now it is not difficult to show that (c.f.[1,2])

$$\left|\Delta\tilde{\varphi}(x)\right| \le c \cdot \operatorname{osc}_{B_x(1)}\varphi.$$

3) Consider the function $g = \mathrm{e}^{-\frac{c}{\sqrt{1+\rho}}\rho} \in C^\infty, \delta(x) = \frac{c}{\sqrt{1+\rho}} = c(1+\rho)^{-\frac{1}{2}}$. Then,

$$\nabla\delta = -\frac{c}{2}(1+\rho)^{-\frac{3}{2}}\nabla\rho,$$

$$\Delta\delta = -\frac{c}{2}\left[-\frac{3}{2}(1+\rho)^{-\frac{5}{2}}|\nabla\rho|^2 + (1+\rho)^{-\frac{3}{2}}\Delta\rho\right] = -\frac{c}{2}\left[-\frac{3}{2}(1+\rho)^{-\frac{5}{2}} + (1+\rho)^{-\frac{3}{2}}\Delta\rho\right].$$

The last equality is due to $|\nabla\rho|^2 \equiv 1$. Hence, we deduce the followings:

$$\nabla g = -\mathrm{e}^{-\delta(x)\rho(x)}\left[\nabla\delta(x)\rho(x) + \delta(x)\nabla\rho(x)\right],$$

$$\begin{aligned}
\Delta g &= \mathrm{e}^{-\delta(x)\rho(x)}\left[\nabla\delta(x)\rho(x) + \delta(x)\nabla\rho(x)\right]^2 - \mathrm{e}^{-\delta(x)\rho(x)}\left(\Delta\delta(x)\rho(x) + 2\nabla\delta(x)\cdot\nabla\rho(x) + \delta(x)\Delta\rho(x)\right) \\
&= \mathrm{e}^{-\delta(x)\rho(x)}\left[|\nabla\delta(x)|^2\rho(x) + \delta^2(x) + 2\delta(x)\rho(x)\nabla\delta(x)\cdot\nabla\rho(x) - \Delta\delta(x)\rho(x) - 2\nabla\delta(x)\cdot\nabla\rho(x) - \delta(x)\Delta\rho(x)\right] \\
&= \mathrm{e}^{-\delta\rho}c^2\left[\frac{1}{4}(1+\rho)^{-3}\rho + (1+\rho)^{-1} - (1+\rho)^{-2}\rho\right] + c\mathrm{e}^{-\delta\rho}\left[-\frac{3}{4}(1+\rho)^{-\frac{5}{2}}\rho + (1+\rho)^{-\frac{3}{2}} + \frac{1}{2}\rho(1+\rho)^{-\frac{3}{2}}\Delta\rho - (1+\rho)^{-\frac{1}{2}}\Delta\rho\right].
\end{aligned} \tag{4}$$

For any fixed point $p \in M$, denote $\rho_0 = d(o,p)$, and denote $\rho = d(o,x)$ for any $x \in \overline{B(o,\rho_0)}$. Then

$$-b^2 \le K_M^{\min}(\rho) \le K_M^{\max}(\rho) \le -\frac{a^2}{1+\rho_0}, \tag{5}$$

which means that

$$\frac{(n-1)a}{\sqrt{1+\rho_0}} \le \frac{(n-1)a}{\sqrt{1+\rho_0}} \coth \frac{a(n-1)}{\sqrt{1+\rho_0}}\rho \le \Delta\rho \le \frac{n-1}{\rho}(1+b), \tag{6}$$

by (4), we have

$$\begin{aligned}
\Delta g &\le c^2\mathrm{e}^{-\delta\rho}\left[\frac{\rho}{4(1+\rho)^3} + \frac{1}{1+\rho} - \frac{\rho}{(1+\rho)^2}\right] \\
&\quad + c\mathrm{e}^{-\delta\rho}\left[-\frac{3\rho}{4(1+\rho)^{\frac{5}{2}}} + \frac{1}{(1+\rho)^{\frac{3}{2}}} + \frac{(n-1)(1+b)}{2(1+\rho)^{\frac{3}{2}}} - \frac{1}{(1+\rho)^{\frac{1}{2}}}\frac{(n-1)a}{\sqrt{1+\rho_0}}\coth\frac{a(n-1)}{\sqrt{1+\rho_0}}\rho\right] \\
&\le \mathrm{e}^{-\delta\rho}\left[\frac{\rho c^2}{4(1+\rho)^3} + \frac{c^2}{1+\rho} - \frac{\rho c^2}{(1+\rho)^2}\right] + ce^{-\delta\rho}\left[-\frac{3\rho}{4(1+\rho)^{\frac{5}{2}}} + \frac{1}{(1+\rho)^{\frac{3}{2}}} + \frac{(n-1)(1+b)}{2(1+\rho)^{\frac{3}{2}}} - \frac{(n-1)a}{(1+\rho)}\right] \\
&= \mathrm{e}^{-\delta\rho}\left[\frac{\rho c^2}{4(1+\rho)^2}\left(\frac{1}{4(1+\rho)} - 1\right) + \frac{c^2}{1+\rho} - \frac{3(1+\rho)c}{4(1+\rho)^{\frac{5}{2}}} + \frac{3c}{4(1+\rho)^{\frac{5}{2}}} + \frac{c}{(1+\rho)^{\frac{3}{2}}} + \frac{(n-1)(1+b)c}{2(1+\rho)^{\frac{3}{2}}} - \frac{(n-1)ac}{(1+\rho)}\right] \\
&\le \mathrm{e}^{-\delta\rho}\left[\frac{c^2}{1+\rho} + \frac{7 + 2(n-1)(1+b) - 4(n-1)ac}{4(1+\rho)}\right] < 0
\end{aligned}$$

provided that c is small enough and by the conditions

$$1+b<2a,\ 2(n-1)+(2a-1-b)>7.$$

Hence,

$$\Delta g<0.$$

It is obvious that $\mathrm{e}^{-\delta(x)\rho(x)} \geq \mathrm{e}^{-\frac{c}{\sqrt{1+\rho}}\rho(x)} \frac{1}{1+\rho}$ so that there exists a constant c_1 such that

$$\Delta(c_1 g) \leq -|\Delta\tilde{\varphi}|.$$

According to the well-known Perron canonical harmonic function theorem, the barrier functions $\tilde{\varphi}+cg$ and $\tilde{\varphi}-cg$ assure that there exists a harmonic function u satisfying

$$\tilde{\varphi}+c_1 g \geq u \geq \tilde{\varphi}-c_1 g.$$

Now, it is easy to verify that u satisfies the boundary conditions. Thus, Theorem 4 is proved.

3. References

[1] S. T. Yau and R. Scheon, "Differential Geometry," Science Publishing Company of China, Beijing, 1988, p. 37.

[2] M. T. Anderson and R. Scheon, "Positive Harmonic Functions on Complete Manifolds of Negative Curvature," *Annals of Mathematics*, Vol. 121, No. 3, 1985, pp. 429- 461. doi:10.2307/1971181

[3] D. Sullivan, "The Ditchless Problem at Infinity for a Negatively Curved Manifold," *Journal of Differential Geometry*, Vol. 18, 1983, pp. 723-732.

[4] W. Klingenberg, "Riemannian Geometry," De Grunter, Berlin, 1982.

[5] Y. Macrognathia, "Manifolds with Pinched Radial Curvature," *Proceedings of the American Mathematical Society*, Vol. 118, No. 3, 1993, pp. 975-985.

[6] Y. Macrognathia, "Complete Open Manifolds of Nonnegative Radial Curvature," *Pacific Journal of Mathematics*, Vol. 165, No. 1, 1994, pp. 153-160.

[7] C. Y. Xia, "Open Manifolds with Nonnegative Ricci Curvature and Large Volume Growth," *Commentarii Mathematici Helvetici*, Vol. 74, No. 3, 1999, pp. 456-466. doi:10.1007/s000140050099

