On the Increments of Stable Subordinators

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Abstract

Let \( \{X(t), t \geq 0\} \) be a stable subordinator defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( a_t \) for \( t > 0 \) be a non-negative valued function. In this paper, it is shown that under varying conditions on \( a_t \), there exists a function \( \lambda_p(t) \) such that

\[
\liminf_{t \to \infty} \frac{X(t + a_t) - X(t)}{\lambda_p(t)} = 1 \quad \text{a.s.},
\]

where \( \lambda_p(t) = \theta_a a_t^{\frac{1}{\alpha}} \left( \log \frac{t}{a_t} + \beta \log \log t + (1 - \beta) \log \log a_t \right)^{\frac{\alpha - 1}{\alpha}} \), \( 0 \leq \beta \leq 1 \),

\[
\theta_{\alpha} = \left( B(\alpha) \right)^{\frac{1 - \alpha}{\alpha}} \quad \text{and} \quad B(\alpha) = (1 - \alpha) a^{\frac{\alpha}{1 - \alpha}} \left( \cos \left( \frac{\pi \alpha}{2} \right) \right)^{\frac{1}{\alpha - 1}}.
\]

Keywords

Increments, Stable Subordinators, Iterated Logarithm Laws

1. Introduction

Let \( \{X(t), t \geq 0\} \) be a stableordinator with exponent \( \alpha \) with \( 0 < \alpha < 1 \), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \( a_t \) for \( t > 0 \) be a non-negative valued function and \( Y(t) = X(t + a_t) - X(t), t > 0 \). Define

\[
\lambda_p(t) = \theta_a a_t^{\frac{1}{\alpha}} \left( \log \frac{t}{a_t} + \beta \log \log t + (1 - \beta) \log \log a_t \right)^{\frac{\alpha - 1}{\alpha}},
\]

where \( 0 \leq \beta \leq 1 \),

\[
\theta_{\alpha} = \left( B(\alpha) \right)^{\frac{1 - \alpha}{\alpha}} \quad \text{and} \quad B(\alpha) = (1 - \alpha) a^{\frac{\alpha}{1 - \alpha}} \left( \cos \left( \frac{\pi \alpha}{2} \right) \right)^{\frac{1}{\alpha - 1}}.
\]
For any value of $t$, the characteristic function of $X(t)$ is of the form

$$E(e^{iuX(t)}) = \exp\left(-\frac{1}{2} - \frac{u}{\tan \left(\frac{\pi \alpha}{2}\right)}\right), \quad 0 < \alpha < 1.$$ 

Limit theorems on the increments of stable subordinators have been investigated in various directions by many authors [1]-[6]. Among the above many results, we are interested in Fristedt [4] and Vasudeva and Divanji [3] whose results are the following limit theorems on the increments of stable subordinators.

**Theorem 1** ([4])

$$\liminf_{t \to \infty} \frac{1}{t} \left(\log \log t\right)^{-1/\alpha} \frac{X(t)}{t} = 1 \quad \text{almost surely (a.s.).}$$

**Theorem 2** ([3]) Let $0 < a$, for $t > 0$, be a non-decreasing function of $t$ such that

(i) $0 < a \leq t$ for $t > 0$,

(ii) $a \to \infty$ as $t \to \infty$, and

(iii) $a/t$ is non-increasing. Then

$$\liminf_{t \to \infty} \frac{X(t + a) - X(t)}{\xi(t)} = 1 \quad \text{a.s.,}$$

where $\xi(t) = \theta_a a^{-\alpha} \left(\log \frac{t}{a} + \log \log t\right)^{-\alpha/\alpha}.$

In this paper, our aim is to investigate Liminf behaviors of the increments of $Y$. We establish that, under certain conditions on $a$,

$$\liminf_{t \to \infty} \frac{Y(t)}{\lambda_\beta(t)} = 1 \quad \text{a.s.,}$$

where $Y(t) = X(t + a) - X(t)$.

Throughout the paper $c$ and $k$ (integer), with or without suffix, stand for positive constants. i.o. means infinitely often. We shall define for each $u \geq 0$ the functions $\log u = \log (\max(u, 1))$ and $\log \log u = \log \log (\max(u, 3))$.

### 2. Main Result

In this section, we reformulate the result obtained in Theorem 2 and establish our main result using $\lambda_\beta(t)$ with $0 \leq \beta \leq 1$ instead of $\xi(t)$.

**Theorem 3** Let $a, \ t > 0$, be a non-decreasing function of $t$ such that

(i) $0 < a \leq t$ for $t > 0$,

(ii) $a \to \infty$ as $t \to \infty$, and

(iii) $a/t$ is non-increasing. Then

$$\liminf_{t \to \infty} \frac{Y(t)}{\lambda_\beta(t)} = 1 \quad \text{a.s.}$$

**Remark 1** Let us mention some particular cases

1. For $a = t$ we obtain Fristedt’s iterated logarithm laws (see Theorem 1).
2. If $\beta = 1$ we have Vasudeva and Divanji theorem (see Theorem 2).
3. If $\beta = 0$ under assumptions (i), (ii) and (iii) of Theorem 3 we also have

$$\liminf_{t \to \infty} \frac{Y(t)}{\lambda_{\beta}(t)} = 1 \quad \text{a.s.}$$

In order to prove Theorem 3, we need the following Lemma

**Lemma 1** (see [3] or [7]) Let $X$ be a positive stable random variable with characteristic function

$$E\left(\exp \{iuX\} \right) = \exp \left\{ -\left|\mu\right|^\alpha \left( 1 - \frac{iu}{|\mu|} \tan \left( \frac{\pi\alpha}{2} \right) \right) \right\}, \quad 0 < \alpha < 1.$$ 

Then, as $x \to 0$,

$$P\left(X \leq x\right) = \frac{\chi_{\alpha(1-\alpha)}}{\sqrt{2\pi B}\left(\alpha\right)} \exp \left\{ -B\left(\alpha\right) \frac{x^\alpha}{\alpha-1}\right\}$$

where

$$B\left(\alpha\right) = (1-\alpha)\alpha^{-\alpha} \left( \cos \left( \frac{\pi\alpha}{2} \right) \right)^{1-\alpha}.$$ 

**Proof of Theorem 3.** Firstly, we show that for any given $\varepsilon > 0$, as $t \to \infty$,

$$P\left(Y(t) \leq (1+\varepsilon)\lambda_{\beta}(t) \ i\alpha\right) = 1. \quad (3)$$

Let $u_k$ be a number such that $a_{u_k} > 1$. Define a sequence $(u_k)$ through $u_{k+1} = u_k + a_{u_k}$, for $k = 1, 2, \cdots$. Now we show that

$$P\left(Y(u_k) \leq (1+\varepsilon)\lambda_{\beta}(u_k) \ i\alpha\right) = 1.$$

From Mijhneer [8], we have

$$P\left(Y(u_k) \leq (1+\varepsilon)\lambda_{\beta}(u_k) \right) = P\left( X(1) \leq \frac{(1+\varepsilon)\lambda_{\beta}(u_k)}{a_{u_k}^{\alpha-1}} \right). \quad (4)$$

But

$$\frac{\lambda_{\beta}(u_k)}{a_{u_k}^{\alpha-1}} = \theta_x \left( \log \frac{u_k}{a_{u_k}} + \beta \log\log u_k + (1-\beta) \log\log a_{u_k} \right)^{\frac{\alpha-1}{\alpha}}.$$

Applying Lemma 1 with

$$x = (1+\varepsilon)\theta_x \left( \log \frac{u_k}{a_{u_k}} + \beta \log\log u_k + (1-\beta) \log\log a_{u_k} \right)^{\frac{\alpha-1}{\alpha}},$$

one can find a $k_0$ such that, for all $k \geq k_0$,.
\[ P \left( X(1) \leq \frac{(1+\varepsilon)\lambda_\beta(u_k)}{a_{n_k}} \right) \]

\[ \geq \frac{c_0}{2 \left( \log \left( \frac{u_k}{a_{n_k}} \right) \left( \log a_{n_k} \right)^{1-\beta} \right)^{1/2}} \]

\[ \times \exp \left\{ -(1+\varepsilon)^{(a-1)} \log \left( \frac{u_k}{a_{n_k}} \right) \left( \log a_{n_k} \right)^{1-\beta} \right\}, \]

where \( c_0 \) is some positive constant. Notice that

\[ (1+\varepsilon)^{\frac{\alpha}{a-1}} = (1-\varepsilon_1) < 1 \] for some \( \varepsilon_1 > 0. \)

Hence

\[ P \left( X(1) \leq \frac{(1+\varepsilon)\lambda_\beta(u_k)}{a_{n_k}} \right) \]

\[ \geq \frac{c_0}{2 \left( \log \left( \frac{u_k}{a_{n_k}} \right) \left( \log a_{n_k} \right)^{1-\beta} \right)^{1/2}} \left( \frac{a_{n_k}}{u_k} \right) \]

\[ \times \left( \frac{u_k}{a_{n_k}} \right)^{\varepsilon_k} \frac{1}{\left( \log u_k \right)^{\beta} \left( \log a_{n_k} \right)^{1-\beta}^{(1-\varepsilon)}} \]

\[ = \frac{c_0}{2 \left( \log \left( \frac{u_k}{a_{n_k}} \right) \left( \log a_{n_k} \right)^{1-\beta} \right)^{1/2}} \left( \frac{u_{k+1} - u_k}{u_k} \right) \]

\[ \times \left( \frac{u_k}{a_{n_k}} \right)^{\varepsilon_k} \frac{1}{\left( \log u_k \right)^{\beta} \left( \log a_{n_k} \right)^{1-\beta}^{(1-\varepsilon)}}. \]

Let \( l_k = \frac{u_k}{a_{n_k}} \) and \( m_k = \left( \log u_k \right)^{\beta} \left( \log a_{n_k} \right)^{1-\beta} \). Note that \( l_k \) is non-decreasing and \( m_k \rightarrow \infty \) as \( k \rightarrow \infty \). In turn one finds a \( k_i \geq k_{n_i} \) such that

\[ \frac{l_{k_i}^{m_{k_i}}}{\left( \log l_{k_i} m_{k_i} \right)^{1/2}} \geq 1, \] whenever \( k \geq k_i. \)

Therefore, for all \( k \geq k_i \), we have
Observe that

\[
\int_{t_i}^\infty \frac{dr}{r \log t} \leq \sum_{k=i}^\infty \frac{(u_{k+1} - u_k)}{u_k \log u_k}.
\]  

(6)

From the fact that \( \int_{t_i}^\infty \frac{dr}{r \log t} = \infty \) and from (4), (5), and (6) one gets

\[
\sum_{k=1}^\infty P(Y(u_k) \leq (1 + \varepsilon) \lambda_\beta(u_k)) = \infty.
\]

Observe that \( \{Y(u_k)\} \) is a sequence of mutually independent random variables (for, \( u_{k+1} = u_k + a_{u_k} \)) and by applying Borel-Cantelli lemma, we get

\[
P(Y(u_k) \leq (1 + \varepsilon) \lambda_\beta(u_k) \ i.o. = 1
\]

which establishes (3).

Now we complete the proof by showing that, for any \( \varepsilon \in (0, 1) \),

\[
P(Y(t) \leq (1 - \varepsilon) \lambda_\beta(t) \ i.o. = 0.
\]  

(7)

Define a subsequence \( \{t_k\} \), such that

\[
a_{u_k} = (t_{k+1} - t_k) \left( \frac{(1 - \beta)(1 + \varepsilon)}{\log t_k} \right)^{1-\beta}, \quad k = 1, 2, \ldots
\]

(8)

and the events \( A_k \) and \( B_k \) as

\[
A_k = \{Y(t) \leq (1 - \varepsilon) \lambda_\beta(t)\}
\]

and

\[
B_k = \left\{ \inf_{t_k \leq t \leq t_{k+1}} Y(t) \leq (1 - \varepsilon) \lambda_\beta(t_{k+1}) \right\}, \quad k = 1, 2, \ldots
\]

Note that

\[
(A_k \ i.o., t \to \infty) \subset (B_k \ i.o., k \to \infty).
\]

Further, define

\[
C_k = \left\{ X(t_k + a_{u_k}) - X(t_{k+1}) \leq (1 - \varepsilon) \lambda_\beta(t_{k+1}) \right\}
\]

and observe that

\[
(B_k \ i.o., k \to \infty) \subset \left( C_k \ i.o., k \to \infty \right)
\]

Hence in order to prove (7) it is enough to show that

\[
P(C_k \ i.o.) = 0.
\]  

(9)

We have
\[ P\left(X(t_k + a_k) - X(t_{k+1})\right) \leq (1 - \epsilon) \lambda_p(t_{k+1}) = P\left(X(1) \leq \frac{(1 - \epsilon) \lambda_p(t_{k+1})}{(a_k + t_k - t_{k+1})^{\alpha/\alpha}}\right) \]

and

\[ \frac{(1 - \epsilon) \lambda_p(t_{k+1})}{(a_k + t_k - t_{k+1})^{\alpha/\alpha}} = (1 - \epsilon) \theta_a \left( \frac{a_{k+1}}{a_k} \right)^{1/\alpha} \left( \log \left( \frac{t_{k+1} \left( \log t_{k+1} \right)^{\beta} \left( \log a_k \right)^{1 - \beta}}{a_k} \right) \right)^{(\alpha - 1)/\alpha}. \]

The fact that \( a_k t_k \) is non-increasing as \( t \to \infty \) implies that

\[ \frac{a_{k+1}}{t_{k+1}} \leq \frac{a_k}{t_k} \quad \text{or} \quad \frac{a_{k+1}}{a_k} \leq \frac{t_{k+1}}{t_k}. \]

Hence for a given \( \epsilon > 0 \) satisfying \( (1 - \epsilon)(1 + \epsilon)^{\alpha/\alpha} < 1 \), there exists a \( k_2 \) such that

\[ a_{k+1}/a_k \leq (1 + \epsilon_k), \quad \text{for all } k \geq k_2. \]

Let \( (1 - \epsilon)(1 + \epsilon)^{\alpha/\alpha} = (1 - \epsilon_2). \) Then, for \( k \geq k_2, \)

\[ P(C_k) \leq P\left(X(1) \leq (1 - \epsilon_2) \theta_a \left( \frac{a_{k+1}}{a_k} \right)^{1/\alpha} \left( \log \left( \frac{t_{k+1} \left( \log t_{k+1} \right)^{\beta} \left( \log a_k \right)^{1 - \beta}}{a_k} \right) \right)^{(\alpha - 1)/\alpha}. \right) \]

From lemma 1, we can find a \( k_3 \geq k_2 \) such that for all \( k \geq k_3, \)

\[ P(C_k) \leq c_1 \left( \log \left( \frac{t_{k+1} \left( \log t_{k+1} \right)^{\beta} \left( \log a_k \right)^{1 - \beta}}{a_k} \right)^{1/\alpha} \right)^{1/2} \times \exp \left( (1 - \epsilon_2)^{\alpha/\alpha} \left( \log \left( \frac{t_{k+1} \left( \log t_{k+1} \right)^{\beta} \left( \log a_k \right)^{1 - \beta}}{a_k} \right)^{1/\alpha} \right) \right), \]

where \( c_1 \) is a positive constant.

Let \( (1 - \epsilon_2)^{1/\alpha} = (1 + \epsilon_3), \quad \epsilon_3 > 0. \) Then, for all \( k \geq k_3, \)

\[ P(C_k) \leq c_1 \left( \log \left( \frac{t_{k+1} \left( \log t_{k+1} \right)^{\beta} \left( \log a_k \right)^{1 - \beta}}{a_k} \right)^{1/\alpha} \right)^{1/2} \left( \frac{a_{k+1}}{a_k} \right)^{(1 + \epsilon_3)} \times \left( \left( \log t_{k+1} \right)^{\beta} \left( \log a_k \right)^{1 - \beta} \right)^{(1 + \epsilon_3)} \]

Since

\[ \left( a_{k+1}/t_{k+1} \right)^{(1 + \epsilon_3)} \leq \left( a_k/t_k \right)^{(1 + \epsilon_3)} \leq a_k/t_k, \]

then from (8) and for all \( k \geq k_3, \) we have

\[ P(C_k) \leq c_1 \left( \log \left( \frac{t_k \left( \log t_k \right)^{\beta} \left( \log a_k \right)^{1 - \beta}}{a_k} \right)^{1/\alpha} \right)^{1/2} \left( \frac{a_k}{t_k} \right)^{(1 + \epsilon_3)} \times \left( \left( \log t_k \right)^{\beta} \left( \log a_k \right)^{1 - \beta} \right)^{(1 + \epsilon_3)}. \]
\[
P(C_i) \leq c_i \left( \frac{t_{k+1} - t_k}{\log t_k} \right)^\beta \left( \frac{\log a_{k+1}}{\log a_k} \right)^{1/2} \left( \frac{t_{k+1} - t_k}{t_k} \right)
\]

\[
\leq c_i \left( \frac{t_{k+1} - t_k}{t_k} \right) \frac{1}{(\log t_k)^{1/\lambda}}.
\]

Observe that
\[
\int_{t_k}^\infty \frac{dr}{r^{1/\lambda}(\log r)^{1/\lambda}} \geq \sum_{k=0}^\infty \frac{(t_{k+1} - t_k)}{t_{k+1}(\log t_{k+1})^{1/\lambda}},
\]

and
\[
\frac{(t_{k+1} - t_k)}{t_{k+1}(\log t_{k+1})^{1/\lambda}} = \frac{(t_{k+1} - t_k)}{t_k (\log t_k)^{1/\lambda}}.
\]

Hence
\[
\sum_{k=0}^\infty \frac{(t_{k+1} - t_k)}{t_k (\log t_k)^{1/\lambda}} < \infty.
\]

Now we get \( \sum_{k=0}^\infty P(C_i) < \infty \), which in turn establishes (9) by applying to the Borel-Cantelli lemma. The proof of Theorem 3 is complete.

### 3. Conclusion

In this paper, we developed some limit theorems on increments of stable subordinators. We reformulated the result obtained by Vasudeva and Divanji [3], and established our result by using \( \lambda_P(t) \).

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