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Complete Semigroups of Binary Relations Defined by Semilattices of the Class $\Sigma_1(X,10)$

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Abstract

In this paper we give a full description of idempotent elements of the semigroup $B_X(D)$, which are defined by semilattices of the class $\Sigma_1(X, 10)$. For the case where X is a finite set we derive formulas by means of which we can calculate the numbers of idempotent elements of the respective semigroup.

Keywords

Semilattice, Semigroup, Binary Relation

1. Introduction

Let X be an arbitrary nonempty set, D be an X-semilattice of unions, *i.e.* such a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping of the set X in the set D. To each such a mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition

$$\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$$

The set of all such α_f $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an X-semilattice of unions D.

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Recall that we denote by $\varnothing$ an empty binary relation or empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further let $x, y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$, $\breve{D}=\cup D$ and $t\in\breve{D}$. Then by symbols we denoted the following sets:

$$y\alpha=\{x\in X\,|\,y\alpha x\},\ \ Y\alpha=\bigcup\nolimits_{y\in Y}y\alpha,\ \ 2^X=\{Y\,|\,Y\subseteq X\},\ \ X^*=2^X\setminus\{\varnothing\},$$
$$V(D,\alpha)=\{Y\alpha\,|\,Y\in D\},\ \ D'_T=\{T'\in D'\,|\,T\subseteq T'\},\ \ \ddot{D}'_T=\{T'\in D'\,|\,T'\subseteq T\},$$
$$D'_t=\{Z'\in D'\,|\,t\in Z'\},\ \ l(D',T)=\cup(D'\setminus D'_T).$$

By symbol $\Lambda(D,D')$ is denoted an exact lower bound of the set D' in the semilattice D.

Definition 1. *We say that the complete X-semilattice of unions D is an XI-semilattice of unions if it satisfies the following two conditions*:

a) $\Lambda(D,D_t)\in D$ *for any* $t\in\breve{D}$;

b) $Z=\bigcup\nolimits_{t\in Z}\Lambda(D,D_t)$ *for any nonempty element Z of the semilattice D.*

Definition 2. *We say that a nonempty element T is a nonlimiting element of the set D′ if* $T\setminus l(D',T)\neq\varnothing$ *and a nonempty element T is a limiting element of the set D′ if* $T\setminus l(D',T)=\varnothing$.

Definition 3. *Let* $\alpha\in B_X(D)$, $T\in V(X^*,\alpha)$, $Y_T^\alpha=\{y\in X\,|\,y\alpha=T\}$. *A representation of a binary relation* α *of the form* $\alpha=\bigcup\nolimits_{T\in V(X^*,\alpha)}(Y_T^\alpha\times T)$ *is called quasinormal.*

Note that, if $\alpha=\bigcup\nolimits_{T\in V(X^*,\alpha)}(Y_T^\alpha\times T)$ is a quasinormal representation of the binary relation α, then the following conditions are true:

1) $X=\bigcup\nolimits_{T\in V(X^*,\alpha)}Y_T^\alpha$;

2) $Y_T^\alpha\cap Y_{T'}^\alpha=\varnothing$ for $T,\ T'\in V(X^*,\alpha)$ and $T\neq T'$.

Let $\sum\nolimits_n(X,m)$ denote the class of all complete X-semilattices of unions where every element is isomorphic to a fixed semilattice D.

The following Theorems are well know (see [1] and [3]).

Theorem 4. *Let X be a finite set; δ and q be respectively the number of basic sources and the number of all automorphisms of the semilattice D. If* $|X|=n\geq\delta$ *and* $|\Sigma_n(X,m)|=s$, *then*

$$s=\frac{1}{q}\cdot\sum_{p=\delta}^{m}\left(\sum_{i=1}^{p+1}\left(\frac{(-1)^{p+i+1}\cdot C_{m-\delta}^{p-\delta}\cdot C_p^\delta\cdot(\delta!)\cdot((p-\delta)!)\cdot i^n}{(i-1)!\cdot(p-i+1)!}\right)\right)$$

where $C_j^k=\dfrac{j!}{(k!)\cdot(j-k)!}$ (*see Theorem* 11.5.1 [1]).

Theorem 5. *Let D be a complete X-semilattice of unions. The semigroup* $B_X(D)$ *possesses right unit iff D is an XI-semilattice of unions* (*see Theorem* 6.1.3 [1]).

Theorem 6. *Let X be a finite set and* $D(\alpha)$ *be the set of all those elements T of the semilattice* $Q=V(D,\alpha)\setminus\{\varnothing\}$ *which are nonlimiting elements of the set* $\ddot{Q}_T$. *A binary relation* α *having a quasinormal representation* $\alpha=\bigcup\nolimits_{T\in V(D,\alpha)}(Y_T^\alpha\times T)$ *is an idempotent element of this semigroup iff*

a) $V(D,\alpha)$ *is complete XI-semilattice of unions*;

b) $\bigcup\nolimits_{T'\in\ddot{D}(\alpha)_T}Y_{T'}^\alpha\supseteq T$ *for any* $T\in D(\alpha)$;

c) $Y_T^\alpha\cap T\neq\varnothing$ *for any nonlimiting element of the set* $\ddot{D}(\alpha)_T$ (*see Theorem* 6.3.9 [1]).

Theorem 7. *Let D,* $\Sigma(D)$, $E_X^{(r)}(D')$ *and I denote respectively the complete X-semilattice of unions, the set of all XI-subsemilatices of the semilattice D, the set of all right units of the semigroup* $B_X(D')$ *and the set of all idempotents of the semigroup* $B_X(D)$. *Then for the sets* $E_X^{(r)}(D')$ *and I the following statements are true*:

1) *if* $\varnothing\in D$ *and* $\Sigma_\varnothing(D)=\{D'\in\Sigma(D)\,|\,\varnothing\in D'\}$ *then*

a) $E_X^{(r)}(D')\cap E_X^{(r)}(D'')=\varnothing$ for any elements D' and D'' of the set $\Sigma_\varnothing(D)$ that satisfy the condition $D'\neq D''$;

b) $I=\bigcup\nolimits_{D'\in\Sigma_\varnothing(D)}E_X^{(r)}(D')$

c) *the equality* $|I|=\sum\nolimits_{D'\in\Sigma_\varnothing(D)}\left|E_X^{(r)}(D')\right|$ *is fulfilled for the finite set X.*

2) if $\varnothing \notin D$, then

a) $E_X^{(r)}(D') \cap E_X^{(r)}(D'') = \varnothing$ *for any elements* D' *and* D'' *of the set* $\Sigma(D)$ *that satisfy the condition* $D' \neq D''$;

b) $I = \bigcup_{D' \in \Sigma(D)} E_X^{(r)}(D')$

c) *the equality* $|I| = \sum_{D' \in \Sigma(D)} \left| E_X^{(r)}(D') \right|$ *is fulfilled for the finite set X* (*see Theorem* 6.2.3 [1]).

Corollary 1. *Let* $Y = \{y_1, y_2, \cdots, y_k\}$ *and* $D_j = \{T_1, T_2, \cdots, T_j\}$ *be some sets, where* $k \geq 1$ *and* $j \geq 1$. *Then the number* $s(k, j)$ *of all possible mappings of the set Y into any such subset* D'_j *of the set* D_j *that* $T_j \in D'_j$ *can be calculated by the formula* $s(k, j) = j^k - (j-1)^k$ (*see Corollary* 1.18.1 [1]).

2. Idempotent Elements of the Semigroups $B_X(D)$ Defined by Semilattices of the Class $\Sigma_1(X, 10)$

Let X and $\Sigma_1(X, 10)$ be respectively an arbitrary nonempty set and a class X-semilattices of unions, where each element is isomorphic to some X-semilattice of unions $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ that satisfies the conditions:

$$\begin{aligned}
& Z_9 \subset Z_4 \subset Z_1 \subset \breve{D},\ Z_9 \subset Z_5 \subset Z_1 \subset \breve{D}, \\
& Z_9 \subset Z_6 \subset Z_1 \subset \breve{D},\ Z_9 \subset Z_6 \subset Z_2 \subset \breve{D}, \\
& Z_9 \subset Z_6 \subset Z_3 \subset \breve{D},\ Z_9 \subset Z_7 \subset Z_3 \subset \breve{D}, \\
& Z_9 \subset Z_8 \subset Z_3 \subset \breve{D},\ Z_1 \setminus Z_2 \neq \varnothing,\ Z_2 \setminus Z_1 \neq \varnothing, \\
& Z_1 \setminus Z_3 \neq \varnothing,\ Z_3 \setminus Z_1 \neq \varnothing,\ Z_2 \setminus Z_3 \neq \varnothing, \\
& Z_3 \setminus Z_2 \neq \varnothing,\ Z_4 \setminus Z_5 \neq \varnothing,\ Z_5 \setminus Z_4 \neq \varnothing, \\
& Z_4 \setminus Z_6 \neq \varnothing,\ Z_6 \setminus Z_4 \neq \varnothing,\ Z_4 \setminus Z_7 \neq \varnothing, \\
& Z_7 \setminus Z_4 \neq \varnothing,\ Z_4 \setminus Z_8 \neq \varnothing,\ Z_8 \setminus Z_4 \neq \varnothing, \\
& Z_5 \setminus Z_6 \neq \varnothing,\ Z_6 \setminus Z_5 \neq \varnothing,\ Z_5 \setminus Z_7 \neq \varnothing, \\
& Z_7 \setminus Z_5 \neq \varnothing,\ Z_5 \setminus Z_8 \neq \varnothing,\ Z_8 \setminus Z_5 \neq \varnothing, \\
& Z_6 \setminus Z_7 \neq \varnothing,\ Z_7 \setminus Z_6 \neq \varnothing,\ Z_6 \setminus Z_8 \neq \varnothing, \\
& Z_8 \setminus Z_6 \neq \varnothing,\ Z_7 \setminus Z_8 \neq \varnothing,\ Z_8 \setminus Z_7 \neq \varnothing, \\
& Z_1 \cup Z_2 = Z_1 \cup Z_3 = Z_2 \cup Z_3 = Z_4 \cup Z_2 \\
& \qquad = Z_4 \cup Z_3 = Z_4 \cup Z_7 = Z_4 \cup Z_8 = Z_5 \cup Z_2 \\
& \qquad = Z_5 \cup Z_3 = Z_5 \cup Z_7 = Z_5 \cup Z_8 = Z_7 \cup Z_1 \\
& \qquad = Z_7 \cup Z_2 = Z_8 \cup Z_1 = Z_8 \cup Z_2 = \breve{D}, \\
& Z_4 \cup Z_5 = Z_4 \cup Z_6 = Z_5 \cup Z_6 = Z_1, \\
& Z_6 \cup Z_7 = Z_6 \cup Z_8 = Z_7 \cup Z_8 = Z_3.
\end{aligned} \tag{1}$$

An X-semilattice that satisfies conditions (1) is shown in **Figure 1**.

Let $C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9\}$ be a family of sets, where P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9

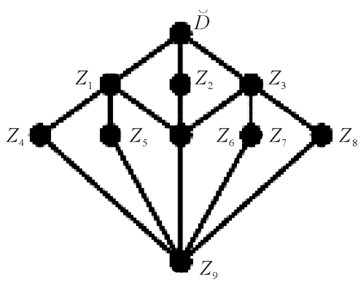

Figure 1. Diagram of D.

are pairwise disjoint subsets of the set X and $\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 & Z_9 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \end{pmatrix}$ be a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

$$\begin{aligned}
\breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9, \\
Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9, \\
Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9, \\
Z_3 &= P_0 \cup P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9, \\
Z_4 &= P_0 \cup P_2 \cup P_3 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9, \\
Z_5 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7 \cup P_8 \cup P_9, \\
Z_6 &= P_0 \cup P_4 \cup P_5 \cup P_7 \cup P_8 \cup P_9, \\
Z_7 &= P_0 \cup P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_8 \cup P_9, \\
Z_8 &= P_0 \cup P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_9, \\
Z_9 &= P_0.
\end{aligned} \tag{2}$$

Here the elements $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$ are basis sources, the elements P_0, P_6, P_9 are sources of completeness of the semilattice D. Therefore $|X| \geq 7$ and $\delta = 7$ (see [2]).

Lemma 1. *Let* $D \in \Sigma_1(X,10)$, $|\Sigma_1(X,10)| = s$ *and* $|X| \geq \delta \geq 7$. *If X is a finite set, then*

$$s = \frac{1}{8}\left((-1) \times 4^n + 7 \times 5^n - 21 \times 6^n + 35 \times 7^n - 35 \times 8^n + 21 \times 9^n + 11^n\right).$$

Proof. In this case we have: $m = 10$, $\delta = 7$. Notice that an X-semilattice given in **Figure 1** has eight automorphims. By Theorem 1.1 it follows that

$$s = \frac{1}{8} \cdot \sum_{p=7}^{10}\left(\sum_{i=1}^{p+1}\left(\frac{(-1)^{p+i+1} \cdot C_3^{p-7} \cdot C_p^7 \cdot (7!) \cdot ((p-7)!) \cdot i^n}{(i-1)! \cdot (p-i+1)!}\right)\right),$$

where $C_j^k = \dfrac{j!}{k! \cdot (j-k)!}$ and that

$$s = \frac{1}{8}\left((-1) \times 4^n + 7 \times 5^n - 21 \times 6^n + 35 \times 7^n - 35 \times 8^n + 21 \times 9^n + 11^n\right).$$

Example 8. *Let* $n = 7, 8, 9, 10$ *Then*:

$$|B_X(D)| = 10^7,\ 10^8,\ 10^9,\ 10^{10}.$$

Lemma 2. *Let* $D \in \Sigma_1(X,10)$. *Then the following sets are all proper subsemilattices of the semilattice* $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$:

1) $\{Z_9\}$, $\{Z_8\}$, $\{Z_7\}$, $\{Z_6\}$, $\{Z_5\}$, $\{Z_4\}$, $\{Z_3\}$, $\{Z_2\}$, $\{Z_1\}$, $\{\breve{D}\}$

(*see diagram* 1 *of the* ***Figure 2***);

2) $\{Z_9, Z_8\}$, $\{Z_9, Z_7\}$, $\{Z_9, Z_6\}$, $\{Z_9, Z_5\}$, $\{Z_9, Z_4\}$, $\{Z_9, Z_3\}$, $\{Z_9, Z_2\}$, $\{Z_9, Z_1\}$, $\{Z_9, \breve{D}\}$, $\{Z_8, Z_3\}$, $\{Z_8, \breve{D}\}$, $\{Z_7, Z_3\}$, $\{Z_7, \breve{D}\}$, $\{Z_6, Z_3\}$, $\{Z_6, Z_2\}$, $\{Z_6, Z_1\}$, $\{Z_6, \breve{D}\}$, $\{Z_5, Z_1\}$, $\{Z_5, \breve{D}\}$, $\{Z_4, Z_1\}$, $\{Z_4, \breve{D}\}$, $\{Z_3, \breve{D}\}$, $\{Z_2, \breve{D}\}$, $\{Z_1, \breve{D}\}$

(*see diagram* 2 of the ***Figure 2***);

3) $\{Z_9, Z_8, Z_3\}$, $\{Z_9, Z_8, \breve{D}\}$, $\{Z_9, Z_7, Z_3\}$, $\{Z_9, Z_7, \breve{D}\}$, $\{Z_9, Z_6, Z_3\}$, $\{Z_9, Z_6, Z_2\}$, $\{Z_9, Z_6, Z_1\}$, $\{Z_9, Z_6, \breve{D}\}$, $\{Z_9, Z_5, Z_1\}$, $\{Z_9, Z_5, \breve{D}\}$, $\{Z_9, Z_4, Z_1\}$, $\{Z_9, Z_4, \breve{D}\}$, $\{Z_9, Z_3, \breve{D}\}$, $\{Z_9, Z_2, \breve{D}\}$,

$\{Z_9, Z_1, \breve{D}\}$, $\{Z_8, Z_3, \breve{D}\}$, $\{Z_7, Z_3, \breve{D}\}$, $\{Z_6, Z_3, \breve{D}\}$, $\{Z_6, Z_2, \breve{D}\}$, $\{Z_6, Z_1, \breve{D}\}$, $\{Z_5, Z_1, \breve{D}\}$, $\{Z_4, Z_1, \breve{D}\}$

(see *diagram* 3 of the ***Figure 2***);

4) $\{Z_9, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_5, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_2, \breve{D}\}$, $\{Z_9, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_7, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_3, \breve{D}\}$

(*see diagram* 4 *of the* ***Figure 2***);

5) $\{Z_9, Z_5, Z_4, Z_1\}$, $\{Z_9, Z_6, Z_4, Z_1\}$, $\{Z_9, Z_6, Z_5, Z_1\}$, $\{Z_9, Z_7, Z_6, Z_3\}$, $\{Z_9, Z_8, Z_6, Z_3\}$, $\{Z_9, Z_8, Z_7, Z_3\}$, $\{Z_9, Z_8, Z_4, \breve{D}\}$, $\{Z_9, Z_8, Z_5, \breve{D}\}$, $\{Z_9, Z_7, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_4, \breve{D}\}$, $\{Z_9, Z_7, Z_5, \breve{D}\}$, $\{Z_9, Z_8, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_2, \breve{D}\}$, $\{Z_9, Z_4, Z_2, \breve{D}\}$, $\{Z_9, Z_4, Z_3, \breve{D}\}$, $\{Z_9, Z_5, Z_2, \breve{D}\}$, $\{Z_9, Z_5, Z_3, \breve{D}\}$, $\{Z_9, Z_7, Z_1, \breve{D}\}$, $\{Z_9, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_3, Z_2, \breve{D}\}$, $\{Z_6, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_6, Z_3, Z_2, \breve{D}\}$;

(*see diagram* 5 of the ***Figure 2***);

6) $\{Z_9, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_5, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_3, \breve{D}\}$

(*see diagram* 6 *of the* ***Figure 2***);

7) $\{Z_9, Z_6, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_3, Z_2, \breve{D}\}$

(*see diagram* 7 *of the* ***Figure 2***);

8) $\{Z_9, Z_8, Z_6, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_6, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$

(*see diagram* 8 *of the* ***Figure 2***);

9) $\{Z_8, Z_7, Z_3\}$, $\{Z_8, Z_6, Z_3\}$, $\{Z_8, Z_6, \breve{D}\}$, $\{Z_8, Z_5, \breve{D}\}$, $\{Z_8, Z_4, \breve{D}\}$, $\{Z_8, Z_2, \breve{D}\}$, $\{Z_8, Z_1, \breve{D}\}$, $\{Z_7, Z_6, Z_3\}$, $\{Z_7, Z_5, \breve{D}\}$, $\{Z_7, Z_4, \breve{D}\}$, $\{Z_7, Z_2, \breve{D}\}$, $\{Z_7, Z_1, \breve{D}\}$, $\{Z_6, Z_5, Z_1\}$, $\{Z_6, Z_4, Z_1\}$, $\{Z_5, Z_4, Z_1\}$, $\{Z_5, Z_3, \breve{D}\}$, $\{Z_5, Z_2, \breve{D}\}$, $\{Z_4, Z_3, \breve{D}\}$, $\{Z_4, Z_2, \breve{D}\}$, $\{Z_3, Z_2, \breve{D}\}$, $\{Z_3, Z_1, \breve{D}\}$, $\{Z_2, Z_1, \breve{D}\}$

(*see diagram* 9 *of the* ***Figure 2***);

10) $\{Z_8, Z_6, Z_3, \breve{D}\}$, $\{Z_8, Z_7, Z_3, \breve{D}\}$, $\{Z_7, Z_6, Z_3, \breve{D}\}$, $\{Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_1, \breve{D}\}$, $\{Z_6, Z_5, Z_1, \breve{D}\}$

(*see diagram* 10 *of the* ***Figure 3***);

11) $\{Z_6, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_6, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_6, Z_3, Z_2, \breve{D}\}$, $\{Z_7, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_6, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_6, Z_3, Z_1, \breve{D}\}$

(*see diagram* 11 *of the* ***Figure 2***);

12) $\{Z_6, Z_5, Z_4, Z_1\}$, $\{Z_8, Z_7, Z_6, Z_3\}$, $\{Z_8, Z_2, Z_1, \breve{D}\}$, $\{Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_7, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_4, Z_2, \breve{D}\}$, $\{Z_7, Z_5, Z_2, \breve{D}\}$, $\{Z_8, Z_4, Z_2, \breve{D}\}$, $\{Z_8, Z_5, Z_2, \breve{D}\}$

(*see diagram* 12 *of the* ***Figure 2***);

13) $\{Z_7, Z_5, Z_3, \breve{D}\}$, $\{Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_4, Z_1, \breve{D}\}$, $\{Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_7, Z_3, Z_1, \breve{D}\}$, $\{Z_7, Z_3, Z_2, \breve{D}\}$, $\{Z_7, Z_4, Z_1, \breve{D}\}$, $\{Z_7, Z_4, Z_3, \breve{D}\}$, $\{Z_7, Z_5, Z_1, \breve{D}\}$, $\{Z_8, Z_4, Z_3, \breve{D}\}$, $\{Z_8, Z_5, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_3, \breve{D}\}$

(*see diagram* 13 *of the* ***Figure 2***);

14) $\{Z_9, Z_6, Z_5, Z_4, Z_1\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_3\}$, $\{Z_6, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_4, Z_3, Z_2, \breve{D}\}$,

$\{Z_9, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_4, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_4, Z_3, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_2, \breve{D}\}$
(*see diagram* 14 *of the* **Figure 2**);

15) $\{Z_9, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_3, \breve{D}\}$
(*see diagram* 15 *of the* **Figure 2**);

16) $\{Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_7, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_5, Z_3, \breve{D}\}$, $\{Z_8, Z_7, Z_4, Z_3, \breve{D}\}$
(*see diagram* 16 *of the* **Figure 2**);

17) $\{Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_7, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_4, Z_2, Z_1, \breve{D}\}$
(*see diagram* 17 *of the* **Figure 2**);

18) $\{Z_7, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_7, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 18 *of the* **Figure 2**);

19) $\{Z_6, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_3, \breve{D}\}$.
(*see diagram* 19 *of the* **Figure 2**);

20) $\{Z_8, Z_7, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_7, Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 20 *of the* **Figure 2**);

21) $\{Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_3, Z_2, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 21 *of the* **Figure 2**);

22) $\{Z_9, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_4, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_5, Z_3, \breve{D}\}$
(*see diagram* 22 *of the* **Figure 2**);

23) $\{Z_9, Z_8, Z_7, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_6, Z_5, Z_4, Z_1, \breve{D}\}$
(*see diagram* 23 *of the* **Figure 2**);

24) $\{Z_9, Z_8, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 24 *of the* **Figure 2**);

25) $\{Z_9, Z_8, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 25 *of the* **Figure 2**);

26) $\{Z_9, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 26 *of the* **Figure 2**);

27) $\{Z_8, Z_7, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 27 *of the* **Figure 2**);

28) $\{Z_8, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$, $\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 28 *of the* **Figure 2**);

29) $\{Z_8, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 29 *of the* **Figure 2**);

30) $\{Z_8, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 30 *of the* **Figure 2**);

31) $\{Z_8, Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 31 *of the* **Figure 2**);

32) $\{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 32 *of the* **Figure 2**);

33) $\{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 33 *of the* **Figure 2**);

34) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$
(*see diagram* 34 *of the* **Figure 2**);

35) $\{Z_9, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 35 *of the* **Figure 2**);

36) $\{Z_9, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_3, Z_1, \breve{D}\}$
(*see diagram* 36 *of the* **Figure 2**);

37) $\{Z_9, Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 37 *of the* **Figure 2**);

38) $\{Z_9, Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 38 *of the* **Figure 2**);

39) $\{Z_9, Z_7, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$
(*see diagram* 39 *of the* **Figure 2**);

40) $\{Z_9, Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_4, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_9, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 40 *of the* **Figure 2**);

41) $\{Z_9, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_3, Z_2, \breve{D}\}$
(*see diagram* 41 *of the* **Figure 2**);

42) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$

(*see diagram* 42 *of the* **Figure 2**);

43) $\{Z_8, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_8, Z_7, Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$

(*see diagram* 43 *of the* **Figure 2**);

44) $\{Z_8, Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$

(*see diagram* 44 *of the* **Figure 2**);

45) $\{Z_9, Z_8, Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$

(*see diagram* 45 *of the* **Figure 2**);

46) $\{Z_9, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$

(*see diagram* 46 *of the* **Figure 2**);

47) $\{Z_9, Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$

(*see diagram* 47 *of the* **Figure 2**);

48) $\{Z_9, Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$

(*see diagram* 48 *of the* **Figure 2**);

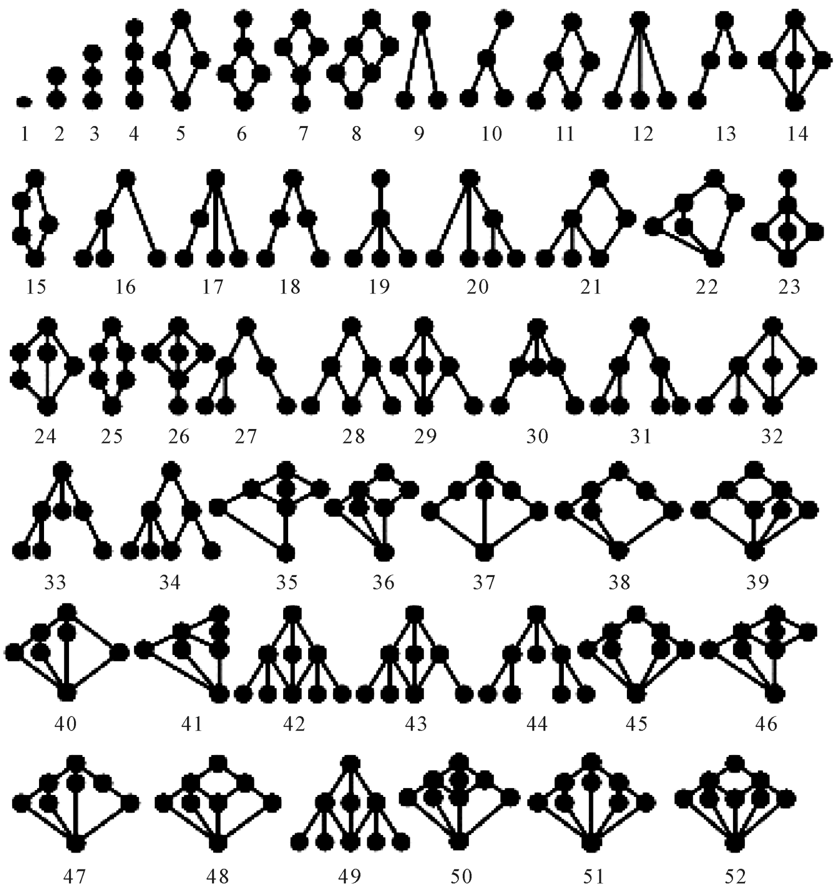

Figure 2. Diagram of all subsemilattices of *D*.

49) $\{Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 49 *of the* ***Figure*** **2**);

50) $\{Z_9, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 50 *of the* ***Figure*** **2**);

51) $\{Z_9, Z_8, Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$
(*see diagram* 51 *of the* ***Figure*** **2**);

52) $\{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$
(*see diagram* 52 *of the* ***Figure*** **2**);

Diagrams of subsemilattices of the semilattice *D*.

Lemma 3. *Let* $D \in \Sigma_1(X,10)$. *Then the following sets are all XI-subsemi-lattices of the given semilattice D*:

1) $\{Z_9\}$, $\{Z_8\}$, $\{Z_7\}$, $\{Z_6\}$, $\{Z_5\}$, $\{Z_4\}$, $\{Z_3\}$, $\{Z_2\}$, $\{Z_1\}$, $\{\breve{D}\}$
(*see diagram* 1 *of the* ***Figure*** **2**);

2) $\{Z_9, \breve{D}\}$, $\{Z_9, Z_8\}$, $\{Z_9, Z_7\}$, $\{Z_9, Z_6\}$, $\{Z_9, Z_5\}$, $\{Z_9, Z_4\}$, $\{Z_9, Z_3\}$, $\{Z_9, Z_2\}$, $\{Z_9, Z_1\}$, $\{Z_8, Z_3\}$, $\{Z_8, \breve{D}\}$, $\{Z_7, Z_3\}$, $\{Z_7, \breve{D}\}$, $\{Z_6, Z_3\}$, $\{Z_6, Z_2\}$, $\{Z_6, Z_1\}$, $\{Z_6, \breve{D}\}$, $\{Z_5, Z_1\}$, $\{Z_5, \breve{D}\}$, $\{Z_4, Z_1\}$, $\{Z_4, \breve{D}\}$, $\{Z_3, \breve{D}\}$, $\{Z_2, \breve{D}\}$, $\{Z_1, \breve{D}\}$
(*see diagram* 2 *of the* ***Figure*** **2**);

3) $\{Z_9, Z_8, \breve{D}\}$, $\{Z_9, Z_7, \breve{D}\}$, $\{Z_9, Z_6, \breve{D}\}$, $\{Z_9, Z_5, \breve{D}\}$, $\{Z_9, Z_4, \breve{D}\}$, $\{Z_9, Z_3, \breve{D}\}$, $\{Z_9, Z_2, \breve{D}\}$, $\{Z_9, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_3\}$, $\{Z_9, Z_7, Z_3\}$, $\{Z_9, Z_6, Z_3\}$, $\{Z_9, Z_6, Z_2\}$, $\{Z_9, Z_6, Z_1\}$, $\{Z_9, Z_5, Z_1\}$, $\{Z_9, Z_4, Z_1\}$, $\{Z_8, Z_3, \breve{D}\}$, $\{Z_7, Z_3, \breve{D}\}$, $\{Z_6, Z_3, \breve{D}\}$, $\{Z_6, Z_2, \breve{D}\}$, $\{Z_6, Z_1, \breve{D}\}$, $\{Z_5, Z_1, \breve{D}\}$, $\{Z_4, Z_1, \breve{D}\}$
(*see diagram* 3 *of the* ***Figure*** **2**);

4) $\{Z_9, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_5, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_2, \breve{D}\}$, $\{Z_9, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_7, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_3, \breve{D}\}$
(*see diagram* 4 *of the* ***Figure*** **2**);

5) $\{Z_9, Z_5, Z_4, Z_1\}$, $\{Z_9, Z_6, Z_4, Z_1\}$, $\{Z_9, Z_6, Z_5, Z_1\}$, $\{Z_9, Z_7, Z_6, Z_3\}$, $\{Z_9, Z_8, Z_6, Z_3\}$, $\{Z_9, Z_8, Z_7, Z_3\}$, $\{Z_9, Z_8, Z_4, \breve{D}\}$, $\{Z_9, Z_8, Z_5, \breve{D}\}$, $\{Z_9, Z_7, Z_2, \breve{D}\}$, $\{Z_9, Z_7, Z_4, \breve{D}\}$, $\{Z_9, Z_7, Z_5, \breve{D}\}$, $\{Z_9, Z_8, Z_1, \breve{D}\}$, $\{Z_9, Z_8, Z_2, \breve{D}\}$, $\{Z_9, Z_4, Z_2, \breve{D}\}$, $\{Z_9, Z_4, Z_3, \breve{D}\}$, $\{Z_9, Z_5, Z_2, \breve{D}\}$, $\{Z_9, Z_5, Z_3, \breve{D}\}$, $\{Z_9, Z_7, Z_1, \breve{D}\}$, $\{Z_9, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_3, Z_2, \breve{D}\}$, $\{Z_6, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_6, Z_3, Z_2, \breve{D}\}$
(*see diagram* 5 *of the* ***Figure*** **2**);

6) $\{Z_9, Z_5, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_4, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_5, Z_1, \breve{D}\}$, $\{Z_9, Z_7, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_6, Z_3, \breve{D}\}$, $\{Z_9, Z_8, Z_7, Z_3, \breve{D}\}$
(*see diagram* 6 *of the* ***Figure*** **2**);

7) $\{Z_9, Z_6, Z_2, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_9, Z_6, Z_3, Z_2, \breve{D}\}$
(*see diagram* 7 *of the* ***Figure*** **2**);

8) $\{Z_9,Z_8,Z_6,Z_3,Z_2,\breve{D}\}$, $\{Z_9,Z_8,Z_6,Z_3,Z_1,\breve{D}\}$, $\{Z_9,Z_7,Z_6,Z_3,Z_2,\breve{D}\}$, $\{Z_9,Z_7,Z_6,Z_3,Z_1,\breve{D}\}$, $\{Z_9,Z_6,Z_5,Z_3,Z_1,\breve{D}\}$, $\{Z_9,Z_6,Z_5,Z_2,Z_1,\breve{D}\}$, $\{Z_9,Z_6,Z_4,Z_3,Z_1,\breve{D}\}$, $\{Z_9,Z_6,Z_4,Z_2,Z_1,\breve{D}\}$
(*see diagram* 8 *of the* **Figure 2**);

Proof. It is well know (see [1]), that the semilattices 1 to 8, which are given by lemma 2 are always *XI*-semilattices. The semilattices 9 and 10 which are given by Lemma 2

$$\{Z_8,Z_7,Z_3\},\ \{Z_8,Z_6,Z_3\},\ \{Z_8,Z_6,\breve{D}\},\ \{Z_8,Z_5,\breve{D}\},\ \{Z_8,Z_4,\breve{D}\},$$
$$\{Z_8,Z_2,\breve{D}\},\ \{Z_8,Z_1,\breve{D}\},\ \{Z_7,Z_6,Z_3\},\ \{Z_7,Z_5,\breve{D}\},\ \{Z_7,Z_4,\breve{D}\},$$
$$\{Z_7,Z_2,\breve{D}\},\ \{Z_7,Z_1,\breve{D}\},\ \{Z_6,Z_5,Z_1\},\ \{Z_6,Z_4,Z_1\},\ \{Z_5,Z_4,Z_1\},$$
$$\{Z_5,Z_3,\breve{D}\},\ \{Z_5,Z_2,\breve{D}\},\ \{Z_4,Z_3,\breve{D}\},\ \{Z_4,Z_2,\breve{D}\},\ \{Z_3,Z_2,\breve{D}\},$$
$$\{Z_3,Z_1,\breve{D}\},\ \{Z_2,Z_1,\breve{D}\}.$$

(see diagram 9 of the **Figure 2**);

$$\{Z_8,Z_6,Z_3,\breve{D}\},\ \{Z_8,Z_7,Z_3,\breve{D}\},\ \{Z_7,Z_6,Z_3,\breve{D}\},\ \{Z_5,Z_4,Z_1,\breve{D}\},\ \{Z_6,Z_4,Z_1,\breve{D}\},\ \{Z_6,Z_5,Z_1,\breve{D}\}$$

(see diagram 10 of the **Figure 2**);
are *XI*-semilattices iff the intersection of minimal elements of the given semilattices is empty set. From the formal equalities (1) of the given semilattice D we have

$$Z_8\cap Z_7=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_9)\cup(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_8\cup P_9)\neq\varnothing$$
$$Z_8\cap Z_6=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_9)\cup(P_0\cup P_4\cup P_5\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_8\cap Z_5=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_4\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_8\cap Z_4=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_8\cap Z_2=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_9)\cup(P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_8\cap Z_1=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_7\cap Z_6=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_8\cup P_9)\cup(P_0\cup P_4\cup P_5\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_7\cap Z_5=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_8\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_4\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_7\cap Z_4=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_8\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_7\cap Z_2=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_8\cup P_9)\cup(P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_7\cap Z_1=(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_8\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_6\cap Z_5=(P_0\cup P_4\cup P_5\cup P_7\cup P_8\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_4\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_6\cap Z_4=(P_0\cup P_4\cup P_5\cup P_7\cup P_8\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_5\cap Z_4=(P_0\cup P_2\cup P_3\cup P_4\cup P_6\cup P_7\cup P_8\cup P_9)\cup(P_0\cup P_2\cup P_3\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_5\cap Z_3=(P_0\cup P_2\cup P_3\cup P_4\cup P_6\cup P_7\cup P_8\cup P_9)\cup(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_5\cap Z_2=(P_0\cup P_2\cup P_3\cup P_4\cup P_6\cup P_7\cup P_8\cup P_9)\cup(P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$
$$Z_4\cap Z_3=(P_0\cup P_2\cup P_3\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\cup(P_0\cup P_1\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\cup P_8\cup P_9)\neq\varnothing$$

$$Z_4 \cap Z_2 = (P_0 \cup P_2 \cup P_3 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \cup (P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \neq \varnothing$$

$$Z_3 \cap Z_2 = (P_0 \cup P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \cup (P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \neq \varnothing$$

$$Z_3 \cap Z_1 = (P_0 \cup P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \cup (P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \neq \varnothing$$

$$Z_2 \cap Z_1 = (P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \cup (P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_9) \neq \varnothing$$

From the equalities given above it follows that the semilattices 9 and 10 are not *XI*-semilattices. □

The semilattices 11

$$\{Z_6, Z_5, Z_3, Z_1, \breve{D}\},\ \{Z_6, Z_5, Z_2, Z_1, \breve{D}\},\ \{Z_6, Z_4, Z_3, Z_1, \breve{D}\},\ \{Z_6, Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_3, Z_2, \breve{D}\},\ \{Z_7, Z_6, Z_3, Z_1, \breve{D}\},\ \{Z_8, Z_6, Z_3, Z_2, \breve{D}\},\ \{Z_8, Z_6, Z_3, Z_1, \breve{D}\}.$$

(see diagram 1-8 of the **Figure 3**);
are not *XI*-semilattice since we have the following inequalities

$$Z_5 \cap Z_3 \neq \varnothing,\ Z_5 \cap Z_2 \neq \varnothing,\ Z_4 \cap Z_3 \neq \varnothing,\ Z_4 \cap Z_2 \neq \varnothing,$$
$$Z_7 \cap Z_2 \neq \varnothing,\ Z_7 \cap Z_1 \neq \varnothing,\ Z_8 \cap Z_2 \neq \varnothing,\ Z_8 \cap Z_1 \neq \varnothing.$$

The semilattices 12 to 52 are never *XI*-semilattices. We prove that the semilattice, diagram 52 of the **Figure 2**, is not an *XI*-semilattice (see **Figure 4**). Indeed, let $Q = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ and

$$C(Q) = \{P_0', P_1', P_2', P_3', P_4', P_5', P_6', P_7', P_8'\}$$

be a family of sets, where $P_0', P_1', P_2', P_3', P_4', P_5', P_6', P_7', P_8'$ are pairwise disjoint subsets of the set *X*. Let

$$\varphi = \begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ P_0' & P_1' & P_2' & P_3' & P_4' & P_5' & P_6' & P_7' & P_8' \end{pmatrix}$$

be a mapping of the semilattice Q onto the family of sets $C(Q)$. Then for the formal equalities of the semilattice Q we have a form:

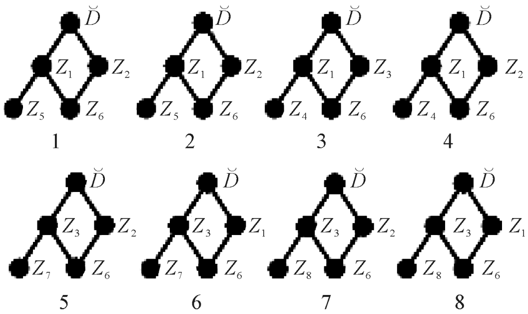

Figure 3. Diagram of all subsemilattices which are isomorphic to 11 in **Figure 2**.

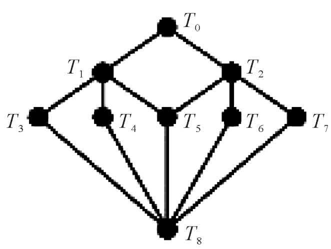

Figure 4. Diagram of subsemilattice 52 in **Figure 2**.

$$
\begin{aligned}
&T_0 = P_0' \cup P_1' \cup P_2' \cup P_3' \cup P_4' \cup P_5' \cup P_6' \cup P_7' \cup P_8',\\
&T_1 = P_0' \cup P_2' \cup P_3' \cup P_4' \cup P_5' \cup P_6' \cup P_7' \cup P_8',\\
&T_2 = P_0' \cup P_1' \cup P_3' \cup P_4' \cup P_5' \cup P_6' \cup P_7' \cup P_8',\\
&T_3 = P_0' \cup P_2' \cup P_4' \cup P_5' \cup P_6' \cup P_7' \cup P_8',\\
&T_4 = P_0' \cup P_2' \cup P_3' \cup P_5' \cup P_6' \cup P_7' \cup P_8',\\
&T_5 = P_0' \cup P_3' \cup P_4' \cup P_6' \cup P_7' \cup P_8',\\
&T_6 = P_0' \cup P_1' \cup P_3' \cup P_4' \cup P_5' \cup P_7' \cup P_8',\\
&T_7 = P_0' \cup P_1' \cup P_3' \cup P_4' \cup P_5' \cup P_6' \cup P_8',\\
&T_8 = P_0'.
\end{aligned} \tag{3}
$$

Here the elements $P_1', P_2', P_3', P_4', P_6', P_7'$ are basis sources, the elements P_0', P_5', P_8' are sources of completeness of the semilattice D. Therefore $|X| \geq 6$ and $\delta = 7$ (see [2]). Then of the formal equalities we have:

$$
Q_t = \begin{cases}
Q, & \text{if } t \in P_0',\\
\{T_7, T_6, T_2, T_0\}, & \text{if } t \in P_1',\\
\{T_4, T_3, T_1, T_0\}, & \text{if } t \in P_2',\\
\{T_7, T_6, T_5, T_4, T_2, T_1, T_0\}, & \text{if } t \in P_3',\\
\{T_7, T_6, T_5, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_4',\\
\{T_7, T_6, T_4, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_5',\\
\{T_7, T_5, T_4, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_6',\\
\{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_7',\\
\{T_8, T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_8'.
\end{cases}
$$

$$
\Lambda(Q, Q_t) = \begin{cases}
T_8, & \text{if } t \in P_0',\\
T_8, & \text{if } t \in P_1',\\
T_8, & \text{if } t \in P_2',\\
T_8, & \text{if } t \in P_3',\\
T_8, & \text{if } t \in P_4',\\
T_8, & \text{if } t \in P_5',\\
T_8, & \text{if } t \in P_6',\\
T_8, & \text{if } t \in P_7',\\
T_8, & \text{if } t \in P_8'.
\end{cases}
$$

We have, that $Q^{\wedge} = \{T_8\}$ and $\Lambda(Q, Q_t) \in Q$ for any $t \in Q$. But elements T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0 are not union of some elements of the set $Q^{\wedge}$. Therefore from the Definition 1 it follows that Q is not an *XI*-semilattice of unions. Statements 12 to 51 can be proved analogously.

We denoted the following semitattices by symbols:

a) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 of the **Figure 5**);

b) $Q_2 = \{T, T'\}$, where T, $T' \in D$ and $T \subset T'$ (see diagram 2 of the **Figure 5**);

c) $Q_3 = \{T, T', T''\}$, where T, T', $T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 of the **Figure 5**);

d) $Q_4 = \{Z_9, T, T', \breve{D}\}$, where T, $T' \in D$ and $Z_9 \subset T \subset T' \subset \breve{D}$ (see diagram 4 of the **Figure 5**);

e) $Q_5 = \{T, T', T'', T' \cup T''\}$ where T, T', $T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, (see diagram 5 of the **Figure 5**);

f) $Q_6 = \{Z_9, T, T', T \cup T', \breve{D}\}$, where T, $T' \in D$, $Z_9 \subset T$, $Z_9 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$ (see diagram 6 of the **Figure 5**);

g) $Q_7 = \{Z_9, Z_6, T, T', \breve{D}\}$, where T, $T' \in D$, $Z_6 \subset T$, $Z_6 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' = \breve{D}$ (see diagram 7 of the **Figure 5**);

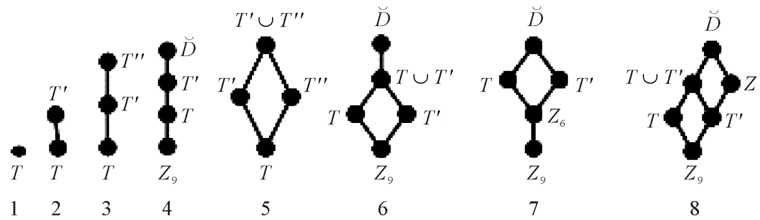

Figure 5. Diagram of all *XI*-subsemilattices of *D*.

h) $Q_8=\left\{Z_9,T,T',T\cup T',Z,\breve{D}\right\}$, where $Z_9\subset T'\subset Z$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$, $\left(T\cup T'\right)\setminus Z\neq\varnothing$, $Z\setminus\left(T\cup T'\right)\neq\varnothing$ (see diagram 8 of the **Figure 5**);

Note that the semilattices in **Figure 5** are all *XI*-semilattices (see [1] and Lemma 1.2.3).

Definition 9. *Let us assume that by the symbol* $\Sigma'_{XI}\left(X,D\right)$ *denote a set of all XI-subsemilatices of X-semilatices of unions D that every element of this set contains an empty set if* $\varnothing\in D$ *or denotes a set of all XI-subsemilatices of D.*

Further, let $D',D''\in\Sigma'_{XI}\left(X,D\right)$ and $\vartheta_{XI}\subseteq\Sigma'_{XI}\left(X,D\right)\times\Sigma'_{XI}\left(X,D\right)$. It is assumed that $D'\vartheta_{XI}D''$ iff there exists some complete isomorphism φ between the semilattices D' and D''. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{XI}\left(X,D\right)$.

By the symbol $Q_i\vartheta_{XI}$ denote the ϑ_{XI}-equivalence class of the set $\Sigma'_{XI}\left(X,D\right)$, where every element is isomorphic to the *X*-semilattice Q_i $\left(i=1,2,\cdots,8\right)$.

Let D' be an *XI*-subsemilattice of the semilattice *D*. By $I\left(D'\right)$ we denoted the set of all right units of the semigroup $B_X\left(D'\right)$, and

$$\left|I^*\left(Q_i\right)\right|=\sum_{D'\in Q_i\vartheta_{XI}}\left|I\left(D'\right)\right|$$

where $i=1,2,\cdots,8$.

Lemma 4. *If X is a finite set, then the following equalities hold*

a) $\left|I\left(Q_1\right)\right|=1$

b) $\left|I\left(Q_2\right)\right|=\left(2^{\left|T'\setminus T\right|}-1\right)\cdot 2^{\left|X\setminus T'\right|}$

c) $\left|I\left(Q_3\right)\right|=\left(2^{\left|T'\setminus T\right|}-1\right)\cdot\left(3^{\left|T''\setminus T'\right|}-2^{\left|T''\setminus T'\right|}\right)\cdot 3^{\left|X\setminus T''\right|}$

d) $\left|I\left(Q_4\right)\right|=\left(2^{\left|T\setminus Z_9\right|}-1\right)\cdot\left(3^{\left|T'\setminus T\right|}-2^{\left|T'\setminus T\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus T'\right|}-3^{\left|\breve{D}\setminus T'\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}$

e) $\left|I\left(Q_5\right)\right|=\left(2^{\left|T'\setminus T''\right|}-1\right)\cdot\left(2^{\left|T''\setminus T'\right|}-1\right)\cdot 4^{\left|X\setminus\left(T'\cup T''\right)\right|}$

f) $\left|I\left(Q_6\right)\right|=\left(2^{\left|T'\setminus T''\right|}-1\right)\cdot\left(2^{\left|T''\setminus T'\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus\left(T\cup T''\right)\right|}-4^{\left|\breve{D}\setminus\left(T\cup T''\right)\right|}\right)\cdot 5^{\left|X\setminus\breve{D}\right|}$

g) $\left|I\left(Q_7\right)\right|=\left(2^{\left|Z_6\setminus Z_9\right|}-1\right)\cdot 2^{\left|\left(T\cap T'\right)\setminus Z_6\right|}\cdot\left(3^{\left|T\setminus T'\right|}-2^{\left|T\setminus T'\right|}\right)\cdot\left(3^{\left|T'\setminus T\right|}-2^{\left|T'\setminus T\right|}\right)\cdot 5^{\left|X\setminus\breve{D}\right|}$

h) $\left|I\left(Q_8\right)\right|=\left(2^{\left|T\setminus Z\right|}-1\right)\cdot\left(2^{\left|T'\setminus T\right|}-1\right)\cdot\left(3^{\left|Z\setminus\left(T\cup T'\right)\right|}-2^{\left|Z\setminus\left(T\cup T'\right)\right|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}$

Proof. This lemma immediately follows from Theorem 13.1.2, 13.3.2, and 13.7.2 of the [1]. □

Theorem 10. *Let* $D\in\Sigma_1\left(X,10\right)$ *and* $\alpha\in B_X\left(D\right)$. *Binary relation* α *is an idempotent relation of the semmigroup* $B_X\left(D\right)$ *iff binary relation* α *satisfies only one conditions of the following conditions*:

a) $\alpha=X\times T$, *where* $T\in D$;

b) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)$, *where* $T,T'\in D$, $T\subset T'$, Y_T^α, $Y_{T'}^\alpha\notin\left\{\varnothing\right\}$, *and satisfies the conditions*: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\cap T'\neq\varnothing$;

c) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)$, *where* $T,T',T''\in D$, $T\subset T'\subset T''$, Y_T^α, $Y_{T'}^\alpha$, $Y_{T''}^\alpha\notin\left\{\varnothing\right\}$, *and satisfies the conditions*: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

d) $\alpha = \left(Y_9^\alpha \times Z_9\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, *where* $T, T' \in D$, $Z_9 \subset T \subset T' \subset \breve{D}$, Y_9^α, Y_T^α, $Y_{T'}^\alpha$, $Y_0^\alpha \notin \{\varnothing\}$, *and satisfies the conditions*: $Y_9^\alpha \supseteq Z_9$, $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

e) $\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T''}^\alpha \times T''\right) \cup \left(Y_{T' \cup T''}^\alpha \times (T' \cup T'')\right)$, *where* T, T', $T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, Y_T^α, $Y_{T'}^\alpha$, $Y_{T''}^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

f) $\alpha = \left(Y_9^\alpha \times Z_9\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T \cup T'}^\alpha \times (T \cup T')\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, *where* $Z_9 \subset T$, $Z_9 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, Y_T^α, $Y_{T'}^\alpha$, $Y_0^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

g) $\alpha = \left(Y_9^\alpha \times Z_9\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, *where* $T, T' \in D$, $Z_6 \subset T$, $Z_6 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' = \breve{D}$, Y_6^α, Y_T^α, $Y_{T'}^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \supseteq Z_9$, $Y_9^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_9^\alpha \cup Y_6^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_6^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

h) $\alpha = \left(Y_9^\alpha \times Z_9\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T \cup T'}^\alpha \times (T \cup T')\right) \cup \left(Y_Z^\alpha \times Z\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, *where* $Z_9 \subset T' \subset Z$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $(T \cup T') \setminus Z \neq \varnothing$, $Z \setminus (T \cup T') \neq \varnothing$, Y_T^α, $Y_{T'}^\alpha$, $Y_Z^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_{T'}^\alpha \supseteq T'$ $Y_9^\alpha \cup Y_{T'}^\alpha \cup Y_Z^\alpha \supseteq Z$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_Z^\alpha \cap Z \neq \varnothing$.

Proof. By Lemma 3 we know that 1 to 8 are an *XI*-semilattices. We prove only statement *g*. Indeed, if

$$\alpha = \left(Y_9^\alpha \times Z_9\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where Y_6^α, Y_T^α, $Y_{T'}^\alpha \notin \{\varnothing\}$, then it is easy to see, that the set $D(\alpha) = \{Z_9, Z_6, T, T_1'\}$ is a generating set of the semilattice $\{Z_9, Z_6, T, T', \breve{D}\}$. Then the following equalities hold

$$\ddot{D}(\alpha)_{Z_9} = \{Z_9\}, \; \ddot{D}(\alpha)_{Z_6} = \{Z_9, Z_6\},$$
$$\ddot{D}(\alpha)_T = \{Z_9, Z_6, T\}, \; \ddot{D}(\alpha)_{T'} = \{Z_9, Z_6, T'\}.$$

By statement *a* of the Theorem 6.2.1 (see [1]) we have:

$$Y_9^\alpha \supseteq Z_9, \; Y_9^\alpha \cup Y_6^\alpha \supseteq Z_6, \; Y_9^\alpha \cup Y_6^\alpha \cup Y_T^\alpha \supseteq T, \; Y_9^\alpha \cup Y_6^\alpha \cup Y_{T'}^\alpha \supseteq T'.$$

Further, one can see, that the equalities are true:

$$l\left(\ddot{D}(\alpha)_{Z_6}, Z_6\right) = \cup\left(\ddot{D}(\alpha)_{Z_6} \setminus \{Z_6\}\right) = Z_9, \; Z_6 \setminus l\left(\ddot{D}(\alpha)_{Z_6}, Z_6\right) = Z_6 \setminus Z_9 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_T, T\right) = \cup\left(\ddot{D}(\alpha)_T \setminus \{T\}\right) = Z_6, \; T \setminus l\left(\ddot{D}(\alpha)_T, T\right) = T \setminus Z_6 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T'}, T'\right) = \cup\left(\ddot{D}(\alpha)_{T'} \setminus \{T'\}\right) = Z_6, \; T' \setminus l\left(\ddot{D}(\alpha)_{T'}, T'\right) = T' \setminus Z_6 \neq \varnothing,$$

We have the elements Z_6, T, T' are nonlimiting elements of the sets $\ddot{D}(\alpha)_{Z_6}$, $\ddot{D}(\alpha)_T$, $\ddot{D}(\alpha)_{T'}$ respectively. By statement *b* of the Theorem 6.2.1 [1] it follows, that the conditions $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$ hold. Therefore, the statement *g* is proved. Rest of statements can be proved analogously.

Lemma 5. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 \neq \varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_1)\right|$ *may be calculated by the formula* $\left|I^*(Q_1)\right| = 10$.

Lemma 6. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 \neq \varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_2)\right|$ *may be calculated by formula*

$$\begin{aligned}\left|I^*(Q_2)\right| = {} & \left(2^{|Z_8 \setminus Z_9|} - 1\right) \cdot 2^{|X \setminus Z_8|} + \left(2^{|Z_7 \setminus Z_9|} - 1\right) \cdot 2^{|X \setminus Z_7|} + \left(2^{|Z_6 \setminus Z_9|} - 1\right) \cdot 2^{|X \setminus Z_6|} + \left(2^{|Z_5 \setminus Z_9|} - 1\right) \cdot 2^{|X \setminus Z_5|} \\ & + \left(2^{|Z_4 \setminus Z_9|} - 1\right) \cdot 2^{|X \setminus Z_4|} + \left(2^{|Z_3 \setminus Z_9|} + 2^{|Z_3 \setminus Z_8|} + 2^{|Z_3 \setminus Z_7|} + 2^{|Z_3 \setminus Z_6|} - 4\right) \cdot 2^{|X \setminus Z_3|} \\ & + \left(2^{|Z_2 \setminus Z_9|} + 2^{|Z_2 \setminus Z_6|} - 2\right) \cdot 2^{|X \setminus Z_2|} + \left(2^{|Z_1 \setminus Z_9|} + 2^{|Z_1 \setminus Z_6|} + 2^{|Z_1 \setminus Z_5|} + 2^{|Z_1 \setminus Z_4|} - 4\right) \cdot 2^{|X \setminus Z_1|} \\ & + \left(2^{|\breve{D} \setminus Z_9|} + 2^{|\breve{D} \setminus Z_8|} + 2^{|\breve{D} \setminus Z_7|} + 2^{|\breve{D} \setminus Z_6|} + 2^{|\breve{D} \setminus Z_5|} + 2^{|\breve{D} \setminus Z_4|} + 2^{|\breve{D} \setminus Z_3|} + 2^{|\breve{D} \setminus Z_2|} + 2^{|\breve{D} \setminus Z_1|} - 9\right) \cdot 2^{|X \setminus \breve{D}|}.\end{aligned}$$

Lemma 7. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 \neq \varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_3)\right|$ *may be calculated by formula*

$$\begin{aligned}
\left|I^*(Q_3)\right| = {} & \left(2^{|Z_8\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_8|}-2^{|\breve{D}\setminus Z_8|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_7\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_7|}-2^{|\breve{D}\setminus Z_7|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_5\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_3\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_2\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_1\setminus Z_9|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_8\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_8|}-2^{|Z_3\setminus Z_8|}\right)\cdot 3^{|X\setminus Z_3|}+\left(2^{|Z_7\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_7|}-2^{|Z_3\setminus Z_7|}\right)\cdot 3^{|X\setminus Z_3|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_6|}-2^{|Z_3\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_3|}+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_2|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_1|}+\left(2^{|Z_5\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_1|}\\
&+\left(2^{|Z_4\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|}+\left(2^{|Z_3\setminus Z_8|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_3\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_2\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_1\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_1\setminus Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus \breve{D}|}+\left(2^{|Z_1\setminus Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus \breve{D}|}.
\end{aligned}$$

Lemma 8. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_4)\right|$ *may be calculated by formula*

$$\begin{aligned}
\left|I^*(Q_4)\right| = {} & \left(2^{|Z_8\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_8|}-2^{|Z_3\setminus Z_8|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_7|}-2^{|Z_3\setminus Z_7|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_6|}-2^{|Z_3\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus \breve{D}|}.
\end{aligned}$$

Lemma 9. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_5)\right|$ *may be calculated by formula*

$$
\begin{aligned}
\left|I^*(Q_5)\right| = {} & \left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}\\
&+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_7\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus Z_3|}\\
&+\left(2^{|Z_8\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus Z_3|}+\left(2^{|Z_8\setminus Z_7|}-1\right)\cdot\left(2^{|Z_7\setminus Z_8|}-1\right)\cdot 4^{|X\setminus Z_3|}\\
&+\left(2^{|Z_8\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+\left(2^{|Z_8\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+\left(2^{|Z_7\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_4|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_5|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_5|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+2\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+2\cdot\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+2\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}.
\end{aligned}
$$

Lemma 10. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9\neq\varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_6)\right|$ *may be calculated by formula*

$$\left|I^*(Q_6)\right| = 1+3+3+3+3+1 = 14$$

Lemma 11. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9\neq\varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_7)\right|$ *may be calculated by formula*

$$
\begin{aligned}
\left|I^*(Q_7)\right| = {} & \left(2^{|Z_6\setminus Z_9|}-1\right)\cdot 2^{|(Z_2\cap Z_1)\setminus Z_6|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot 2^{|(Z_3\cap Z_1)\setminus Z_6|}\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 5^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_6|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 5^{|X\setminus \breve{D}|}.
\end{aligned}
$$

Lemma 12. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9\neq\varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_8)\right|$ *may be calculated by formula*

$$
\begin{aligned}
\left|I^*(Q_8)\right| = {} & \left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}.
\end{aligned}
$$

Figure 6 shows all *XI*-subsemilattices with six elements.

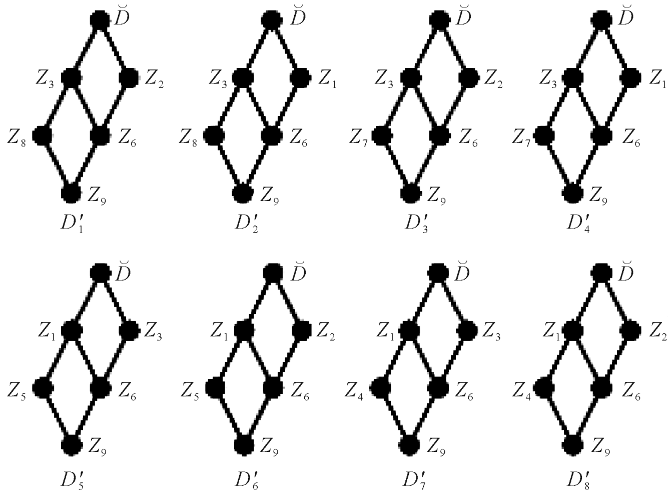

Figure 6. Diagram of all subsemilattices which are isomorphic.

Theorem 11. *Let* $D \in \Sigma_1(X,10)$, $Z_9 \neq \varnothing$. *If X is a finite set and* I_D *is a set of all idempotent elements of the semigroup* $B_X(D)$. *Then* $|I_D| = \sum_{i=1}^{8} |I^*(Q_i)|$.

Example 12. *Let* $X = \{1,2,3,4,5,6,7,8\}$,

$$P_0 = \{6\},\ P_1 = \{1\},\ P_2 = \{2\},\ P_3 = \{3\},\ P_4 = \{4\},$$
$$P_5 = \{5\},\ P_7 = \{7\},\ P_8 = \{8\},\ P_9 = P_6 = \varnothing.$$

Then $\breve{D} = \{1,2,3,4,5,6,7,8\}$, $Z_1 = \{2,3,4,5,6,7,8\}$, $Z_2 = \{1,3,4,5,6,7,8\}$, $Z_3 = \{1,2,4,5,6,7,8\}$, $Z_4 = \{2,3,5,6,7,8\}$, $Z_5 = \{2,3,4,6,7,8\}$, $Z_6 = \{4,5,6,7,8\}$, $Z_7 = \{1,2,4,5,6,8\}$, $Z_8 = \{1,2,4,5,6,7\}$ and $Z_9 = \{6\}$.

$$D = \{\{1,2,3,4,5,6,7,8\},\{2,3,4,5,6,7,8\},\{1,3,4,5,6,7,8\},\{1,2,4,5,6,7,8\},\{2,3,5,6,7,8\},$$
$$\{2,3,4,6,7,8\},\{4,5,6,7,8\},\{1,2,4,5,6,8\},\{1,2,4,5,6,7\},\{6\}\}$$

We have $Z_9 \neq \varnothing$. Where $|I^*(Q_1)| = 10$, $|I^*(Q_2)| = 1169$, $|I^*(Q_3)| = 2154$, $|I^*(Q_4)| = 349$, $|I^*(Q_5)| = 122$, $|I^*(Q_6)| = 14$, $|I^*(Q_7)| = 90$, $|I(Q_8)| = 8$, $|I_D| = 3916$.

3. Results

Lemma 13. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 = \varnothing$. *Then the following sets exhaust all subsemilattices of the semilattice* $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ *which contains the empty set*:

1) $\{\varnothing\}$
(*see diagram* 1 *of the* ***Figure*** **2**);

2) $\{\varnothing,\breve{D}\}$, $\{\varnothing,Z_8\}$, $\{\varnothing,Z_7\}$, $\{\varnothing,Z_6\}$, $\{\varnothing,Z_5\}$, $\{\varnothing,Z_4\}$, $\{\varnothing,Z_3\}$, $\{\varnothing,Z_2\}$, $\{\varnothing,Z_1\}$
(*see diagram* 2 *of the* ***Figure*** **2**);

3) $\{\varnothing,Z_8,\breve{D}\}$, $\{\varnothing,Z_7,\breve{D}\}$, $\{\varnothing,Z_6,\breve{D}\}$, $\{\varnothing,Z_5,\breve{D}\}$, $\{\varnothing,Z_4,\breve{D}\}$, $\{\varnothing,Z_3,\breve{D}\}$, $\{\varnothing,Z_2,\breve{D}\}$, $\{\varnothing,Z_1,\breve{D}\}$, $\{\varnothing,Z_8,Z_3\}$, $\{\varnothing,Z_7,Z_3\}$, $\{\varnothing,Z_6,Z_3\}$, $\{\varnothing,Z_6,Z_2\}$, $\{\varnothing,Z_6,Z_1\}$, $\{\varnothing,Z_5,Z_1\}$, $\{\varnothing,Z_4,Z_1\}$
(*see diagram* 3 *of the* ***Figure*** **2**);

4) $\{\varnothing, Z_4, Z_1, \breve{D}\}$, $\{\varnothing, Z_5, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_2, \breve{D}\}$, $\{\varnothing, Z_6, Z_3, \breve{D}\}$, $\{\varnothing, Z_7, Z_3, \breve{D}\}$, $\{\varnothing, Z_8, Z_3, \breve{D}\}$
(*see diagram* 4 *of the* ***Figure* 2**);

5) $\{\varnothing, Z_5, Z_4, Z_1\}$, $\{\varnothing, Z_6, Z_4, Z_1\}$, $\{\varnothing, Z_6, Z_5, Z_1\}$, $\{\varnothing, Z_7, Z_6, Z_3\}$, $\{\varnothing, Z_8, Z_6, Z_3\}$, $\{\varnothing, Z_8, Z_7, Z_3\}$, $\{\varnothing, Z_8, Z_4, \breve{D}\}$, $\{\varnothing, Z_8, Z_5, \breve{D}\}$, $\{\varnothing, Z_7, Z_2, \breve{D}\}$, $\{\varnothing, Z_7, Z_4, \breve{D}\}$, $\{\varnothing, Z_7, Z_5, \breve{D}\}$, $\{\varnothing, Z_8, Z_1, \breve{D}\}$, $\{\varnothing, Z_8, Z_2, \breve{D}\}$, $\{\varnothing, Z_4, Z_2, \breve{D}\}$, $\{\varnothing, Z_4, Z_3, \breve{D}\}$, $\{\varnothing, Z_5, Z_2, \breve{D}\}$, $\{\varnothing, Z_5, Z_3, \breve{D}\}$, $\{\varnothing, Z_7, Z_1, \breve{D}\}$, $\{\varnothing, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_3, Z_2, \breve{D}\}$
(*see diagram* 5 *of the* ***Figure* 2**);

6) $\{\varnothing, Z_5, Z_4, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_4, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_5, Z_1, \breve{D}\}$, $\{\varnothing, Z_7, Z_6, Z_3, \breve{D}\}$, $\{\varnothing, Z_8, Z_6, Z_3, \breve{D}\}$, $\{\varnothing, Z_8, Z_7, Z_3, \breve{D}\}$
(*see diagram* 6 *of the* ***Figure* 2**);

7) $\{\varnothing, Z_6, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_3, Z_2, \breve{D}\}$
(*see diagram* 7 *of the* ***Figure* 2**);

8) $\{\varnothing, Z_8, Z_6, Z_3, Z_2, \breve{D}\}$, $\{\varnothing, Z_8, Z_6, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_7, Z_6, Z_3, Z_2, \breve{D}\}$, $\{\varnothing, Z_7, Z_6, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_5, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_5, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$
(*see diagram* 8 *of the* ***Figure* 2**);

Theorem 13. *Let* $D \in \Sigma_1(X,10)$, $Z_9 = \varnothing$ *and* $\alpha \in B_X(D)$. *Binary relation* α *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* α *satisfies only one conditions of the following conditions*:

a) $\alpha = \varnothing$;

b) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_T^\alpha \times T)$, *where* $T \in D$, $\varnothing \neq T$, $Y_T^\alpha \neq \varnothing$, *and satisfies the conditions*: $Y_T^\alpha \cap T \neq \varnothing$;

c) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, *where* $T, T' \in D$, $\varnothing \neq T \subset T'$, Y_T^α, $Y_{T'}^\alpha \notin \{\varnothing\}$, *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

d) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$, *where* $T, T' \in D$, $\varnothing \neq T \subset T' \subset \breve{D}$, Y_T^α, $Y_{T'}^\alpha$, $Y_0^\alpha \notin \{\varnothing\}$, *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

e) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T\cup T'}^\alpha \times (T \cup T'))$, *where* $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, Y_T^α, $Y_{T'}^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

f) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T\cup T'}^\alpha \times (T \cup T')) \cup (Y_0^\alpha \times \breve{D})$, *where* $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, Y_T^α, $Y_{T'}^\alpha$, $Y_0^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$.

g) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$, *where* $T, T' \in D$, $Z_6 \subset T$, $Z_6 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' = \breve{D}$, Y_6^α, Y_T^α, $Y_{T'}^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_9^\alpha \cup Y_6^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_6^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

h) $\alpha = (Y_9^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T\cup T'}^\alpha \times (T \cup T')) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D})$, *where* $T' \subset Z$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $(T \cup T') \setminus Z \neq \varnothing$, $Z \setminus (T \cup T') \neq \varnothing$, Y_T^α, $Y_{T'}^\alpha$, $Y_Z^\alpha \notin \{\varnothing\}$ *and satisfies the conditions*: $Y_9^\alpha \cup Y_T^\alpha \supseteq T$, $Y_9^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_9^\alpha \cup Y_{T'}^\alpha \cup Y_Z^\alpha \supseteq Z$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_Z^\alpha \cap Z \neq \varnothing$;

Lemma 14. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 = \varnothing$. *If X is a finite set, then* $|I^*(Q_1)| = 1$.

Lemma 15. *Let* $D \in \Sigma_1(X,10)$ *and* $Z_9 = \varnothing$. *If X is a finite set, then the number* $|I^*(Q_2)|$ *may be calcu-*

lated by formula

$$\left|I^*(Q_2)\right| = \left(2^{|\breve{D}|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|Z_8|}-1\right)\cdot 2^{|X\setminus Z_8|} + \left(2^{|Z_7|}-1\right)\cdot 2^{|X\setminus Z_7|} + \left(2^{|Z_6|}-1\right)\cdot 2^{|X\setminus Z_6|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot 2^{|X\setminus Z_5|} + \left(2^{|Z_4|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3|}-1\right)\cdot 2^{|X\setminus Z_3|}$$
$$+\left(2^{|Z_2|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1|}-1\right)\cdot 2^{|X\setminus Z_1|}.$$

Lemma 16. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_3)\right|$ *may be calculated by formula*

$$\left|I^*(Q_3)\right| = \left(2^{|Z_8|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_8|}-2^{|\breve{D}\setminus Z_8|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_7|}-2^{|\breve{D}\setminus Z_7|}\right)\cdot 3^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_3|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_2|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_8|}-1\right)\cdot\left(3^{|Z_3\setminus Z_8|}-2^{|Z_3\setminus Z_8|}\right)\cdot 3^{|X\setminus Z_3|} + \left(2^{|Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_7|}-2^{|Z_3\setminus Z_7|}\right)\cdot 3^{|X\setminus Z_3|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_6|}-2^{|Z_3\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_3|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_2|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_1|}$$
$$+\left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|}.$$

Lemma 17. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_4)\right|$ *may be calculated by formula*

$$\left|I^*(Q_4)\right| = \left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_6|}-2^{|Z_3\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_7|}-2^{|Z_3\setminus Z_7|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_8|}-1\right)\cdot\left(3^{|Z_3\setminus Z_8|}-2^{|Z_3\setminus Z_8|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}.$$

Lemma 18. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *If X is a finite set, then the number* $\left|I^*(Q_5)\right|$ *may be calcu-*

lated by formula

$$\begin{aligned}\left|I^*\left(Q_5\right)\right| = &\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}\\
&+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_7\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus Z_3|}\\
&+\left(2^{|Z_8\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus Z_3|}+\left(2^{|Z_8\setminus Z_7|}-1\right)\cdot\left(2^{|Z_7\setminus Z_8|}-1\right)\cdot 4^{|X\setminus Z_3|}\\
&+\left(2^{|Z_8\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_8\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_7\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_4|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_5|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_5|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 19. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *If X is a finite set, then the number* $\left|I^*\left(Q_6\right)\right|$ *may be calculated by formula*

$$\begin{aligned}\left|I^*\left(Q_6\right)\right| = &\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_7\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_3|}-4^{|\breve{D}\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_8\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_3|}-4^{|\breve{D}\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_8\setminus Z_7|}-1\right)\cdot\left(2^{|Z_7\setminus Z_8|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_3|}-4^{|\breve{D}\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 20. *Let* $D\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *If X is a finite set, then the number* $\left|I^*\left(Q_7\right)\right|$ *may be calculated by formula*

$$\begin{aligned}\left|I^*\left(Q_7\right)\right| = &\left(2^{|Z_6|}-1\right)\cdot 2^{|(Z_2\cap Z_1)\setminus Z_6|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_6|}-1\right)\cdot 2^{|(Z_3\cap Z_1)\setminus Z_6|}\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_6|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_6|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 21. *Let* $D\in\Sigma_1(X,7)$ *and* $Z_9=\varnothing$. *If X is a finite set, then the number* $\left|I^*\left(Q_8\right)\right|$ *may be calcu-*

lated by formula

$$\begin{aligned}\left|I^*(Q_8)\right| = &\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}.\end{aligned}$$

Theorem 14. *Let* $D\in\Sigma_1(X,10)$, $Z_9=\varnothing$. *If X is a finite set and* I_D *is a set of all idempotent elements of the semigroup* $B_X(D)$, *then* $|I_D|=\sum_{i=1}^{8}\left|I^*(Q_i)\right|$.

Example 15. *Let* $X=\{1,2,3,4,5,6,7\}$,

$$P_1=\{1\},\ P_2=\{2\},\ P_3=\{3\},\ P_4=\{4\},\ P_5=\{5\},\ P_7=\{6\},\ P_8=\{7\},\ P_0=P_6=P_9=\varnothing.$$

Then $\breve{D}=\{1,2,3,4,5,6,7\}$, $Z_1=\{2,3,4,5,6,7\}$, $Z_2=\{1,3,4,5,6,7\}$, $Z_3=\{1,2,4,5,6,7\}$, $Z_4=\{2,3,5,6,7\}$, $Z_5=\{2,3,4,6,7\}$, $Z_6=\{4,5,6,7\}$, $Z_7=\{1,2,4,5,7\}$, $Z_8=\{1,2,4,5,6\}$ and $Z_9=\varnothing$.

$$\begin{aligned}D=\{&\{1,2,3,4,5,6,7\},\{2,3,4,5,6,7\},\{1,3,4,5,6,7\},\{1,2,4,5,6,7\},\{2,3,5,6,7\},\\ &\{2,3,4,6,7\},\{4,5,6,7\},\{1,2,4,5,7\},\{1,2,4,5,6\},\varnothing\}\end{aligned}$$

We have $Z_9=\varnothing$. Where $\left|I^*(Q_1)\right|=1$, $\left|I^*(Q_2)\right|=1121$, $\left|I^*(Q_3)\right|=2141$, $\left|I^*(Q_4)\right|=349$, $\left|I^*(Q_5)\right|=119$, $\left|I^*(Q_6)\right|=14$, $\left|I^*(Q_7)\right|=90$, $\left|I(Q_8)\right|=8$, $\left|I_D\right|=3843$.

It was seen in ([4], Theorem 2) that if α and β are regular elements of $B_X(D)$ then $V(D,\alpha\circ\beta)$ is an *XI*-subsemilattice of *D*. Therefore $\alpha\circ\beta$ is regular elements of $B_X(D)$. That is the set of all regular elements of $B_X(D)$ is a subsemigroup of $B_X(D)$.

References

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