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U-Type Designs via New Generalized Partially Balanced Incomplete Block Designs with *m* = 4, 5 and 7 Associated Classes

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Abstract

The traditional combinatorial designs can be used as basic designs for constructing designs of computer experiments which have been used successfully till now in various domains such as engineering, pharmaceutical industry, etc. In this paper, a new series of generalized partially balanced incomplete blocks PBIB designs with *m* associated classes (*m* = 4, 5 and 7) based on new generalized association schemes with number of treatments *v* arranged in *w* arrays of *n* rows and *l* columns (*w* ≥ 2, *n* ≥ 2, *l* ≥ 2) is defined. Some construction methods of these new PBIB are given and their parameters are specified using the Combinatory Method (*s*). For *n* or *l* even and *s* divisor of *n* or *l*, the obtained PBIB designs are resolvable PBIB designs. So the Fang RBIBD method is applied to obtain a series of particular *U*-type designs $U\left(wnl; \dfrac{wnl^r}{2s}\right)$ (*r* is the repetition number of each treatment in our resolvable PBIB design).

1. Introduction

Designs of computer experiments drew a wide attention in the previous two decades and were still being used

successfully till now in various domains. Among the various construction methods of these designs, the traditional combinatorial designs can be used as basic designs (example: [1] [2]) and particularly the PBIB designs. The association schemes of two or three associated classes have been widely studied, while it is not the case of those over three associated classes. However, some association schemes with five associated classes have been studied (example: [3]). Besides, a method to obtain new association schemes by crossing or nesting other association schemes was given by Bailey [4] using the character tables and strata of the initial association schemes to find the parameters of the obtained association schemes.

In this paper, new association schemes with 4, 5 and 7 associated classes are described starting by a geometric representation. The parameter expressions of these association schemes are given. Moreover, some methods to construct the PBIB designs based on these association schemes are explained using an accessible construction method called the Combinatory Method (*s*) [5], which allows obtaining a series of PBIB designs by only using these association schemes. The parameters expressions of these new designs are given. In addition, for *n* or *l* even

and *s* divisor of *n* or *l*, the obtained PBIB are resolvable PBIB. Then, a series of *U*-type designs $U\left(wnl; \dfrac{wnl^r}{2s}\right)$

(*r* is the repetition number of each treatment in the resolvable PBIB design) is obtained by applying the RBIBD method [6] on these designs.

The paper is organized as follows. In Section 2, we give new definitions of generalized association schemes with *m* (= 4, 5 and 7) associated classes, starting by geometric representation and we give their parameters as properties. Section 3 describes a series of construction method using the Combinatory Method (*s*) for obtaining the PBIB designs associated to our generalized association schemes. We give the series of the *U*-type designs associated to our constructed PBIB designs in Section 4. We achieve our paper with a Conclusion.

Recall some definitions:

Definition 1. *An m-association scheme* ($m \geq 2$) *of v treatments* [7] *is a relation satisfying the following conditions*:

1) *Any two treatments are either* 1st, 2nd, $\cdots$, *or* m^{th} *associates. The relation of association is symmetric, i.e., if the treatment* α *is an* i^{th} *associate of* β, *then* β *is an* i^{th} *associate of* α $(i = 1, 2, \cdots m)$.

2) *Each treatment* α *has* n_i i^{th} *associates, the number* n_i *being independent of* α $(i = 1, 2, \cdots m)$.

3) *If any two treatments* α *and* β *are* i^{th} *associates, then the number of treatments that are* j^{th} *associates of* α *and* k^{th} *associates of* β *is* p^i_{jk} *and is independent of the pair of* i^{th} *associates* α *and* β $(i, j, k = 1, 2, \cdots m)$.

The numbers v, n_i $(i = 1, 2, \cdots m)$ *and* p^i_{jk} $(i, j, k = 1, 2, \cdots m)$ *are called the parameters of the association scheme*.

Definition 2. *A PBIB design* [7], *based on an m-association scheme* ($m \geq 2$), *with parameters v, b, r, k,* λ_i, $i = 1, 2, \cdots m$, *is a block design with v treatments and b blocks of size k each such that every treatment occurs in* r *blocks and any two distinct treatments being* i^{th} *associate occur together in exactly* λ_i *blocks. The number* λ_i *is independent of the pair of* i^{th} *associates* $(i = 1, 2, \cdots m)$.

A parallel class of PBIB is a collection of disjoint blocks from the b blocks whose union is V. A partition of the b blocks into $q = b/r$ *parallel classes is called a resolution, and PBIB design is resolvable if it has at least one resolution and it denoted by RPBIB design*.

The Combinatory Method (*s*) [5]:

Let an array of *n* rows and *l* columns as follows:

a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1l}
a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2l}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{i1}	a_{i2}	$\cdots$	a_{ij}	$\cdots$	a_{il}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{n1}	a_{n2}	$\cdots$	a_{nj}	$\cdots$	a_{nl}

Consider s different elements of the same row i $(2 \leq s < l)$, and associate with them s other elements of a row i' $(i' \neq i)$, respecting the correspondence between the elements a_{ij} and $a_{i'j}$. Bringing together the $2s$ elements in the same block and making all possible combinations, we obtain a partially balanced incomplete block design of size $k = 2s$.

Definition 3. *Let U $(v; q_1, \cdots, q_r)$ denote a design of v runs and r factors with respective $q_1, \cdots, q_r$ levels. This design corresponds to an $v \times r$ matrix $X = (x^1, \cdots, x^r)$ such that the ith column x^i takes values from a set of q_i elements, say $\{1, \cdots, q_i\}$, equally often. The set of all such designs, called U-type designs in the statistical literature, is denoted by U $(v; q_1, \cdots, q_r)$. Obviously, v must be a multiple of q_i $(1 \leq i \leq r)$. When all the q_i $(1 \leq i \leq r)$ are equal to q, we denote it by U $(v; q^r)$. Note that the rows and columns of X are identified with the runs and factors respectively* [6].

2. Generalized Rectangular Right Angular Association Schemes (m) (m = 4, 5 and 7 Associated Classes)

2.1. Generalized Rectangular Right Angular Association Scheme (4)

Let V be a set of $v = wnl$ treatments, $(w \geq 2, n \geq 2, l \geq 2)$, to which we associate a geometrical representation in the following way:

Each treatment of V is associated with a unique triplet of the set $A^1 \cup \cdots \cup A^g \cup \cdots \cup A^w$ where:

$$A^g = \left\{ (x, y, z) \in \mathbb{N}^3 / 1 \leq x \leq l, \ 1 \leq y \leq n, \ z = g \right\}, \quad g = 1, \cdots, w$$

Let α be a treatment of coordinates $(x, y, z) \in A^g$

- $A_1^g = \left\{ (x', y', z') \in A^g / x' \neq x, y' = y \right\}$ corresponds to the treatments 1st associated to α

- $A_2^g = \left\{ (x', y', z') \in A^g / x' = x, y' \neq y \right\}$ corresponds to the treatments 2nd associated to α.

- $A_3^g = \left\{ (x', y', z') \in A^g / x' \neq x, y' \neq y \right\}$ corresponds to the treatments 3rd associated to α.

- $A_4^g = \left\{ (x', y', z') \in A^{g'}, g' \neq g \right\}$ corresponds to the treatments 4th associated to α.

This geometric representation describes a new association scheme, we call it for convenience, generalized rectangular right angular association scheme (4) with four associated classes, to which we give the following equivalent definition:

Definition 4. *A generalized rectangular right angular association scheme (4) is an arrangement of $v = wnl$ ($w \geq 2, n \geq 2, l \geq 2$) treatments in w arrays of n rows and l columns such that, with respect to each treatment α:*

1) *The first associates of α are the other treatments of the same row in the same array.*
2) *The second associates of α are the other treatments of the same column in the same array.*
3) *The third associates of α are the remaining treatments in the same array.*
4) *The fourth associates of α are the other treatments of the other arrays.*

Property 1. *The parameters of generalized rectangular right angular association schemes (4) are:*

$$v = wnl, \quad n_1 = l - 1, \quad n_2 = n - 1, \quad n_3 = (n-1)(l-1), \quad n_4 = (w-1)nl$$

$$P_1 = \begin{pmatrix} l-2 & 0 & 0 & 0 \\ 0 & 0 & n-1 & 0 \\ 0 & n-1 & (l-2)(n-1) & 0 \\ 0 & 0 & 0 & (w-1)ln \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & l-1 & 0 \\ 0 & n-2 & 0 & 0 \\ l-1 & 0 & (l-1)(n-2) & 0 \\ 0 & 0 & 0 & (w-1)ln \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0 & 1 & l-2 & 0 \\ 1 & 0 & n-2 & 0 \\ l-2 & n-2 & (l-2)(n-2) & 0 \\ 0 & 0 & 0 & (w-1)ln \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & l-1 \\ 0 & 0 & 0 & n-1 \\ 0 & 0 & 0 & (l-1)(n-1) \\ l-1 & n-1 & (l-1)(n-1) & (w-2)ln \end{pmatrix}$$

Definition 5. *A PBIB design based on a generalized rectangular right angular association scheme* (4) *is called generalized rectangular right angular GPBIB$_4$ design.*

2.2. Generalized Rectangular Right Angular Association Scheme (5)

Let V be a set of $v = wnl$ treatments, ($w \geq 2$, $n \geq 2$, $l \geq 2$), to which we associate a geometrical representation in the following way:

Each treatment of V is associated with a unique triplet (of coordinates) of the set $A^1 \cup \cdots \cup A^g \cup \cdots \cup A^w$ where:

$$A^g = \left\{ (x, y, z) \in \mathbb{N}^3 / 1 \leq x \leq l,\ 1 \leq y \leq n,\ z = g \right\}, \quad g = 1, \cdots, w$$

Let α be a treatment of coordinates $(x, y, z) \in A^g$

- $A_1^g = \left\{ (x', y', z') \in A^g / x' \neq x, y' = y \right\}$ corresponds to the treatments 1st associated to α

- $A_2^g = \left\{ (x', y', z') \in A^g / x' = x, y' \neq y \right\}$ corresponds to the treatments 2nd associated to α.

- $A_3^g = \left\{ (x', y', z') \in A^g / x' \neq x, y' \neq y \right\}$ corresponds to the treatments 3rd associated to α.

- $A_4^g = \left\{ (x', y', z') \in A^{g'}, g' \neq g / y' = y \right\}$ corresponds to the treatments 4th associated to α.

- $A_5^g = \left\{ (x', y', z') \in A^{g'}, g' \neq g / y' \neq y \right\}$ corresponds to the treatments 5th associated to α.

This geometric representation describes a new association scheme, we call it for convenience, generalized rectangular right angular association scheme (5) with five associated classes, to which we give the following equivalent definition:

Definition 6. *A generalized rectangular right angular association scheme* (5) *is an arrangement of $v = wnl$ ($w \geq 2$, $n \geq 2$, $l \geq 2$) treatments in w ($n \times l$) rectangular arrays such that, with respect to each treatment α:*

1) The first associates of α are the other treatments of the same row in the same array.
2) The second associates of α are the other treatments of the same column in the same array.
3) The third associates of α are the remaining treatments in the same array.
4) The fourth associates of α are the treatments of same row as α, of the other arrays.
5) The fifth associates of α are the remaining treatments in the other arrays.

Property 2. *The parameters of the generalized rectangular right angular association schemes* (5) *are:*

$$v = wnl, \quad n_1 = l-1, \quad n_2 = n-1, \quad n_3 = (n-1)(l-1), \quad n_4 = (w-1)l, \quad n_5 = (w-1)(n-1)l$$

$$P_1 = \begin{pmatrix} l-2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n-1 & 0 & 0 \\ 0 & n-1 & (l-2)(n-1) & 0 & 0 \\ 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & (w-1)(n-1)l \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & l-1 & 0 & 0 \\ 0 & n-2 & 0 & 0 & 0 \\ l-1 & 0 & (l-1)(n-2) & 0 & 0 \\ 0 & 0 & 0 & 0 & l \\ 0 & 0 & 0 & l & (w-1)(n-2)l \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0 & 1 & l-2 & 0 & 0 \\ 1 & 0 & n-2 & 0 & 0 \\ l-2 & n-2 & (l-2)(n-2) & 0 & 0 \\ 0 & 0 & 0 & 0 & l \\ 0 & 0 & 0 & l & (w-2)(n-2)l \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & l-1 & 0 \\ 0 & 0 & 0 & 0 & n-1 \\ 0 & 0 & 0 & 0 & (l-1)(n-1) \\ l-1 & 0 & 0 & (w-2)l & 0 \\ 0 & n-1 & (l-1)(n-1) & 0 & (w-2)(l-1)(n-1) \end{pmatrix}$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & l-1 \\ 0 & 0 & 0 & 1 & n-2 \\ 0 & 0 & 0 & l-1 & (l-1)(n-2) \\ 0 & 1 & l-1 & 0 & (w-2)l \\ l-1 & n-2 & (l-1)(n-2) & (w-2)l & (w-2)(n-2)l \end{pmatrix}$$

Definition 7. *A PBIB design based on a generalized rectangular right angular association scheme* (5) *is called generalized rectangular right angular GPBIB$_5$ design.*

2.3. Generalized Rectangular Right Angular Association Scheme (7)

Let V be a set of $v = wnl$ treatments, ($w \geq 2$, $n \geq 2$, $l \geq 2$), to which we associate a geometrical representation in the following way:

Each treatment of V is associated with a unique triplet of the set $A^1 \cup \cdots \cup A^g \cup \cdots \cup A^w$ where:

$$A^g = \left\{(x, y, z) \in \mathbb{N}^3 / 1 \leq x \leq l,\ 1 \leq y \leq n,\ z = g\right\}, \quad g = 1, \cdots, w$$

Let α be a treatment of coordinates $(x, y, z) \in A^g$

- $A_1^g = \left\{(x', y', z') \in A^g / x' \neq x, y' = y\right\}$ corresponds to the treatments 1st associated to α
- $A_2^g = \left\{(x', y', z') \in A^g / x' = x, y' \neq y\right\}$ corresponds to the treatments 2nd associated to α.
- $A_3^g = \left\{(x', y', z') \in A^g / x' \neq x, y' \neq y\right\}$ corresponds to the treatments 3th associated to α.
- $A_4^g = \left\{(x', y', z') \in A^{g'}, g' \neq g / x' = x, y' = y\right\}$ corresponds to the treatments 4th associated to α.
- $A_5^g = \left\{(x', y', z') \in A^{g'}, g' \neq g / x' \neq x, y' = y\right\}$ corresponds to the treatments 5th associated to α.
- $A_6^g = \left\{(x', y', z') \in A^{g'}, g' \neq g / x' = x, y' \neq y\right\}$ corresponds to the treatments 6th associated to α.
- $A_7^g = \left\{(x', y', z') \in A^{g'}, g' \neq g / x' \neq x, y' \neq y\right\}$ corresponds to the treatments 7th associated to α.

This geometric representation describes a new association scheme, we call it for convenience, generalized rectangular right angular association scheme (7) with seven associated classes, to which we give the following equivalent definition:

Definition 8. *A generalized rectangular right angular association scheme* (7) *is an arrangement of v = wnl*

($w \geq 2$, $n \geq 2$, $l \geq 2$) *treatments in w arrays of n rows and l columns such that, with respect to each treatment* α:

1) *The first associates of* α *are the other treatments of the same row in the same array.*

2) *The second associates of* α *are the other treatments of the same column in the same array.*

3) *The third associates of* α *are the remaining treatments in the same array.*

4) *The fourth associates of* α *are the treatments in the same row and the same column as* α, *of the other arrays.*

5) *The fifth associates of* α *are the treatments of the same row as* α *in the other arrays, that are different from the fourth associates of* α.

6) *The sixth associates of* α *are the treatments of the same column as* α *in the other arrays, that are different from the fourth associates of* α.

7) *The seventh associates of* α *are the remaining treatments in the other arrays.*

Property 3. *the parameters of generalized rectangular right angular association schemes* (7) *are*:

$$v = wnl, \quad n_1 = l-1, \quad n_2 = n-1, \quad n_3 = (n-1)(l-1), \quad n_4 = w-1,$$

$$n_5 = (w-1)(l-1), \quad n_6 = (w-1)(n-1), \quad n_7 = (w-1)(n-1)(l-1)$$

$$P_1 = \begin{pmatrix} l-2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n-1 & 0 & 0 & 0 & 0 \\ 0 & n-1 & (l-2)(n-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w-1 & 0 & 0 \\ 0 & 0 & 0 & w-1 & (w-1)(l-2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (w-1)(n-1) \\ 0 & 0 & 0 & 0 & 0 & (w-1)(n-1) & (w-1)(n-1)(l-2) \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & l-1 & 0 & 0 & 0 & 0 \\ 0 & n-2 & 0 & 0 & 0 & 0 & 0 \\ l-1 & 0 & (l-1)(n-2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (w-1)(l-1) \\ 0 & 0 & 0 & w-1 & 0 & (w-1)(n-2) & 0 \\ 0 & 0 & 0 & 0 & (w-1)(l-1) & 0 & (w-1)(l-1)(n-2) \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0 & 1 & l-2 & 0 & 0 & 0 & 0 \\ 1 & 0 & n-2 & 0 & 0 & 0 & 0 \\ l-2 & n-2 & (l-2)(n-2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w-1 \\ 0 & 0 & 0 & 0 & 0 & w-1 & (w-1)(l-2) \\ 0 & 0 & 0 & 0 & w-1 & 0 & (w-1)(n-2) \\ 0 & 0 & 0 & w-1 & (w-1)(l-2) & (w-1)(n-2) & (w-1)(l-2)(n-2) \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & l-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (l-1)(n-1) \\ 0 & 0 & 0 & w-2 & 0 & 0 & 0 \\ l-1 & 0 & 0 & 0 & (w-2)(l-1) & 0 & 0 \\ 0 & n-1 & 0 & 0 & 0 & (w-2)(n-1) & 0 \\ 0 & 0 & (l-1)(n-1) & 0 & 0 & 0 & (w-2)(l-1)(n-1) \end{pmatrix}$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & l-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n-1 \\ 0 & 0 & 0 & 0 & 0 & n-1 & (l-2)(n-1) \\ 1 & 0 & 0 & 0 & w-2 & 0 & 0 \\ l-2 & 0 & 0 & w-2 & (w-2)(l-2) & 0 & 0 \\ 0 & 0 & n-1 & 0 & 0 & 0 & (w-2)(n-1) \\ 0 & n-1 & (l-2)(n-1) & 0 & 0 & (w-2)(n-1) & (w-2)(l-2)(n-1) \end{pmatrix}$$

$$P_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & l-1 \\ 0 & 0 & 0 & 1 & 0 & n-2 & 0 \\ 0 & 0 & 0 & 0 & l-1 & 0 & (l-1)(n-2) \\ 0 & 1 & 0 & 0 & 0 & w-2 & 0 \\ 0 & 0 & l-1 & 0 & 0 & 0 & (w-2)(l-1) \\ 0 & n-2 & 0 & w-2 & 0 & (w-2)(n-2) & 0 \\ l-1 & 0 & (l-1)(n-2) & 0 & (w-2)(l-1) & 0 & (w-2)(l-1)(n-2) \end{pmatrix}$$

$$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & l-2 \\ 0 & 0 & 0 & 0 & 1 & 0 & n-2 \\ 0 & 0 & 0 & 1 & l-2 & n-2 & (l-2)(n-2) \\ 0 & 0 & 1 & 0 & 0 & 0 & w-2 \\ 0 & 1 & l-2 & 0 & 0 & w-2 & (w-2)(l-2) \\ 1 & 0 & n-2 & 0 & w-2 & 0 & (w-2)(n-2) \\ l-2 & n-2 & (l-2)(n-2) & w-2 & (w-2)(l-2) & (w-2)(n-2) & (w-2)(n-2)(l-2) \end{pmatrix}$$

Definition 9. *A PBIB design based on a generalized rectangular right angular association scheme* (7) *is called generalized rectangular right angular GPBIB$_7$ design.*

3. Construction Method of *GPBIB$_m$* Designs (*m* = 4, 5 and 7 Associated Classes)

Let $v = wnl$ ($w \geq 2$, $n \geq 2$, $l \geq 2$) treatments be arranged in w arrays of n rows and l columns $A^{(1)}$, ..., $A^{(w)}$ and written as follows $\forall g = 1, \cdots, w$:

$$
\begin{array}{c|cccccc}
 & C_1^{(g)} & C_2^{(g)} & \cdots & C_j^{(g)} & \cdots & C_l^{(g)} \\
\hline
R_1^{(g)} & a_{11}^{(g)} & a_{12}^{(g)} & \cdots & a_{1j}^{(g)} & \cdots & a_{1l}^{(g)} \\
R_2^{(g)} & a_{21}^{(g)} & a_{22}^{(g)} & \cdots & a_{2j}^{(g)} & \cdots & a_{2l}^{(g)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_i^{(g)} & a_{i1}^{(g)} & a_{i2}^{(g)} & \cdots & a_{ij}^{(g)} & \cdots & a_{il}^{(g)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_n^{(g)} & a_{n1}^{(g)} & a_{n2}^{(g)} & \cdots & a_{nj}^{(g)} & \cdots & a_{nl}^{(g)} \\
\hline
 & & & A^{(g)} & & &
\end{array}
$$

3.1. Construction Method of *GPBIB$_4$* Designs

3.1.1. First Construction Method of *GPBIB$_4$* Designs

Applying the Combinatory Method (*s*) on each of the w arrays, with chosen $s \in \{2, \cdots, l-1\}$, then we obtain w rectangular PBIB designs. The set of all blocks gives a PBIB design with 4 associated classes.

Theorem 1. *The partially balanced incomplete block designs with the parameters*:

$$v = wnl, \quad b = wn(n-1)C_s^l/2, \quad r = (n-1)C_{s-1}^{l-1}, \quad k = 2s, \quad \lambda_1 = (n-1)C_{s-2}^{l-2}, \quad \lambda_2 = C_{s-1}^{l-1}, \quad \lambda_3 = C_{s-2}^{l-2}, \quad \lambda_4 = 0$$

are generalized rectangular right angular GPBIB$_4$ designs.

Proof. For each array of the *w* arrays, we obtain a rectangular design with parameters: $v^* = nl$,
$b^* = n(n-1)C_s^l/2$, $r = (n-1)C_{s-1}^{l-1}$, $k = 2s$, $\lambda_1 = (n-1)C_{s-2}^{l-2}$, $\lambda_2 = C_{s-1}^{l-1}$, $\lambda_3 = C_{s-2}^{l-2}$. (see [5]).

For the *w* arrays we obtain a generalized rectangular right angular GPBIB$_4$ design with parameters: $v = wnl$,
$b = wn(n-1)C_s^l/2$, $r = (n-1)C_{s-1}^{l-1}$, $k = 2s$, $\lambda_1 = (n-1)C_{s-2}^{l-2}$, $\lambda_2 = C_{s-1}^{l-1}$, $\lambda_3 = C_{s-2}^{l-2}$.

λ_4: Two treatments $a_{ij}^{(g)}$ and $a_{i'j'}^{(g')}$ from the arrays $A^{(g)}$ and $A^{(g')}$ respectively $\left(g \neq g' \in \{1, \cdots, w\} \right)$ they never appear together in the same block thus $\lambda_4 = 0$. $\square$

Lemma 1. *For the special case s = l, the previous method can also be used for the construction of nested group divisible designs, with parameters*:

$$v = wnl, \quad b = wn(n-1)/2, \quad r = (n-1), \quad k = 2l, \quad \lambda_1 = n-1, \quad \lambda_2 = 1 = \lambda_3, \quad \lambda_4 = 0$$

Remark 1.

- *For w = 1, the GPBIB$_4$ design of Theorem* 1 *is a rectangular design with parameters as in the Theorem* 1 *of* [5].
- *For w = 2, the GPBIB$_4$ design of Theorem* 1 *is a rectangular right angular PBIB$_4$ design with parameters as in Proposition* 1 *of* [8].

Proposition 2. *Let GPBIB$_4$ be a design with parameters*:

$$v = wnl, \quad b = wn(n-1)C_s^l/2, \quad r = (n-1)C_{s-1}^{l-1}, \quad k = 2s, \quad \lambda_1 = (n-1)C_{s-2}^{l-2}, \quad \lambda_2 = C_{s-1}^{l-1}, \quad \lambda_3 = C_{s-2}^{l-2}, \quad \lambda_4 = 0.$$

For n or l even and s divisor of l or s, the GPBIB$_4$ design is a resolvable PBIB designs (RGPBIB$_4$) with r parallel classes where each parallel classes contain $q = w\dfrac{nl}{2s}$ blocks.

Proof.

$$q = b/r = \frac{wn(n-1)C_s^l}{2(n-1)C_{s-1}^{l-1}} = \frac{wn(n-1)lC_{s-1}^{l-1}}{2s(n-1)C_{s-1}^{l-1}} = \frac{wnl}{2s}$$

n or l is even and s is divisor of n or l, then $q \in \mathbf{N}$.

Example 1. *Let $v = 3 \times 4 \times 4$ treatments be arranged in the three following arrays*:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32

33	34	35	36
37	38	39	40
41	42	43	44
45	46	47	48

The construction method for (*s* = 2), give the following resolvable generalized rectangular right angular GPBIB$_4$ design, with the parameters:

$$v = 48, \quad b = 108, \quad r = 9, \quad k = 4, \quad q = 12$$

PC₁			
1	2	5	6
9	10	13	14
3	4	7	8
11	12	15	16
17	18	21	22
25	26	29	30
19	20	23	24
27	28	31	32
33	34	37	38
35	36	39	40
41	42	45	46
43	44	47	48

PC₂			
1	2	9	10
5	6	13	14
3	4	11	12
7	8	15	16
17	18	25	26
21	22	29	30
19	20	27	28
23	24	31	32
33	34	41	42
37	38	45	46
35	36	43	44
39	40	47	48

PC₃			
1	2	13	14
5	6	9	10
3	4	15	16
7	8	11	12
17	18	29	30
21	22	25	26
19	20	31	32
23	24	27	28
33	34	45	46
37	38	41	42
35	36	47	48
39	40	43	44

PC$_4$			
1	3	5	7
9	11	13	15
2	4	6	8
10	12	14	16
17	19	21	23
25	27	29	31
18	20	22	24
26	28	30	32
33	35	37	39
34	37	38	40
41	43	45	47
42	44	46	48

PC$_5$			
1	3	9	11
5	7	13	15
2	4	10	12
6	8	14	16
17	19	25	27
21	23	29	31
18	20	26	28
22	24	30	32
33	35	41	43
37	39	45	47
34	36	42	44
38	40	46	48

PC$_6$			
1	3	13	15
5	7	9	11
2	4	14	16
6	8	10	12
17	19	29	31
21	23	25	27
18	20	30	32
22	24	26	28
33	35	45	47
37	39	41	43
34	36	46	48
38	40	42	44

PC$_7$			
1	4	5	8
9	12	13	16
2	3	6	7
10	11	14	15
17	20	21	24
25	28	29	32
18	19	22	23
26	27	30	31
33	36	37	40
34	35	38	39
41	44	45	48
42	43	46	47

PC$_8$			
1	4	9	12
5	8	13	16
2	3	10	11
6	7	14	15
17	20	25	28
21	24	29	32
18	19	26	27
22	23	30	31
33	36	41	44
37	40	45	48
34	35	42	43
38	39	46	47

PC$_9$			
1	4	13	16
5	8	9	12
2	3	14	15
6	7	10	11
17	20	29	32
21	24	25	28
18	19	30	31
22	23	26	27
33	36	45	48
37	40	41	44
34	35	46	47
38	39	42	43

3.1.2. Second Construction Method of *GPBIB$_4$* Designs with $\lambda_4 \neq 0$

Let $C_j^{(g)} = \left(a_{1j}^{(g)}, a_{2j}^{(g)}, \cdots, a_{nj}^{(g)}\right)'$, $C_j^{(g)1} = C_j^{(g)}$ and $C_j^{(g)h} = \left(a_{hj}^{(g)}, a_{(h+1)j}^{(g)}, \cdots, a_{nj}^{(g)}, a_{1j}^{(g)}, \cdots, a_{(h-1)j}^{(g)}\right)'$, $h = 2, 3, \cdots, n$.

Applying the Combinatory Method (s) with chosen $s \in \{3, \cdots, l\}$ on each array of the form $C_j^{(g)h} \cup A^{(g')}$ $(g' \neq g)$ for $j = 1, \cdots, l$, $h = 1, \cdots, n$ and $g = 1, \cdots, w$, by only considering the combinations of s treatments that always contain a component of the vector $C_j^{(g)h}$, the set of all the obtained blocks provides a $_{\text{PBIP}}$ with 4 associated classes.

Theorem 3. *The partially balanced incomplete block designs with the parameters*:

$$v = wnl, \quad b = w(w-1)ln^2(n-1)C_{s-1}^l/2, \quad r = s(w-1)n(n-1)C_{s-1}^l, \quad k = 2s, \quad \lambda_1 = (w-1)ln(n-1)C_{s-3}^{l-2},$$

$$\lambda_2 = s(w-1)nC_{s-1}^l, \quad \lambda_3 = (w-1)nlC_{s-3}^{l-2}, \quad \lambda_4 = 4(n-1)C_{s-2}^{l-1}$$

are generalized rectangular right angular GPBIB$_4$ designs.

Proof.

- The v and k values are obvious.
- r: For each treatment $a_{ij_0}^{(g)}$ of the array $A^{(g)}$ $g = 1, \cdots, w$, we have:
  - On an array $C_{j_0}^{(g)h} \cup A^{(g')}$ $(g' \neq g)$, applying the procedure with the l other elements of the same row. There is C_{s-1}^l possibilities, each one being repeated $(n-1)$ times, with n permutations $(h = 1, \cdots, n)$, then we have $n(n-1)C_{s-1}^l$ repetitions. Therefore, we have $w-1$ arrays of the form $C_{j_0}^{(g)h} \cup A^{(g')}$ $(g' \neq g)$; so we have $(w-1)n(n-1)C_{s-1}^l$ repetitions of the treatment $a_{ij_0}^{(g)}$.
  - On an array $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$, $j = 1, \cdots, l$ and $h = 1, \cdots, n$, applying the procedure with the $l-1$ other elements of the same row. There is C_{s-2}^{l-1} possibilities each one being repeated $(n-1)$ times, with n permutations $(h = 1, \cdots, n)$, then we have $(n-1)C_{s-2}^{l-1}$ appearances repeated themselves n times. Therefore, for an array $A^{(g)} \cup C_j^{(g')h}$ we have $n(n-1)C_{s-2}^{l-1}$ repetitions of a_{ij_0}. Considering all the arrays $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$ for $j = 1, \cdots, l$ and $g' = 1, \cdots, w$; so we have $(w-1)ln(n-1)C_{s-2}^{l-1}$ repetitions of a_{ij_0}.

  Thus: $r = (w-1)n(n-1)\left[C_{s-1}^l + lC_{s-2}^{l-1}\right] = s(w-1)n(n-1)C_{s-1}^l$.

- λ_1: Consider two treatments $a_{ij_0}^{(g)}$ and $a_{ij_0'}^{(g)}$ from the array $A^{(g)}$ ($g = 1, \cdots, w$ and $j_0 \neq j_0'$), they appear together $(n-1)C_{s-3}^{l-2}$ times with the other $l-2$ elements of the same row i of the arrays $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$, with n permutations $(h = 1, \cdots, n)$, we obtain $n(n-1)C_{s-3}^{l-2}$ times in which the two treatments appear together. Considering all the arrays $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$ for $j = 1, \cdots, l$ and $g' = 1, \cdots, w$; then we have $(w-1)ln(n-1)C_{s-3}^{l-2}$ times where the two treatments a_{ij_0} and $a_{ij_0'}$ appear together.

- λ_2: Consider two treatments $a_{ij_0}^{(g)}$ and $a_{i'j_0}^{(g)}$ from the array $A^{(g)}$, ($g = 1, \cdots, w$ and $i \neq i'$), we have:
  - In an array $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$: the two treatments appear together C_{s-2}^{l-1} times, with the n permutations $(h = 1, \cdots, n)$ they appear together nC_{s-2}^{l-1} times. Considering all the arrays of the form $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$ for $j = 1, \cdots, l$ and $g' = 1, \cdots, w$, they appear together $(w-1)ln C_{s-2}^{l-1}$ times.
  - In the array $C_{j_0}^{(g)h} \cup A^{(g')}$: both treatments appear together C_{s-1}^l times, with the n permutations $(h = 1, \cdots, n)$ they appear nC_{s-1}^l times. Considering all the arrays of the form $C_{j_0}^{(g)h} \cup A^{(g')}$ $(g' \neq g)$ for $g' = 1, \cdots, w$; they appear together $(w-1)nC_{s-1}^l$ times.

  In total $\lambda_2 = (w-1)n\left[lC_{s-2}^{l-1} + C_{s-1}^l\right] = sn(w-1)C_{s-1}^l$.

- λ_3: Consider two treatments $a_{ij_0}^{(g)}$ and $a_{i'j_0'}^{(g)}$ from the array $A^{(g)}$ ($g = 1, \cdots, w$, $i \neq i'$ and $j_0 \neq j_0'$), they appear together C_{s-3}^{l-2} times for each array of the form $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$, with the n permutations $(h = 1, \cdots, n)$ they appear together nC_{s-3}^{l-2} times. Considering all the arrays $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$ for $j = 1, \cdots, l$ and $g' = 1, \cdots, w$, then the two treatments $a_{ij_0}^{(g)}$ and $a_{i'j_0'}^{(g)}$ appear together $(w-1)ln C_{s-3}^{l-2}$ times.

- λ_4: Consider two treatments $a_{ij}^{(g)}$ and $a_{i'j'}^{(g')}$ from the arrays $A^{(g)}$ and $A^{(g')}$ respectively (g, $g' = 1, \cdots, w$):
  - If $i = i'$, for the array $A^{(g)} \cup C_j^{(g')h}$ $(g' \neq g)$ and $h = 1$ the two treatments appear together $(n-1)C_{s-2}^{l-1}$ times. For the array $C_j^{(g)h} \cup A^{(g')}$ $(g' \neq g)$ and $h = 1$ they also appear together $(n-1)C_{s-2}^{l-1}$ time, so we have $2(n-1)C_{s-2}^{l-1}$ times. On the other hand, for $h = 2, \cdots, n$, the two treatments appear together C_{s-2}^{l-1} times for the array $A^{(g)} \cup C_j^{(g')h}$ and C_{s-2}^{l-1} times for the array $C_j^{(g)h} \cup A^{(g')}$, so we have $2C_{s-2}^{l-1}$ for one value of h. Taking all the values of $h = 2, \cdots, n$, we obtain $2(n-1)C_{s-2}^{l-1}$ times where the two treatments appear together.

In total $\lambda_4 = 2(n-1)C_{s-2}^{l-1} + 2(n-1)C_{s-2}^{l-1} = 4(n-1)C_{s-2}^{l-1}$.

○ If $i \neq i'$, then the two treatments appear together C_{s-2}^{l-1} times for the array of the form $A^{(g)} \cup C_{j'}^{(g)h}$ $(g' \neq g)$ and appear together C_{s-2}^{l-1} times for the array of the form $C_j^{(g)h} \cup A^{(g')}$ $(g' \neq g)$ for $h = 1$, so we have $2C_{s-2}^{l-1}$ times. For $h = 2, \cdots, n$, among the $(n-1)$ permutations of the vector $C_{j'}^{(g)h}$, and for a given value of h, the treatment $a_{i'j'}^{(g')}$ takes the same row as $a_{ij}^{(g)}$ then the two treatments appear together $(n-1)C_{s-2}^{l-1}$ times, for the remaining values the two treatments appear together C_{s-2}^{l-1} times. For $h = 2, \cdots, n$, among the $(n-1)$ permutations of the vector $C_j^{(g)}$ and for a given value of h, the treatment $a_{ij}^{(g)}$ takes the same row as $a_{i'j'}^{(g')}$ then the two treatments appear together $(n-1)C_{s-2}^{l-1}$ times, for the remaining values the two treatments appear together C_{s-2}^{l-1} times.

In total $\lambda_4 = 2C_{s-2}^{l-1} + (n-1)C_{s-2}^{l-1} + (n-1)C_{s-2}^{l-1} + (n-1)C_{s-2}^{l-1} + (n-1)C_{s-2}^{l-1} = 4(n-1)C_{s-2}^{l-1}$.

- b: Using the above construction method on each array of the form: $A^{(g)} \cup C_{j'}^{(g)h}$ $(g' \neq g)$, we obtain $n(n-1)C_{s-1}^l/2$ blocks. So for the l arrays of the form $A \cup C_j^{(g)h}$ we have $ln(n-1)C_{s-1}^l/2$ blocks, with the n permutations $(h = 1, \cdots, n)$ we obtain $ln^2(n-1)C_{s-1}^l/2$ blocks. Considering all the arrays $A^{(g)} \cup C_j^{(g')h}$ for $g' = 1, \cdots, w$, then in total we have $(w-1)ln^2(n-1)C_{s-1}^l/2$ blocks. Considering all the arrays $A^{(g)} \cup C_j^{(g')h}$ for $g = 1, \cdots, w$, then in total we have $w(w-1)ln^2(n-1)C_{s-1}^l/2$ blocks. □

Remark 2. *For $w = 2$, the GPBIB$_4$ design of Theorem 3 is a rectangular right angular PBIB$_4$ design with parameters as in Proposition 2 of* [8].

Proposition 4 *Let GPBIB$_4$ be a design with parameters*:

$$v = wnl, \quad b = w(w-1)ln^2(n-1)C_{s-1}^l/2, \quad r = s(w-1)n(n-1)C_{s-1}^l, \quad k = 2s, \quad \lambda_1 = (w-1)ln(n-1)C_{s-3}^{l-2},$$

$$\lambda_2 = s(w-1)nC_{s-1}^l, \quad \lambda_3 = (w-1)nlC_{s-3}^{l-2}, \quad \lambda_4 = 4(n-1)C_{s-2}^{l-1}$$

For n or l even and s divisor of l or s, the GPBIB$_4$ design is a resolvable PBIB designs (RGPBIB$_4$) with r parallel classes where each parallel classes contain $q = w\dfrac{nl}{2s}$ blocks.

3.2. Construction Method of *GPBIB$_5$* Designs

Let $C_j^{(g)} = \left(a_{1j}^{(g)}, a_{2j}^{(g)}, \cdots, a_{nj}^{(g)}\right)'$ be the jth column of the gth array $g \in \{1, \cdots, w\}$. Applying the Combinatory Method (s) with s chosen $s \in \{3, \cdots, l\}$, on each array in the form $C_j^{(g)} \cup A^{(g')}$ $(g \neq g')$ for $j = 1, \cdots, l$ and $j = 1, \cdots, w$ by only considering the combinations of s treatments that contain a component of the vector $C_j^{(g)}$, then the set of all the blocks obtained, gives a PBIB design with 5 associated classes.

Theorem 5. *The incomplete block designs with parameters*:

$$v = wnl, \quad b = w(w-1)ln(n-1)C_{s-1}^l/2, \quad r = s(w-1)(n-1)C_{s-1}^l, \quad k = 2s, \quad \lambda_1 = (w-1)l(n-1)C_{s-3}^{l-2},$$

$$\lambda_2 = (w-1)sC_{s-1}^l, \quad \lambda_3 = (w-1)lC_{s-3}^{l-2}, \quad \lambda_4 = 2(n-1)C_{s-2}^{l-1}, \quad \lambda_5 = 2C_{s-2}^{l-1}$$

are generalized rectangular right angular GPBIB$_5$ designs.

Remark 3. *For $w = 2$, the GPBIB$_5$ design of Theorem 5 is a rectangular right angular PBIB$_5$ design with parameters as in Proposition 3 of* [8].

Proposition 6. *Let GPBIB$_5$ be a design with parameters*:

$$v = wnl, \quad b = w(w-1)ln(n-1)C_{s-1}^l/2, \quad r = s(w-1)(n-1)C_{s-1}^l, \quad k = 2s, \quad \lambda_1 = (w-1)l(n-1)C_{s-3}^{l-2},$$

$$\lambda_2 = (w-1)sC_{s-1}^l, \quad \lambda_3 = (w-1)lC_{s-3}^{l-2}, \quad \lambda_4 = 2(n-1)C_{s-2}^{l-1}, \quad \lambda_5 = 2C_{s-2}^{l-1}$$

For n or l even and s divisor of l or s, the GPBIB$_5$ design is a resolvable PBIB designs (RGPBIB$_5$) with r parallel classes where each parallel classes contain $q = w\dfrac{nl}{2s}$ blocks.

Example 2. *Let $v = 3 \times 2 \times 3$ treatments be arranged in the two following arrays*:

1	2	3
4	5	6

7	8	9
10	11	12

13	14	15
16	17	18

The construction method for ($s = 3$), give the following generalized rectangular right angular $GPBIB_5$ design, with parameters:

$$v = 18, \quad b = 54, \quad r = 18, \quad k = 6, \quad q = 6$$

PC$_1$					
1	7	8	4	10	11
9	13	14	12	16	17
15	2	3	18	5	6

PC$_2$					
1	7	9	4	10	12
8	13	15	11	16	18
14	2	3	17	5	6

PC$_3$					
1	8	9	4	11	12
7	14	15	10	17	18
13	2	3	16	5	6

PC$_4$					
2	7	8	5	10	11
9	13	15	12	16	18
14	1	3	17	4	6

PC$_5$					
2	7	9	5	10	12
13	1	3	16	4	6
8	14	15	11	17	18

PC$_6$					
2	8	9	5	11	12
7	13	14	10	16	17
15	1	3	18	4	6

PC₇					
3	7	8	6	10	11
9	14	15	12	17	18
13	1	2	16	4	5

PC₈					
3	7	9	6	10	12
8	13	14	11	16	17
15	1	2	18	4	5

PC₉					
3	8	9	6	11	12
7	13	15	10	16	18
14	1	2	17	4	5

PC₁₀					
13	7	8	16	10	11
9	1	2	12	4	5
3	14	15	6	17	18

PC₁₁					
13	7	9	16	10	12
8	1	3	11	4	6
2	14	15	5	17	18

PC₁₂					
13	8	9	16	11	12
7	2	3	10	5	6
1	14	15	4	17	18

PC₁₃					
14	7	8	17	10	11
9	2	3	12	5	6
1	13	15	4	16	18

PC₁₄					
14	7	9	17	10	12
8	1	2	11	4	5
3	13	15	6	16	18

PC$_{15}$					
14	8	9	17	11	12
7	1	3	10	4	6
2	13	15	5	16	18

PC$_{16}$					
15	7	8	18	10	11
9	1	3	12	4	6
2	13	14	5	16	17

PC$_{17}$					
15	7	9	18	10	12
8	2	3	11	5	6
1	13	14	4	16	17

PC$_{18}$					
15	8	9	18	11	12
7	1	2	10	4	5
3	13	14	6	16	17

3.3. Construction Method of *GPBIB$_7$* Designs

3.3.1. First Construction Method of *GPBIB$_7$* Designs

Applying the Combinatory Method (*s*) on each of the *w* arrays, with chosen and fixed $s \in \{2, \cdots, l-1\}$, then we obtain *w* rectangular PBIB designs. The juxtaposition of the blocks of the *w* rectangular PBIB designs, such that the blocks containing treatment $a_{ij}^{(g)}$ and $a_{ij}^{(g')}$ $\left(g \neq g' \in \{1, \cdots, w\}\right)$ are put side by side, gives a *PBIB* design with 7 associated classes.

Theorem 7. *The incomplete block designs with parameters*:

$$v = wnl, \quad b = \frac{n(n-1)C_s^l}{2}, \quad \lambda_0 = r = (n-1)C_{s-1}^{l-1} = \lambda_4, \quad k = 2sw,$$

$$\lambda_1 = (n-1)C_{s-2}^{l-2} = \lambda_5, \quad \lambda_2 = C_{s-1}^{l-1} = \lambda_6, \quad \lambda_3 = C_{s-2}^{l-2} = \lambda_7$$

are generalized rectangular right angular GPBIB$_7$ designs.

Proof. The design parameters are deduced from the construction method. □

Remark 4.

- *For w = 1, the GPBIB$_7$ design of Theorem 7 is a rectangular design with parameters as in Theorem 1 of* [5].
- *For w = 2, the GPBIB$_7$ design of Theorem 7 is a rectangular right angular PBIB$_7$ design with parameters as in Proposition 4 of* [8].

Proposition 8. *Let GPBIB$_7$ be a design with parameters*:

$$v = wnl, \quad b = \frac{n(n-1)C_s^l}{2}, \quad \lambda_0 = r = (n-1)C_{s-1}^{l-1} = \lambda_4, \quad k = 2sw,$$

$$\lambda_1 = (n-1)C_{s-2}^{l-2} = \lambda_5, \quad \lambda_2 = C_{s-1}^{l-1} = \lambda_6, \quad \lambda_3 = C_{s-2}^{l-2} = \lambda_7$$

For n or l even and s divisor of l or s, the GPBIB$_7$ design is a resolvable PBIB designs (RGPBIB$_7$) with r

parallel classes where each parallel classes contain $q = \dfrac{nl}{2s}$ *blocks.*

Example 3. *Let $v = 3 \times 4 \times 4$ treatments be arranged in the three following arrays*:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32

33	34	35	36
37	38	39	40
41	42	43	44
45	46	47	48

The construction method for ($s = 2$), give the following resolvable generalized rectangular right angular *GPBIB$_7$* design, with the parameters:

$$v = 48, \quad b = 36, \quad r = 9, \quad k = 12, \quad q = 4$$

PC$_1$											
1	2	5	6	17	18	21	22	33	34	37	38
9	10	13	14	25	26	29	30	41	42	45	46
3	4	7	8	19	20	23	24	35	36	39	40
11	12	15	16	27	28	31	32	43	44	47	48

PC$_2$											
1	2	9	10	17	18	25	26	33	34	41	42
5	6	13	14	21	22	29	30	37	38	45	46
3	4	11	12	19	20	27	28	35	36	43	44
7	8	15	16	23	24	31	32	39	40	47	48

PC$_3$											
1	2	13	14	17	18	29	30	33	34	45	46
5	6	9	10	21	22	25	26	37	38	41	42
3	4	15	16	19	20	31	32	35	36	47	48
7	8	11	12	23	24	27	28	39	40	43	44

PC₄											
1	3	5	7	17	19	21	23	33	35	37	39
9	11	13	15	25	27	29	31	34	37	38	40
2	4	6	8	18	20	22	24	41	43	45	47
10	12	14	16	26	28	30	32	42	44	46	48

PC₅											
1	3	9	11	17	19	25	27	33	35	41	43
5	7	13	15	21	23	29	31	37	39	45	47
2	4	10	12	18	20	26	28	34	36	42	44
6	8	14	16	22	24	30	32	38	40	46	48

PC₆											
1	3	13	15	17	19	29	31	33	35	45	47
5	7	9	11	21	23	25	27	37	39	41	43
2	4	14	16	18	20	30	32	34	36	46	48
6	8	10	12	22	24	26	28	38	40	42	44

PC₇											
1	4	5	8	17	20	21	24	33	36	37	40
9	12	13	16	25	28	29	32	34	35	38	39
2	3	6	7	18	19	22	23	41	44	45	48
10	11	14	15	26	27	30	31	42	43	46	47

PC₈											
1	4	9	12	17	20	25	28	33	36	41	44
5	8	13	16	21	24	29	32	37	40	45	48
2	3	10	11	18	19	26	27	34	35	42	43
6	7	14	15	22	23	30	31	38	39	46	47

PC₉											
1	4	13	16	17	20	29	32	33	36	45	48
5	8	9	12	21	24	25	28	37	40	41	44
2	3	14	15	18	19	30	31	34	35	46	47
6	7	10	11	22	23	26	27	38	39	42	43

3.3.2. Second Construction Method of *GPBIB₇* Designs with $\lambda_i \neq \lambda_{i+4}$ $(i = 0, \cdots, 4)$

Let $C_j^{(g)} = \left(a_{1j}^{(g)}, a_{2j}^{(g)}, \cdots, a_{nj}^{(g)} \right)'$ be the jth column of the gth array and let $R_i^{(g)} = \left(a_{i1}^{(g)}, a_{i2}^{(g)}, \cdots, a_{il}^{(g)} \right)$ be the ith row of the gth array $\left(g \in \{1, \cdots, w\} \right)$. Then applying the Combinatory Method (*s*) with chosen $s \in \{2, \cdots, l\}$ on each

array of the form $C_j^{(g)} \cup \left[A^{(g')} \setminus C_j^{(g')} \right]$ $(g' \neq g)$ for $j = 1, \cdots, l$ and g, $g' = 1, \cdots, w$ (respectively $R_i^{(g)} \cup \left[A^{(g')} \setminus R_i^{(g')} \right]$ $(g' \neq g)$ for $i = 1, \cdots, n$ and g, $g' = 1, \cdots, w$), by only considering the combinations of s treatments that always contain a component of the column $C_j^{(g)h}$ (respectively the row $R_i^{(g)}$), the set of all the obtained blocks provides a PBIP with 7 associated classes.

Theorem 9. *The partially balanced incomplete block designs with the parameters*:

$$v = wnl, \quad b = w(w-1)n(n-1)(s+2)C_s^l/2, \quad \lambda_0 = r = (w-1)(n-1)C_{s-1}^{l-1}(s+2), \quad k = 2s,$$

$$\lambda_1 = s(n-1)(w-1)C_{s-2}^{l-2}, \quad \lambda_2 = s(w-1)C_{s-1}^{l-1}, \quad \lambda_3 = (w-1)(l-2)C_{s-3}^{l-3}, \quad \lambda_4 = 0,$$

$$\lambda_5 = 2(n-1)C_{s-2}^{l-2}, \quad \lambda_6 = 2C_{s-1}^{l-1}, \quad \lambda_7 = 4C_{s-2}^{l-2}$$

are generalized rectangular right angular GPBIB$_7$ designs.

Proof. The design parameters are deduced from the construction method. $\square$

Remark 5. *For w = 2, the GPBIB$_7$ design of Theorem 9 is a rectangular right angular PBIB$_7$ design with parameters as in Proposition 5 of* [8].

Proposition 10. *Let GPBIB$_7$ be a design with parameters*:

$$v = wnl, \quad b = w(w-1)n(n-1)(s+2)C_s^l/2, \quad \lambda_0 = r = (w-1)(n-1)C_{s-1}^{l-1}(s+2), \quad k = 2s,$$

$$\lambda_1 = s(n-1)(w-1)C_{s-2}^{l-2}, \quad \lambda_2 = s(w-1)C_{s-1}^{l-1}, \quad \lambda_3 = (w-1)(l-2)C_{s-3}^{l-3}, \quad \lambda_4 = 0,$$

$$\lambda_5 = 2(n-1)C_{s-2}^{l-2}, \quad \lambda_6 = 2C_{s-1}^{l-1}, \quad \lambda_7 = 4C_{s-2}^{l-2}$$

For n or l even and s divisor of l or s, the GPBIB$_7$ design is a resolvable PBIB designs (RGPBIB$_7$) with r parallel classes where each parallel classes contain $q = w\dfrac{nl}{2s}$ *blocks.*

Example 4. *Let v = 3 × 3 × 3 treatments be arranged in the three following arrays*:

1	2	3
4	5	6
7	8	9

10	11	12
13	14	15
16	17	18

19	20	21
22	23	24
25	2	27

The construction method for (s = 3), give a generalized rectangular right angular *GPBIB$_7$* design with parameters:

$$v = 27, \quad b = 90, \quad r = 20, \quad k = 6, \quad \lambda_1 = 12, \quad \lambda_2 = 6,$$
$$\lambda_3 = 4, \quad \lambda_4 = 0, \quad \lambda_5 = 4, \quad \lambda_6 = 2, \quad \lambda_7 = 4$$

To illustrate the method, we applying the construction method for the columns and rows of the first array, where each column represents a block:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
1	1	4	2	2	5	3	3	6	1
11	11	14	10	10	13	10	10	13	20
12	12	15	12	12	15	11	11	14	21
4	7	7	5	8	8	6	9	9	4
14	17	17	13	16	16	13	16	16	23
15	18	18	15	18	18	14	17	17	24

b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}
1	4	2	2	5	3	3	6	1	1
20	23	19	19	22	19	19	22	2	2
21	24	21	21	24	20	20	23	3	3
7	7	5	8	8	6	9	9	13	16
26	26	22	25	25	22	25	25	14	17
27	27	24	27	27	23	26	26	15	18

b_{21}	b_{22}	b_{23}	b_{24}	b_{25}	b_{26}	b_{27}	b_{28}	b_{29}	b_{30}
4	4	7	7	1	1	4	4	7	7
5	5	8	8	2	2	5	5	8	8
6	6	9	9	3	3	6	6	9	9
10	16	10	13	22	25	19	25	19	22
11	17	11	14	23	26	20	26	20	23
12	18	12	15	24	27	21	27	21	24

4. Construction of the U-Type Designs Based on Resolvable *GPBIB$_m$* Designs (*m* = 4, 5 and 7)

In this section we apply the RBIBD method (see [6]) on our resolvable rectangular right angular RGPBIB$_m$ designs (*m* = 4, 5 and 7) to obtain a series of *U*-type designs $U\left(v; q^r\right)$.

Let a resolvable GPBIB$_m$ designs (*m* = 4, 5 and 7) with r parallel classes $PC_1, \cdots, PC_r$ where each j-th class contains $q = b/r$ blocks ($1 \leq j \leq r$). Then we can construct a *U*-type design $U\left(v; q^r\right)$ from resolvable *GPBIB$_m$* designs (*m* = 4, 5 and 7) as follows:

Algorithm *RGPBIB$_m$ − UD*

- Step 1. Give a natural order $1, \cdots, q$ to the q blocks in each parallel class PC_j, $j = 1, \cdots, r$.
- Step 2. For each PC_j, construct a q-level column $x^j = \left(x_{\alpha j}\right)$ as follows: Set $x_{\alpha j} = u$, if treatment α is contained in the u-th block of PC_j, $u = 1, 2, \cdots, q$
- Step 3. The r q-level columns constructed from PC_j, $j = 1, \cdots, r$ form a $U\left(v; q^r\right)$.

  Proposition 11. *For $v = wnl$ runs ($w \geq 2$, $n \geq 2$, $l \geq 2$), a series of U-type $U\left(v; q^r\right)$ designs exist*:

- $U\left(wnl; \dfrac{wnl^{(n-1)C_{s-1}^{l-1}}}{2s}\right)$, n or l even and s divisor of n or l.

- $U\left(wnl; \dfrac{wnl^{s(w-1)n(n-1)C_{s-1}^{l}}}{2s}\right)$, n or l even and s divisor of n or l.

- $U\left(wnl; \dfrac{wnl^{s(w-1)(n-1)C_{s-1}^{l}}}{2s}\right)$, n or l even and s divisor of n or l.

- $U\left(wnl; \dfrac{nl^{(n-1)C_{s-1}^{l-1}}}{2s}\right)$, n or l even and s divisor of n or l.

- $U\left(wnl; \dfrac{wnl^{(s+2)(w-1)(n-1)C_{s-1}^{l-1}}}{2s}\right)$, n or l even and s divisor of n or l.

Proof. applying the $RGPBIB_m - UD$ Algorithm on each resolvable rectangular right angular $GPBIB_m$ ($m = 4, 5$ and 7) of the Proposition 1, 5. □

Example 5. *Applying the $RGPBIB_m - UD$ Algorithm on the resolvable rectangular right angular $GPBIB_7$ of Example* 1, *we obtain the following U-type U* $(48, 4^9)$ *with* 48 *runs and nine* 4-*level factors.*

Runs	Factor$_1$	Factor$_2$	Factor$_3$	Factor$_4$	Factor$_5$	Factor$_6$	Factor$_7$	Factor$_8$	Factor$_9$
1	1	1	1	1	1	1	1	1	1
2	1	1	1	3	3	3	3	3	3
3	3	3	3	1	1	1	3	3	3
4	3	3	3	3	3	3	1	1	1
5	1	2	2	1	2	2	1	2	2
6	1	2	2	3	4	4	3	4	4
7	3	4	4	1	2	2	3	4	4
8	3	4	4	3	4	4	1	2	2
9	2	1	2	2	1	2	2	1	2
10	2	1	2	4	3	4	4	3	4
11	4	3	4	2	3	4	4	3	4
12	4	3	4	4	3	4	2	1	2
13	2	2	1	2	2	1	2	2	1
14	2	2	1	4	4	3	4	4	3
15	4	4	3	2	2	1	4	4	3
16	4	4	3	4	4	3	2	2	1
17	1	1	1	1	1	1	1	1	1
18	1	1	1	3	3	3	3	3	3
19	3	3	3	1	1	1	3	3	3
20	3	3	3	3	3	3	1	1	1
21	1	2	2	1	2	2	1	2	2
22	1	2	2	3	4	4	3	4	4
23	3	4	4	1	2	2	3	4	4
24	3	4	4	3	4	4	1	2	2
25	2	1	2	2	1	2	2	1	2
26	2	1	2	4	3	4	4	3	4

Continued

27	4	3	4	2	3	4	4	3	4
28	4	3	4	4	3	4	2	1	2
29	2	2	1	2	2	1	2	2	1
30	2	2	1	4	4	3	4	4	3
31	4	4	3	2	2	1	4	4	3
32	4	4	3	4	4	3	2	2	1
33	1	1	1	1	1	1	1	1	1
34	1	1	1	3	3	3	3	3	3
35	3	3	3	1	1	1	3	3	3
36	3	3	3	3	3	3	1	1	1
37	1	2	2	1	2	2	1	2	2
38	1	2	2	3	4	4	3	4	4
39	3	4	4	1	2	2	3	4	4
40	3	4	4	3	4	4	1	2	2
41	2	1	2	2	1	2	2	1	2
42	2	1	2	4	3	4	4	3	4
43	4	3	4	2	3	4	4	3	4
44	4	3	4	4	3	4	2	1	2
45	2	2	1	2	2	1	2	2	1
46	2	2	1	4	4	3	4	4	3
47	4	4	3	2	2	1	4	4	3
48	4	4	3	4	4	3	2	2	1

5. Conclusions

New association schemes with $m = 4$, 5 and 7 associated classes called generalized rectangular right angular association schemes for $v = wnl$ treatments arranged in $w \geq 2$ ($n \times l$) arrays were described and their parameters expressions were given exactly and directly. Some construction methods of *PBIB* designs based on these association schemes accommodated by accessible method called the Combinatory Method (*s*) which facilitates the construction application were explained. Moreover, a series of *U*-type designs $U\left(wnl; \dfrac{wnl^{\,r}}{2s}\right)$, by applying the Fang *RBIBD* method on resolvable generalized rectangular right angular $GPBIB_m$ designs ($m = 4$, 5 and 7) was constructed.

We note that all the construction methods described in this article were programmed with the R-package "CombinS" [9] (the ameliorated version).

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