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Method of Successive Approximations for a Fluid Structure Interaction Problem

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Abstract

In this paper, we present a method for solving coupled problem. This method is mainly based on the successive approximations method. The external force acting on the structure is replaced by $\lambda = p\left(x_1, H + u\left(x_1, \lambda\right)\right)$. Then we have a nonlinear equation of unknown λ to solve by successive approximations method. By this method, we obtain easily the analytic expression of the displacement. In addition, good results are obtained with only a few iterations.

Keywords

Beam Equation, Stokes Equation, Finite Elements Method, Successive Approximations Method

1. Introduction

Problem involved in fluid structure interactions occurs in a wide variety of engineering problem and therefore has attracted the interest of many investigations from different engineering disciplines. As a result, much effort has gone into the development of general computational method for fluid structure system by Osses, Fernandez, quarteroni, Blouza, Mbaye [1]-[7].

In this paper, successive approximations method is applied to solve a fluid-structure interaction problem. We replace the external force acting on the interface between fluid and structure by $\lambda = p\left(x_1, H + u\left(x_1, \lambda\right)\right)$.

Then we introduce a nonlinear equation to solve by successive approximations such that the coupled problem is achievable.

By this method, we obtain good approximate solutions. In addition, the analytic solution of beam equation can be computed easily.

The fluid is modeled by two dimensional Stokes equations for steady flow and the structure is represented by the one dimensional beam equation.

2. Presentation of the Problem

We denote by Ω_F the two dimensional domain occupied by the fluid, Γ_u the elastic interface between fluid and structure and $\Gamma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ be the remaining external boundaries of the fluid as depicted in **Figure 1**.

$\Omega_0 = [0, L] \times [0, H]$ defines the reference domain.

3. Governing Equations

3.1. Structure Equation

We start from the simple equation that governs the structure. The simplified beam equation is:

$$Du^{(4)}(x_1) = f^S(x_1) \quad \text{for} \quad 0 < x_1 < L, \tag{1}$$

$$u(0) = u(L) = u^{(1)}(0) = u^{(1)}(L) = 0. \tag{2}$$

where u is the displacement of the structure, f^S is the external force of the structure, $D = \dfrac{Eh^3}{12}$, E is the young modulus, h is the thickness of the structure This equation is good representation of the structure for small deformation.

3.2. Fluid Equation

We suppose that the fluid is governing by the Stokes equations for steady flow in Ω_F:

$$-\mu\Delta v + \nabla p = f^F \quad \text{in} \quad \Omega_F, \tag{3}$$

$$\nabla \cdot v = 0 \quad \text{in} \quad \Omega_F, \tag{4}$$

$$v = g \quad \text{on} \quad \Sigma_1, \tag{5}$$

$$v \cdot n = 0 \quad \text{on} \quad \Sigma_2, \tag{6}$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{on} \quad \Sigma_2, \tag{7}$$

$$-pIn + \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Sigma_3, \tag{8}$$

$$v = 0 \quad \text{on} \quad \Gamma_u, \tag{9}$$

where v denotes the fluid velocity, p denotes the pressure, f^F denotes the volume force of the fluid, μ the viscosity of the fluid and g denotes the inflow velocity profile of the fluid, n is the unit outward normal vector, I is the identity matrix, on the symmetric axis Σ_2 we have the non-penetration condition $v \cdot n = v_2 = 0$.

4. Formulation of Coupled Problem

The problem is to find u, v and p such that:

$$Du^{(4)}(x_1) = p(x_1, H + u(x_1)) \quad \text{for} \quad 0 < x_1 < L \tag{10}$$

$$u(0) = u(L) = u^{(1)}(0) = u^{(1)}(L) = 0, \tag{11}$$

$$-\mu\Delta v + \nabla p = f^F \quad \text{in} \quad \Omega_F, \tag{12}$$

$$\nabla \cdot v = 0 \quad \text{in} \quad \Omega_F, \tag{13}$$

$$v = g \quad \text{on} \quad \Sigma_1, \tag{14}$$

$$v \cdot n = 0 \quad \text{on} \quad \Sigma_2 \tag{15}$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{on} \quad \Sigma_2, \tag{16}$$

$$-pIn + \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Sigma_3, \tag{17}$$

$$v = 0 \quad \text{on} \quad \Gamma_u, \tag{18}$$

We have a fluid structure interaction problem. The domain of the fluid depends on the displacement and the displacement depends on the velocity and the pressure of the fluid.

5. Successive Approximations Method

We assume that $\lambda = p(x_1, H + u(x_1, \lambda))$. Corresponding to each λ, we consider the coupled problem:

$$Du^{(4)}(x_1) = \lambda \quad \text{for} \quad 0 < x_1 < L \tag{19}$$

$$u(0) = u(L) = u^{(1)}(0) = u^{(1)}(L) = 0, \tag{20}$$

$$-\mu\Delta v + \nabla p = f^F \quad \text{in} \quad \Omega_F, \tag{21}$$

$$\nabla \cdot v = 0 \quad \text{in} \quad \Omega_F, \tag{22}$$

$$v = g \quad \text{on} \quad \Sigma_1, \tag{23}$$

$$v \cdot n = 0 \quad \text{on} \quad \Sigma_2, \tag{24}$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{on} \quad \Sigma_2, \tag{25}$$

$$-pIn + \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Sigma_3, \tag{26}$$

$$v = 0 \quad \text{on} \quad \Gamma_u, \tag{27}$$

To solve this coupled problem, we need to solve a nonlinear equation of unknown λ define as: $\lambda = p(x_1, H + u(x_1, \lambda))$ by the successive approximations method. Then we will find u, v and p.

Description of the Method

The weak formulation of the fluid and the structure is given by Grandmont, Murea [8] [9].
We summarize step by step our computational method to find (λ^n) such that:

$$\lambda^0, \text{ the initial value is done} \tag{28}$$

$$\lambda^{n+1} = p(x_1, H + u(x_1, \lambda^n)), \tag{29}$$

- Step 1: We give λ^0, the initial displacement and the fluid domain are compute.
- Step 2: We solve the Stokes equation by finite elements methods in the reference domain. We find $(v, p) := (v_0, p_0)$,
- Step 3: we fine $\lambda^1 = p(x_1, H + u(x_1, \lambda^0))$,
- Step 4: Do:

- $\lambda^0 = \lambda^1$,
- compute $u(x_1, \lambda^1)$ and $\Omega_F(\lambda^1)$,
- solve the stokes equation in $\Omega_F(\lambda^1)$, we find
- $(v, p) := (v_1, p_1)$,

- we compute $\lambda^1 = p\left(x_1, H + u\left(x_1, \lambda^0\right)\right)$,
- While ($\left|\lambda^1 - \lambda^0\right| \geq tol$),
- Step 5: Give λ, u, (v, p).

6. Numerical Results

For each λ, the analytic solution of the beam equations is

$$u = \frac{\lambda}{D}\left(\frac{x^4}{24} - \frac{Lx^3}{12} + \frac{L^2 x^2}{24}\right).$$

We assume that the velocity on Σ_1 is $v = g = (v_1, v_2)$ such that $v_1(x_1, x_2) = 30\left(1 - x_2^2/H^2\right)$ and $v_2(x_1, x_2) = 0$ for all (x_1, x_2) in Σ_1 Murea [10].

The parameter values of the fluid and the structure are:

Parameter related to fluid: The fluid velocity is $\mu = 0.035\ \text{g/cm}\cdot\text{s}$, the fluid density is $\rho^F = 1\ \text{g/cm}^3$, the channel length is $L = 3\ \text{cm}$, the channel width $H = 0.5\ \text{cm}$ $f^F = (0,0)$.

Parameter related to structure: The structure thickness $h = 0.1\ \text{cm}$, Young's modulus is $E = 0.75 \times 10^6\ \text{g/cm}\cdot\text{s}^2$, $tol = 10^{-4}$.

We use the $P1b$ Lagrange finite element to approach the velocities and $P1$ Lagrange finite element is used to approach the pressure. FreeFem++ Hecht [11] is using for the numerical tests.

7. Conclusion

In this work, we applied successive approximations method to solve fluid structure interaction problem. This method gives good results when the displacement is small. After 11 iterations, we found a good approximate solution of the nonlinear equation and also we obtained the solution of coupled problem.

Table 1 and **Figure 2** show that Error $= \left|\lambda^1 - \lambda^0\right|$ decreases to zero when iterations increase. **Figure 3** and **Figure 4** display the behavior of the fluid flow and the pressure wave respectively after 11.

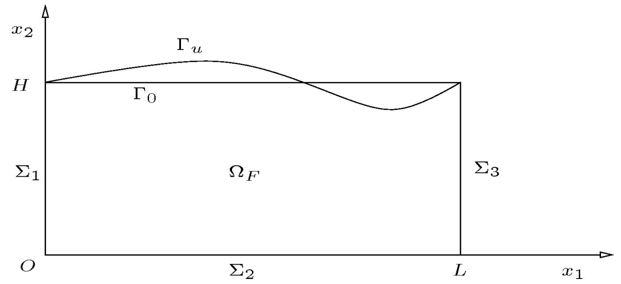

Figure 1. Sets appearing to the fluid structure problem.

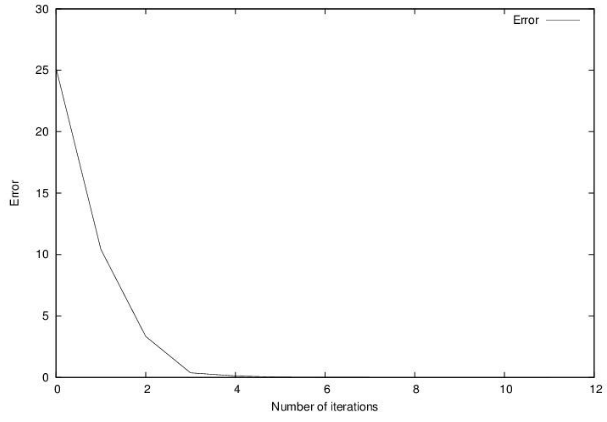

Figure 2. Error with respect to number iterations.

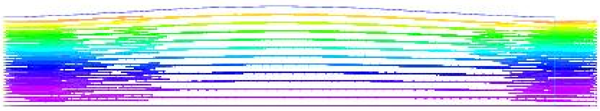

Figure 3. Fluid velocity and structure displacement.

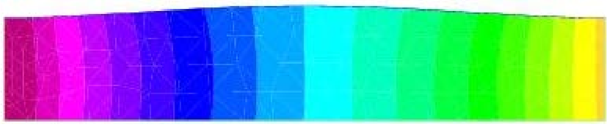

Figure 4. Fluid pressure.

Table 1. Different values of λ after 11 iterations.

λ^1	λ^0	Error $= \lvert\lambda^1 - \lambda^0\rvert$	Iterations
25.1986	0	25.1986	0
14.776	25.1986	10.4227	1
18.1165	14.776	3.34049	2
17.7406	18.1165	0.375886	3
17.8601	17.7406	0.119541	4
17.8368	17.8601	0.0233231	5
17.8456	17.8368	0.0087835	6
17.8423	17.8456	0.00330984	7
17.8435	17.8423	0.00124704	8
17.843	17.8435	0.000468658	9
17.8432	17.843	0.000173383	10
17.8432	17.8432	6.21913e−005	11

In a forthcoming work, we will be showed the theoretical convergence of $\lambda^{n+1} = p\left(x_1, H + u\left(x_1, \lambda^n\right)\right)$ and also successive approximations method will be used to solve an unsteady fluid structure interaction problem.

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