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Discrete Chaos in Fractional Henon Map

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Abstract

In this study, a discrete fractional Henon map is proposed in the Caputo discrete delta's sense. The results show that the discrete fractional calculus is an efficient tool and the maps derived in this way have simpler forms but hold rich dynamical behaviors.

Keywords

Fractional Henon Map, Bifurcation, Difference Scheme, Chaos

1. Introduction

The chaotic behavior is one important aspect of dynamical systems. Much attention has been paid to the topic on fractional differential equations in the past decades [1]-[6]. Recently, the chaotic fractional difference systems start to attract increasing attention due to its potential applications in secure communication and control process. As we all know, the difference models can reveal the nonlinear phenomenon more accurately since their structure is of discrete or discontinuous dynamics.

Generally speaking, the map $x_{n+1} = f(x_n)$ does not have any memory, as the state x_{n+1} only depends on x_n. The memory means that the discrete state x_{n+1} explicitly depends on the previous values $x_n, x_{n-1}, \cdots, x_1$. There are many methods designed for the fractional difference models to prove that the discrete fractional calculus is an efficient tool to discrete the chaotical systems with a memory effect [7]-[9]. Recently Xiao, Li, *et al.* [10] focused on chaotification of the derived fractional difference maps by using the practical controllers and computed the Lyapunov exponents of the controlled fractional difference maps. Wu and Baleanu [11] [12] concentrated on applications of the discrete fractional calculus on an arbitrary time scale and used the theories of delta difference equations to reveal the discrete chaos behavior. The results showed that the Caputo discrete delta's sense has simpler forms but effective. Tarasov and Edelman [13] demonstrated how the attractors of fractional maps were different from the attractors of the dissipative standard map and stated that the evolution is depended on all past states with the weight functions.

This work looks similar, but the essential difference is that it adopts different definitions of fractional differ-

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ence. In our research, on the basis of the Caputo-like delta difference [8] [11] [12], we obtain the fractionalized Henon map. The calculation results show that the discrete fractional calculus is an efficient tool and the maps derived in this way have simpler forms but hold rich dynamical behaviors. The remainder of this paper is organized as follows. Section 2 introduces the definitions and the properties of the discrete fractional calculus. Section 3 presents the fractional odd logistic map on time scales and shows the discrete chaotic solutions while the difference orders and the coefficients are changing. Section 4 is the conclusion.

2. Preliminaries

Concerning nonlinear fractional differential equation of the form

$$ {}_C D_{0,t}^{\alpha} x(t) = f\left(t, x(t)\right), \quad x^{(k)}(0) = x_0^{(k)}, \quad k = 0,1,\cdots,\lceil \alpha \rceil - 1, \tag{1} $$

here $n = \lceil \alpha \rceil - 1$ is the first integer not less than α and ${}_C D_{0,t}^{\alpha}$ is Caputo fractional derivative. It is well known that the initial value problem (1) is equivalent to the Volterra integral system [4]-[7] [10]-[12],

$$ x(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\left(\tau, x(\tau)\right) \mathrm{d}\tau \tag{2} $$

in the sense that if a continuous function solves (2) if and only if it solves (1).

Considering the discrete fractional calculus, we can get the corresponding fractional difference equation. We start with some necessary definitions from discrete fractional calculus theory and preliminary results so that this paper is self-contained.

Definition 1. (See [10].) Let νth fractional sum of f is defined by

$$ \Delta_a^{-\nu} u(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \left(t - \sigma(s)\right)^{(\nu-1)} u(s), $$

where a is the starting point, $\sigma(s) = s+1$ and u is defined for $s = a \bmod (1)$ and $\Delta_a^{-\nu} u(t)$ is defined for $t = (a+\nu) \bmod (1)$. In particular $\Delta_a^{-\nu}$ maps a functions defined on N_a to functions defined on $N_{a+\nu}$, where $N_a = a, a+1, a+2, \cdots$. In addition,

$$ t^{\nu} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} $$

Definition 2. (See [11].) For $\alpha > 0$, $u : N_a \to R$ and α be given, the Caputo-like delta difference is defined by

$$ {}_c\Delta_{\nu}^{\alpha} u(t) := \Delta_a^{n-\nu} \Delta^n u(t) = \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} \left(t - \sigma(s)\right)^{(n-\nu-1)} u(s), \ t \in N_{a+n-\nu}, n = [\nu] + 1, $$

where ν is the difference order.

Theorem 1. (See [12].) For the delta fractional difference equation

$$ {}_c\Delta_a^{\alpha} u(t) = f\left(t + \alpha - 1, u(t + \alpha - 1)\right), $$

$$ \Delta^k u(a) = u_k, n = [\alpha] + 1, k = 0,1,\cdots,n-1. $$

the equivalent discrete integral equation can be obtained as

$$ u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \sum_{a+n-\alpha}^{t-\alpha} \left(t - \sigma(s)\right)^{(\alpha-1)} f\left(s + \alpha - 1, u(s + \alpha - 1)\right), \tag{3} $$

where the initial iteration reads

$$ u(0) = \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} \Delta^k u(a). $$

The complex difference equation with long-term memory is obtained here. It can reduce to the classical one when the difference order $\alpha = 1$.

3. Fractional Henon Map

The Henon map is given by the following pair of first-order difference equations

$$\begin{cases} x_{n+1} = y_n + 1 - \alpha x_n^2 \\ y_{n+1} = \beta x_n. \end{cases}$$

where α and β are (positive) bifurcation parameters, and the Henon map is the most general two-dimensional quadratic map with the property that the contraction is independent of x and y. We can rewrite above equation:

$$\begin{cases} \nabla x_{n+1} = y_n + 1 - \alpha x_n^2 - x_n \\ \nabla y_{n+1} = \beta x_n - y_n. \end{cases}$$

From the discrete fractional calculus, we modify the standard map as a fractional one

$$\begin{cases} {}_c\Delta_a^{\mu} x(t) = y(t+\mu-1) + 1 - \alpha x(t+\mu-1)^2 - x(t+\mu-1), \\ {}_c\Delta_a^{\mu} y(t) = \beta x(t+\mu-1) - y(t+\mu-1). \quad 0 < \mu \le 1, t \in N_{a+1-\mu}. \end{cases}$$

From (3), we can obtain the following discrete integral form from $0 < \mu \le 1$,

$$u(t) = u(a) + \frac{1}{\Gamma(\mu)} \sum_{a+1-\mu}^{t-\mu} (t-\sigma(s))^{(\mu-1)} f(s+\mu-1, u(s+\mu-1)), \; t \in N_{a+1}, \tag{4}$$

where $\frac{(t-\sigma(s))^{(\mu-1)}}{\Gamma(\mu)}$ is a discrete kernel function and $(t-\sigma(s))^{(\mu-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s+1-\mu)}$. As a result, the numerical formula can be presented explicitly. As a result, the numerical formula can be presented explicitly

$$u(n) = u(a) + \frac{1}{\Gamma(\mu)} \sum_{1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} f(j-1, u(j-1)). \tag{5}$$

For the fractional Henon map, an explicit numerical formula can be given as

$$\begin{cases} x(n+1) = x(a) + \frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (y_n + 1 - a x_n^2 - x_n) \\ y(n+1) = y(a) + \frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\beta x_n - y_n), \end{cases} \tag{6}$$

when $\nu = 1$ the above numerical system is the classical one. Using the numerical formula (6), set the step size $\triangle\alpha = 0.002$ and the bifurcation diagrams are plotted in **Figure 1**. For the classical map, an initial point

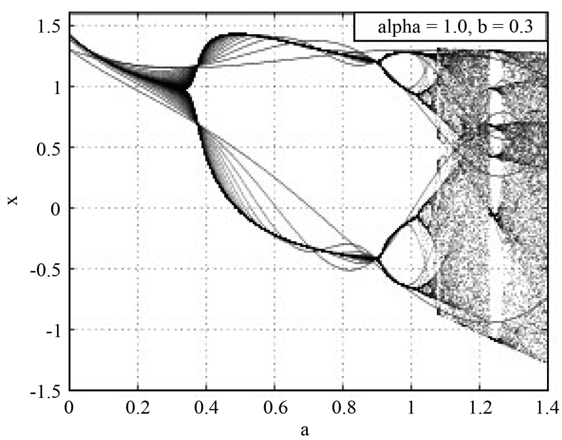

Figure 1. The bifurcation diagram for fractional discrete Henon map when $\alpha = 1$, $x(0) = 0$, $y(0) = 0$.

$x_0 = 0, y_0 = 0, \alpha = 1, \beta = 0.3$ of the plane will approach a set of points known as the Henon strange attractor, see **Figure 2**.

Using the numerical formula (6), set the step size $\triangle\alpha = 0.002$ and different fractional difference order μ, the bifurcation diagrams are plotted in **Figures 3-6**. We can readily obtain the intervals of μ where the chaos

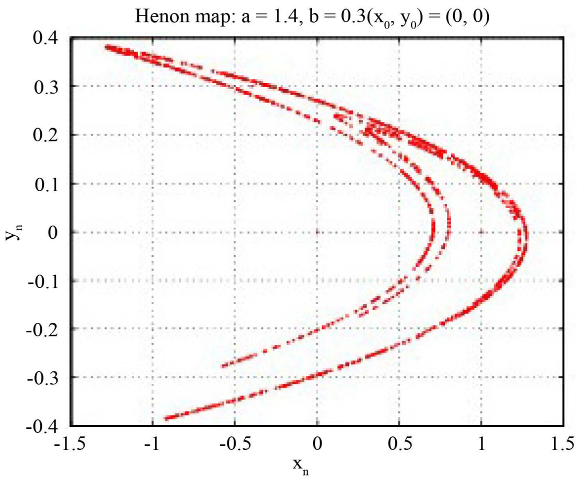

Figure 2. Henon strange attractor for fractional discrete Henon map when $\alpha = 1$, $x(0) = 0$.

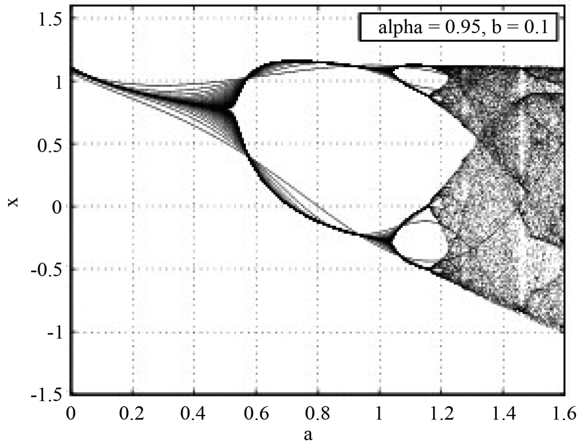

Figure 3. The bifurcation diagram of the fractional discrete Henon map when $\mu = 0.95$.

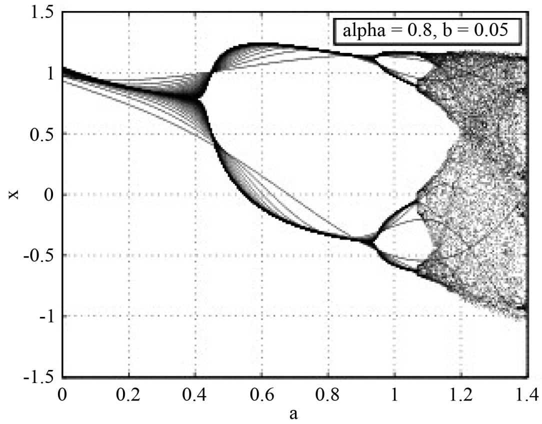

Figure 4. The bifurcation for fractional discrete Henon map diagram when $\mu = 0.8$.

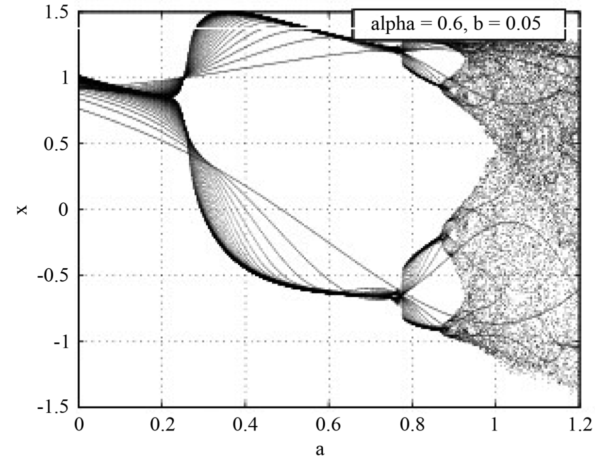

Figure 5. The bifurcation diagram for fractional discrete Henon map when $\mu = 0.6$.

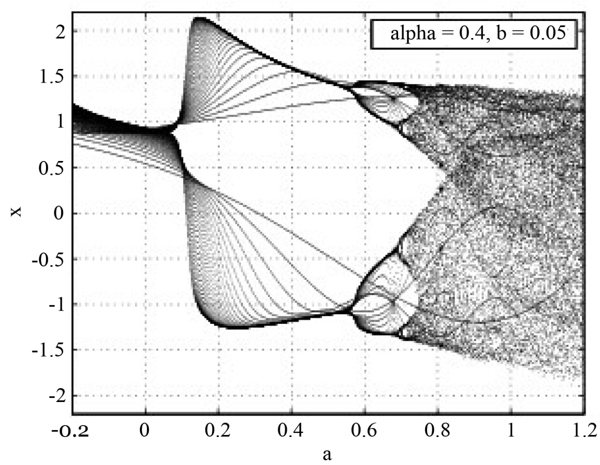

Figure 6. The bifurcation for fractional discrete Henon map diagram when $\mu = 0.4$.

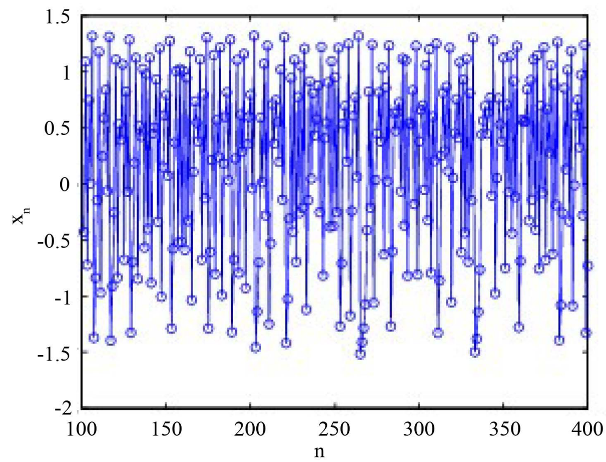

Figure 7. Chaos of the fractional discrete Henon map when $\mu = 0.1$.

happens. It can be concluded that the chaos zones are clearly different when we change the difference order μ, moreover, when $\mu = 0.1$, the chaos still happened, see **Figure 7**.

4. Conclusion

In this paper, the suggested fractional Henon map demonstrates a chaotic behavior with a new type of attractors. The interesting property of the fractional map is the long term memory. Computer simulations of the fractional discrete maps with memory prove that the nonlinear dynamical systems, which are described by the equations with Caputo discrete delta's sense, exhibit a new type of chaotic motion. Through a discrete fractional Henon map it reveals that the dynamical behavior holds the discrete memory even the difference order is very small.

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