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A Note on the Nullity of Unicyclic Graphs

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Abstract

The nullity of a graph is the multiplicity of the eigenvalue zero in its spectrum. In this paper we show the expression of the nullity and nullity set of unicyclic graphs with *n* vertices and girth *r*, and characterize the unicyclic graphs with extremal nullity.

Keywords

Eigenvalues (of Graphs), Nullity, Unicyclic Graphs

1. Introduction

Let $G=(V,E)$ be a simple undirected graph with n vertices. The disjoint union of two graphs G_1 and G_2 is denoted by $G_1 \cup G_2$. The *null graph* of order n is the graph with n vertices and no edges. As usual, the star, path, cycle and the complete graph of order n are denoted by S_n, P_n, C_n and K_n, respectively. An isolated vertex is sometimes denoted by K_1.

Let $A(G)$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of G, and to form the spectrum of this graph. The number of zero eigenvalues in the spectrum of the graph G is called its nullity and is denoted by $\eta(G)$. Let $r(G)$ be the rank of $A(G)$. Clearly, $\eta(G) = n - r(G)$.

A graph is said to be *singular* (*nonsingular*) if its adjacency matrix $A(G)$ is a singular (nonsingular) matrix.

In [1], L. Collatz and U. Sinogowitz first posed the problem of characterizing all graphs which satisfying $\eta(G) > 0$. This question is of great interest in chemistry, because, as has been shown in [2], for a bipartite graph G (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates the molecule which such a graph represents is unstable. The nullity of a graph is also important in mathematics, since it is related to the singularity of $A(G)$. The problem has not yet been solved completely. Some results on trees and it's line graphs, bipartite graphs, unicyclic graphs, bicyclic graphs and tricyclic graphs are known (see [3]-[14]). For details and further references we see [15] [16].

A *unicyclic graph* is a simple connected graph in which the number of edges equals the number of vertices.

The length of the shortest cycle in a graph G is called the *girth* of G, denoted by $g(G)$. If G is a unicyclic graph, then the girth of G is the length of the only cycle in G.

Let U_n be the set of all unicyclic graph with n vertices and let $U(n, r)$ be the set of all unicyclic graphs with n vertices and girth r. A subset N of $\{0, 1, 2, ..., n\}$ is said to be the *nullity set* of $U(n, r)$ provided that for any $k\in N$, there exists at least one graph $U \in \mathcal{U}(n,r)$ such that $\eta(U)=k$, and no $k \notin N$ satisfies this property.

A *matching* of G is a set of independent edges of G, a *maximal matching* is a matching with maximum possible number of edges. The collection of all maximal matching is denoted by $M(G)$, for any $M \in \mathcal{M}(G)$, the size of M, *i.e.*, the maximum number of independent edges in G, is denoted by $m=m(G)$. If n is even and $m=n/2$, then we call the maximal matching a *perfect matching* of G, shot for *PM*.

It is difficult to give an expression of the nullity of a graph, so many papers give that the upper bound of the nullity of some specific graphs and characterized the extremal graphs attaining the upper bound (see [6] [9] [11] [12] [14] [17]). For the trees we know the following concise formula:

Theorem 1.1 [3] If t is a tree with n vertices and m is the size of its maximal matchings, then its nullity is equal to $\eta(T)=n-m$.

Theorem 1.1 implies to $\eta(T)=0$ if and only if T is a *PM*-tree.

In this paper we show the expression of the nullity and nullity set of unicyclic graphs with n vertices and girth r, and characterize the unicyclic graphs with extremal nullity. For terminology and notation not defined here we refer to [3].

2. Some Lemmas

The following lemmas are needed, Lemmas 2.1 and Lemma 2.3 are clear.

Lemma 2.1 Let H be an induced subgraph of G. Then $r(H)\le r(G)$,

Lemma 2.2 Let H be an induced subgraph of G. Then $\eta(G)\le\eta(H)$.

Proof. $\eta(G)=n-r(G)\le n-r(H)=\eta(H)$.

Lemma 2.3 Let $G=G_1\cup G_2\cup\cdots\cup G_t$, then $\eta(G)=\sum_{i=1}^{t}\eta(G_i)$,

where $G_1,G_2,\cdots,G_t$ are connected components of G.

Lemma 2.4 [14]

$$r(C_p)=\begin{cases} n-2, & \text{if } p\equiv 0 \pmod 4;\\ n, & \text{if } p\ne 0 \pmod 4).\end{cases}$$

Let $U\in\mathcal{U}(n,r)$, if $r = n$, then by Lemma 2.4 we have

Lemma 2.5

$$\eta(C_n)=n-r(C_n)=\begin{cases} 2, & \text{if } n\equiv 0 \pmod 4;\\ 0, & \text{if } n\ne 0 \pmod 4).\end{cases}$$

So we discuss that $r < n$ in the following unicyclics. Let $U_0(n, r)$ be the set of all unicyclic graphs with n vertices and girth r and $r < n$, let $U_{0,1}(n, r)$ be the subset of $U_0(n, r)$ with odd girth r and let $U_{0,2}(n, r)$ be the subset of $U_0(n, r)$ with even girth r, clearly $\mathcal{U}(n,r)=\mathcal{U}_0(n,r)\cup\{C_n\}$ and $\mathcal{U}_0(n,r)=\mathcal{U}_{0,1}(n,r)\cup\mathcal{U}_{0,2}(n,r)$.

Lemma 2.6 [3] For a graph G containing a vertex of degree 1, if the induced subgraph H (of G) is obtained by deleting this vertex together with the vertex adjacent to it, then the relation $\eta(H)=\eta(G)$ holds.

The characteristic polynomial of graph G is denoted by

$$\phi(G,x)=\det(xI-A(G))=\sum_{i-0}^{n}c_i x^{n-i} \tag{1}$$

Lemma 2.7 [3] Let $\phi(G,x)=\sum_{i-0}^{n}c_i x^{n-i}$. Then the coefficient of x^{n-i} is

$$c_i=\sum_H(-1)^{k(H)}2^{c(H)}. \tag{2}$$

where the sum is over all subgraphs *H* of *G* consisting of disjoint edges and cycles, and having *i* vertices. If *H* is such a subgraph then *k*(*H*) is the number of components in it and *c*(*H*) is the number of cycles.

Let $i = n$ in (2), then $c_n = \sum_H (-1)^{k(H)} 2^{c(H)}$, where *H* is spanning subgraphs of *G* consisting of disjoint edges and cycles.

3. Main Results

In [18], *Ashraf* and *Bamdad* considered the opposite problem: which graphs have nullity zero? Clearly, for a graph *G*, $\eta(G) = 0$ if and only if $c_n \neq 0$ and $\eta(G) > 0$ if and only if $c_n = 0$ in (1). So by (1) we have following theorem, that is

Theorem 3.1 For a graph *G*,

1) $\eta(G) = 0$ if and only if $\sum_H (-1)^{k(H)} 2^{c(H)} \neq 0$,

2) $\eta(G) > 0$ if and only if $\sum_H (-1)^{k(H)} 2^{c(H)} = 0$.

where the sum is over all spanning subgraphs *H* of *G* consisting of disjoint edges and cycles.

Proof. By (1) it is clear.

By (1) we know also that $\eta(G) = n - i$ if and only if there exist $i \in \{2,3,\cdots,n\}$, such that $c_i \neq 0$ and $c_{i+1} = c_{i+2} = \cdots = c_n = 0$ (Note that $c_0 = 1$ and $c_1 = 0$). So we have

Corollary 3.1 For a graph *G*, $\eta(G) = n - i = n - |V(H)|$ if and only if $\sum_H (-1)^{k(H)} 2^{c(H)} \neq 0$ for $|V(H)| = i$ and $\sum_H (-1)^{k(H)} 2^{c(H)} = 0$ for $|V(H)| > i$ in (2).

Let *U* be a unicyclic graph with girth *r*, Let *H* be a subgraphs of *U* consisting of disjoint edges and cycles with maximum possible number of vertices. Let *H* be the collection of all *H*. Since *U* is unicyclic graph, then *H* have two types: $C_r \cup m(U - V(C_r))P_2$ and $m(U)P_2$, where C_r is induced subgraph of *U* and mP_2 is disjoint union of *m* edges P_2. Let $\mathcal{H}_1 = \{C_r \cup m(U - V(C_r))P_2\} \subset \mathcal{H}$ and $\mathcal{H}_2 = \{m(U)P_2\} \subset \mathcal{H}$, clearly $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{M}(U)$. If $r \equiv 0 \pmod 2$, then $|V(C_r \cup m(U - V(C_r))P_2)| = |V(m(U)P_2)| = 2m(U)$.

Since *U* doesn't contains a subgraph G_1 consisting of disjoint edges and cycles, such that $|V(G_1)| > \max\{r + 2m(U - V(C_r)), 2m(U)\}$, hence for $|V(G_1)| > \max\{r + 2m(U - V(C_r)), 2m(U)\}$, $\sum_{G_1} (-1)^{k(H)} 2^{c(H)} = 0$. So we have

Corollary 3.2 Let *U* be a unicyclic graph with girth *r*, then $\eta(G) = n - |V(H)|$ if and only if $\sum_{H \in \mathcal{H}} (-1)^{k(H)} 2^{c(H)} \neq 0$, where $|V(H)| = \max\{r + 2m(U - V(C_r)), 2m(U)\}$.

Theorem 3.2 Let $U \in \mathcal{U}_0(n, r)$, then

$$\eta(U) = \begin{cases} n - \max\{r + 2m(U - V(C_r)), 2m(U)\}, & \text{if } r \equiv 1 \pmod 2; \\ n - 2m(U), & \text{if } r \equiv 2 \pmod 4; \\ n - 2m(U), & \text{if } r \equiv 0 \pmod 4 \text{ and satisfies (i)}; \\ n - 2m(U) + 2, & \text{if } r \equiv 0 \pmod 4 \text{ and satisfies (ii)}. \end{cases}$$

1) there exist $M \in \mathcal{M}(U)$, for any *r*/2 edges in *M*, such that they not all belong to $E(C_r)$;

2) for any $M \in \mathcal{M}(U)$, there exist *r*/2 edges in *M*, such that they all belong to $E(C_r)$.

Where C_r is induced subgraph of *U*.

Proof. Let $U \in \mathcal{U}_0(n, r)$ and let C_r be an induced subgraph of *U*. By Corollary 2.2, we only need to discuss that $\sum_{H \in \mathcal{H}} (-1)^{k(H)} 2^{c(H)}$ whether equals zero. We give a sign $e_1, e_2, \cdots, e_r$ for the edges of C_r, in nature order.

Case 1. $r \equiv 1 \pmod 2$. Since $|C_r \cup m(U - V(C_r))P_2| = r + 2m(U - V(C_r))$ is odd and $|m(U)P_2| = 2m(U)$

is even, $r+2m(U-V(C_r)) \neq 2m(U)$, hence either $H \in \mathcal{H}_1$ or $H \in \mathcal{H}_2$. If $r+2m(U-V(C_r)) > 2m(U)$, then $H \in \mathcal{H}_1$ and $H \notin \mathcal{H}_2$. Since for all $H \in \mathcal{H}_1$, they have the same number of component, hence $\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)} \neq 0$, where $|V(H)| = r+2m(U-V(C_r))$. If $r+2m(U-V(C_r)) < 2m(U)$, then $H \in \mathcal{H}_2$ and $H \notin \mathcal{H}_1$. Similarly, $\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)} \neq 0$, where $|V(H)| = 2m(U)$. Thus $\eta(U) = n - \max\{r+2m(U-V(C_r)), 2m(U)\}$.

Case 2. $r \equiv 2 \pmod 4$.

Subcase 2.1 There exist $H_0 \in \mathcal{H}_1$, where $H_0 = C_r \cup m(U-V(C_r))P_2$. In this case, the $H_1 = e_1 \cup e_3 \cup \cdots \cup e_{r-1} \cup m(U-V(C_r))P_2 = m(U)P_2 \in \mathcal{H}_2 \subset \mathcal{H}$ and $H_2 = e_2 \cup e_4 \cup \cdots \cup e_r \cup m(U-V(C_r))P_2 = m(U)P_2 \in \mathcal{H}_2 \subset \mathcal{H}$, where the $m(U-V(C_r))P_2$ in H_0, H_1 and H_2 are same, and we call H_1 and H_2 are conjugate subgraph of H_0. Since $r/2$ is odd, hence for any $H \in$ H, the number of component of H have the same odevity, hence $\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)} \neq 0$, where $|V(H)| = 2m(U)$.

Subcase 2.2 There doesn't exist $H \in \mathcal{H}_1$. In this case, since all $H \in \mathcal{H}_2 \subset \mathcal{H}$ and they have the same edges, hence $\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)} \neq 0$, where $|V(H)| = 2m(U)$. So $\eta(U) = n - 2m(U)$.

Case 3. $r \equiv 0 \pmod 4$ and there exist $M \in \mathcal{M}(U)$, for any $r/2$ edges in M, such that they not all belong to $E(C_r)$.

Subcase 3.1 There exist $H_0 \in$ H$_1$, where $H_0 = C_r \cup m(U-V(C_r))P_2$. In this case, the $H_1 = e_1 \cup e_3 \cup \cdots \cup e_{r-1} \cup m(U-V(C_r))P_2 = m(U)P_2 \in \mathcal{H}_2 \subset \mathcal{H}$ and $H_2 = e_2 \cup e_4 \cup \cdots \cup e_r \cup m(U-V(C_r))P_2 = m(U)P_2 \in \mathcal{H}_2 \subset \mathcal{H}$. Let $\mathcal{H}' = \{H_0, H_1, H_2\} \subset \mathcal{H}$. For $H \in \mathcal{H}'$, we have

$$\begin{aligned}\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)} &= (-1)^{1+m(U-V(C_r))}2 + (-1)^{r/2+m(U-V(C_r))} + (-1)^{r/2+m(U-V(C_r))} \\ &= (-1)^{1+m(U-V(C_r))}2 + (-1)^{r/2+m(U-V(C_r))}2 = 0\end{aligned}.$$

Since we know that there exist $M \in \mathcal{M}(U)$, for any $r/2$ edges in M, such that they not all belong to $E(C_r)$, hence we assume that $M = H_3 (= m(U)P_2) \in \mathcal{H}_2$ and for any $r/2$ edges in H_3, such that they not all belong to $E(C_r)$. Except H_3, if there exist others $H_i \in \mathcal{H}_2$ $(i \geq 4)$ and for any $r/2$ edges in H_i $(i \geq 4)$, such that they not all belong to $E(C_r)$, then we have

$$\begin{aligned}\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)} &= (-1)^{1+m(U-V(C_r))}2 + (-1)^{r/2+m(U-V(C_r))} + (-1)^{r/2+m(U-V(C_r))} \\ &\quad + (-1)^{m(U)} + (-1)^{m(U)} + \cdots \\ &= (-1)^{m(U)} + (-1)^{m(U)} + \cdots \neq 0\end{aligned}$$

and $|V(H)| = 2m(U)$, so $\eta(U) = n - 2m(U)$.

Subcase 3.2 There aren't exist $H \in \mathcal{H}_1$. In this case, similar to Subcase 2.2 of Case 2, we have $\eta(U) = n - 2m(U)$.

Case 4. $r \equiv 0 \pmod 4$ and for any $M \in \mathcal{M}(U)$, there exist $r/2$ edges in M, such that they all belong to $E(C_r)$. In this case, for any $M = H_1 \in \mathcal{H}_2$, let $H_1 = e_1' \cup e_2' \cup \cdots \cup e_{r/2}' \cup m(U-V(C_r))P_2$, where e_i' $(i = 1, 2, \cdots, r/2)$ is independent edges in C_r. For the same $m(U-V(C_r))P_2$ with H_1, let $H_2 = e_{r/2+1}' \cup e_{r/2+2}' \cup \cdots \cup e_r' \cup m(U-V(C_r))P_2$ and $H_0 = C_r \cup m(U-V(C_r))P_2$, where $e_{r/2+i}'$ $(i = 1, 2, \cdots, r/2)$ is also independent edges in C_r, then $H_2 \in \mathcal{H}_2$ and $H_0 \in \mathcal{H}_1$. In fact, in this case for any one $H' \in \mathcal{H}_2$, there exist a conjugate graph $H''(\in \mathcal{H}_2)$ of H', such that $H \in \mathcal{H}_1$, where H' and H'' are conjugate subgraphs of H, that is $V(H) = V(H') = V(H'')$ and $E(H) = E(H') \cup E(H'')$. Similarly, for any one $H \in \mathcal{H}_1$, it corresponding two conjugate subgraphs $H', H'' \in \mathcal{H}_2$. So

$$\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)}=(-1)^{1+m(U-V(C_r))}2+(-1)^{r/2+m(U-V(C_r))}+(-1)^{r/2+m(U-V(C_r))}+\cdots=0,$$

where $|V(H)|=2m(U)$. Since $\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)}=0$ if $|V(H)|=2m(U)$, thus we consider the subgraph H of U consisting of disjoint edges and cycles, and having $m(U)-1$ edges. Clearly there exist a $(m(U)-1)$-matching, such that there exist $r/2-1$ edges belong in $E(C_r)$ and $m(U-V(C_r))$ edges belong in $U-V(C_r)$. Similar to Case 3, $\sum_{H\in\mathcal{H}}(-1)^{k(H)}2^{c(H)}\neq 0$, where $|V(H)|=2(m(U)-1)$. So $\eta(U)=n-2m(U)+2$.

Let C_r be a cycle and let P_{n-r} be a path. Suppose that v is a vertex of C_r and u is a pendant vertex of P_{n-r}. Joining v and u by an edge, the resulting graph (**Figure 1**) is denoted by $U(r, n-r)$.

Corollary 3.3 Let $U\in\mathcal{U}_0(n,r)$, then $\eta(U)\le\begin{cases} n-r-1, & \text{if } r=1(\bmod 2);\\ n-r, & \text{if } r=0(\bmod 2).\end{cases}$

Proof. Since $r<n$, hence U contains an induced subgraph $U(r, 1)$ (see **Figure 1**).

Case 1. $r\equiv 1(\bmod 2)$. In this case, by Theorem 2.2 we have $\eta(U(r,1))=n-\max\{r+2m(U(r,1)-V(C_r)),2m(U(r-1))\}=n-2m(U(r-1))=n-r-1$, by Lemma 2.2 we have $\eta(U)\le n-r-1$.

Case 2. $r\equiv 0(\bmod 2)$. In this case, if $r\equiv 2(\bmod 4)$, by Theorem 2.2 we have $\eta(U(r,1))=n-2m(U(r,1))=n-r$. If $r\equiv 0(\bmod 4)$, then there exist $M\in\mathcal{M}(U(r,1))$, such that the pendant edge belong to M, that is for any $r/2$ edges in M, it not all belong to $E(C_r)$, so $\eta(U(r,1))=n-2m(U(r,1))=n-r$, by Lemma 2.2 we have $\eta(U)\le n-r$. □

Let $r=3$ if r is odd and let $r=4$ if r is even in Corollary 2.3, and combine to Lemma 2.7 we have

Corollary 3.4 [18] For any $U\in\mathcal{U}_n$ $(n\ge 5)$, $\eta(U)\le n-4$.

Corollary 3.5 Let $U\in\mathcal{U}_{0,1}(n,r)$, then $\eta(U)=0$ if and only if n is even and U contains PM or n is odd and $U-V(C_r)$ contains PM.

Proof. Let $U\in\mathcal{U}_{0,1}(n,r)$, where r is odd.

"⇒" If $\eta(U)=0$, then by Theorem 2.2 we have $\max\{r+2m(U-V(C_r)),2m(U)\}=n$.

Case 1. If n is even, then $2m(U)=n$, U contains PM.

Case 2. If n is odd, then $r+2m(U-V(C_r))=n$, $2m(U-V(C_r))=n-r$, $U-V(C_r)$ contains PM.

"⇐"

Case 1. If n is even and U contains PM, then $\max\{r+2m(U-V(C_r)),2m(U)\}=2m(U)=n$, by Theorem 2.2, $\eta(U)=0$.

Case 2. If n is odd and $U-V(C_r)$ contains PM, then

$\max\{r+2m(U-V(C_r)),2m(U)\}=r+2m(U-V(C_r))=r+(n-r)=n$, by Theorem 2.2, $\eta(U)=0$. □

Corollary 3.6 Let $U\in\mathcal{U}_{0,2}(n,r)$, then $\eta(U)=0$ if and only if $n\equiv 2(\bmod 4)$ and U contains PM or $n\equiv 0(\bmod 4)$ and U contains PM, and for any $r/2$ edges in the PM, such that they not all belong to $E(C_r)$.

Figure 1. The unicyclic graph $U(r, n-r)$ and $U(r, 1)$.

Proof. Let $U \in \mathcal{U}_{0,2}(n,r)$, where *r* is even.

"⇒" If $\eta(U)=0$, then by theorem 2.2 we have $n-2m(U)=0$ or $n-2m(U)+2=0$. If $n-2m(U)+2=0$, then $m(U)=n/2+1$, a contradiction. So we have $n-2m(U)=0$, *U* contains *PM*. Since *r* is even, hence $r\equiv 2(\mathrm{mod}\,4)$ or $r\equiv 0(\mathrm{mod}\,4)$. If $r\equiv 0(\mathrm{mod}\,4)$, then there exist *PM*, for any *r*/2 edges in the *PM*, such that they not all belong to $E(C_r)$. Otherwise, by Theorem 2.2 we have $n-2m(U)+2=0$, a contradiction.

"⇐"

Case 1. If $r\equiv 2(\mathrm{mod}\,4)$ and *U* contains *PM*, then by Theorem 2.2 we have $\eta(U)=n-2m(U)=0$.

Case 2. If $r\equiv 0(\mathrm{mod}\,4)$ and *U* contains *PM*, and for any *r*/2 edges in the *PM*, such that it not all belong to $E(C_r)$, then by Theorem 2.2 we have $\eta(U)=n-2m(U)=0$.

An edge belonging to a matching of a graph *G* is said to *cover* its two end-vertices. A vertex *v* is said to be *perfectly covered* (*PC*) if it is covered in all maximal matching of *G* [7].

Any vertex adjacent to a pendent vertex is a *PC*-vertex. However, there may be exist *PC*-vertices adjacent to no pendent vertex. For instance, the central vertex in the path on an odd number of vertices is *PC*.

Let $v_i\ (i=1,2,\cdots,\lceil r/2\rceil)$ be the *PC*-vertices of C_r. Let U_r' be a graph is obtained from C_r, by adding r_i $(0\le r_i\le n-r)$ pendant edges in the *PC*-vertex $v_i\ (i=1,2,\cdots,\lceil r/2\rceil)$ of C_r, respectively. Where $\sum_{i=1}^{\lceil r/2\rceil} r_i = n-r>0$. The degree of *PC*-vertices of U_r' needn't equality, even for some *PC*-vertices, no pendant vertex joint to the *PC*-vertex, but the sum of number of all pendant vertices is $n-r$. For $r=5$ and 6, an U_5' and U_6' see **Figure 2**, the *PC*-vertices are indicated by numbers 1, 2, 3.

Let $\mathcal{U}_1'(n,r)$ be the set of all U_r', where *r* is odd and let $\mathcal{U}_2'(n,r)$ be the set of all U_r', where *r* is even. Clearly $\mathcal{U}_1'(n,r)\subset\mathcal{U}_{0,1}(n,r)$ and $\mathcal{U}_2'(n,r)\subset\mathcal{U}_{0,2}(n,r)$. For any $U\in\mathcal{U}_1'(n,r)$ (i = 1, 2), the *PC*-vertices of C_r is also the *PC*-vertices of *U*, where C_r is inducted subgraph of *U*.

Let $d(v, G)$ denote the *distance* from a vertex *v* to the graph *G*, if $v\in V(G)$, then $d(v,G)=0$.

Corollary 3.7 Let $U\in\mathcal{U}_{0,1}(n,r)$, then $\eta(U)=n-r-1$ if and only if $U\in\mathcal{U}_1'(n,r)$.

Proof. Since $U\in\mathcal{U}_{0,1}(n,r)$, hence *r* is odd.

"⇒" Let $U\in\mathcal{U}_{0,1}(n,r)$, if $\eta(U)=n-r-1$, by Theorem 2.1 we have $\max\{r+2m(U-V(C_r)),2m(U)\}=r+1$. Since *r* is odd, hence $2m(U)=r+1$, $m(U)=(r+1)/2$, so for any pendant *v* of *U*, $d(v,C_r)\le 2$. Otherwise, $m(U)\ge(r+3)/2$, a contradiction. If there exist at least one pendant vertex *v* in *U*, such that $d(v,C_r)=2$, then there exist at least one independent edge in $U-V(C_r)$, so $\max\{r+2m(U-V(C_r)),2m(U)\}\ge r+2m(U-V(C_r))\ge r+2$, $\eta(U)\le n-r-2<n-r-1$, a contradiction. So for any pendant vertex of *U*, $d(v,C_r)=1$. Since there exist $(r+1)/2$ *PC*-vertices in C_r, if there exist pendant edges for every vertices of C_r in *U*, then $\max\{r+2m(U-V(C_r)),2m(U)\}=2m(U)=2r>r+1$, a contradiction. Hence there exist pendant edges for part of vertices of C_r in *U*. If there exist $(r+1)/2+1$ vertices in C_r such that every vertex have pendant edges, then $\max\{r+2m(U-V(C_r)),2m(U)\}\ge 2[(r+1)/2+1]>r+1$, a contradiction. So there exist at most $(r+1)/2$ vertices, such that every vertex have pendant edges, that is all pendant vertices of *U* joint to at most $(r+1)/2$ vertices in C_r. In the neighbor vertices of all pendant vertices of *U*, if there exist $(r-1)/2$ *PC*-vertices and one non *PC*-vertex of C_r, then

$$\max\{r+2m(U-V(C_r)),2m(U)\}\ge 2m(U)\ge 2\left(m(U(r,1))+1\right)=2\left((r+1)/2+1\right)>r+1,$$ a contradiction.

Thus all pendant vertices of *U* are joint to the *PC*-vertices of C_r, thus $U\in\mathcal{U}_1'(n,r)$.

"⇐" Let $U\in\mathcal{U}_1'(n,r)$ (see **Figure 2**), since *r* is odd, $r+2m(U-V(C_r))=r$ and $2m(U)=r+1$, hence $\max\{r+2m(U-V(C_r)),2m(U)\}=r+1$, by Theorem 2.1, we have

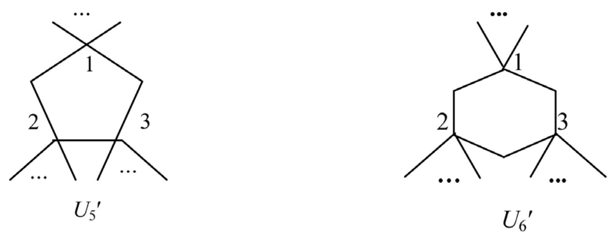

Figure 2. An U_5' and an U_6', its *PC*-vertices are indicated by numbers 1, 2, 3..

$\eta(U) = n - \max\{r + 2m(U - V(C_r)), 2m(U)\} = n - r - 1$. □

Let u be a vertex of C_r, and let v be a k-degree vertex of $K_{1,k+1}$. Joining u and v by a path P_l, the resulting graph is denoted by $U(r, l, k + 1)$, where $r + l + k = n$. When $l = 2$, we get $U(r, 2, k + 1)$ (**Figure 3**).

For convenience, we call the star in $U(r, 2, k + 1)$ is pendant star. Let $U'(r, l, k)$ be a unicyclic graph come from $U(r, l, k + 1)$, by removing a pendant edge and adding it to another vertex of C_r, where $r + l + k = n$ (See **Figure 4**).

Corollary 3.8 Let $U \in \mathcal{U}_{0,2}(n,r)$, then $\eta(U) = n - r$ if and only if $U \in \mathcal{U}_2'(n,r)$ or $U \cong U(r,2,k+1)$ and $r \equiv 0 \pmod 4$

Proof. Since $U \in \mathcal{U}_{0,2}(n,r)$, hence r is even.

"⇒" Let $U \in \mathcal{U}_{0,2}(n,r)$, if $\eta(U) = n - r$, by Theorem 2.2 we have $2m(U) = r$ or $2m(U) - 2 = r$.

Case 1. $2m(U) = r$. In this case, since r is even, hence for any pendant v of U, $d(v, C_r) \le 1$. Otherwise, $m(U) \ge r/2 + 1$, a contradiction. For an edge $uv \in E(C_r)$, If u and v both have at lest one pendant edge in U, respectively. Then $m(U) \ge r/2 + 1$, a contradiction. So all pendant vertices of U join to some *PC*-vertices of U, thus $U \in \mathcal{U}_2'(n,r)$.

Case 2. $2m(U) - 2 = r$. In this case $m(U) = r/2 + 1$, since r is even, hence for any one pendant v of U, $d(v, C_r) \le 3$. Otherwise, $m(U) \ge r/2 + 2$, a contradiction.

Subcase 2.1. There exist $v \in U$, such that $d(v, C_r) = 3$. In this case, $U(n, 3)$ (see **Figure 1**) is an induced subgraph of U, then there exist $M \in \mathcal{M}(U(n,3))$, such that the pendant edge belong to M, so for any $r/2$ edges in M, it not all belong to $E(C_r)$, by Theorem 2.1 we have $\eta(U(n,3)) = n - 2m(U(n,3)) = n - r - 2$, by Lemma 2.2, $\eta(U) \le \eta(U(n,3)) = n - r - 2$, a contradiction.

Subcase 2.2. There exist $v \in U$, such that $d(v, C_r) = 2$. In this case, $U(r, 2, k + 1)$ (see **Figure 3**, specially take $k=0$) is an induced subgraph of U, and only one vertex of U have only one pendant star. Otherwise $m(U) \ge r/2 + 2$, a contradiction. If there exist at lest one pendant edge in other one vertex of C_r, the resulting graph is denoted by $U'(r, 2, k)$ (see **Figure 4**). Since there exist $M \in \mathcal{M}(U'(r,2,k))$, such that the two independent pendant edges in $(U'(r,2,k))$ belong to M, we know that $m(U'(r,2,k)) = r/2 + 1$, hence for any $r/2$ edges in M, they not all belong to $E(C_r)$, by Lemma 2.2 and Theorem 2.2 we have $\eta(U) \le \eta(U'(r,2,k)) = n - 2m(U'(r,2,k)) = n - r - 2$, a contradiction. So $U \cong U(r,2,k+1)$ (see **Figure 3**). If $r \equiv 2 \pmod 4$, by Theorem 2.2 we have $\eta(U(r,2,k+1)) = n - 2m(U(r,2,k+1)) = n - r - 2$, a contradiction. So $U \cong U(r,2,k+1)$ and $r \equiv 0 \pmod 4$.

"⇐" **Case 1**. Let $U \in \mathcal{U}_2'(n,r)$ (see **Figure 2**), since r is even, hence $2m(U) = r$. If $r \equiv 2 \pmod 4$, then by Theorem 2.1 we have $\eta(U) = n - 2m(U) = n - r$. If $r \equiv 0 \pmod 4$, since $r < n$, hence U contains a induced subgraph $U(r, 1)$ (see **Figure 1**), for a $M \in \mathcal{M}(U(r,1))$, let the pendant edge of $U(r, 1)$ belong to the M, then the $r/2$ edges in M, not all belong to $E(C_r)$, by Theorem 2.2 we have $\eta(U) = n - 2m(U) = n - r$.

Case 2. Let $U \cong U(r,2,k+1)$ and $r \equiv 0 \pmod 4$, then $m(U(r,2,k+1)) = r/2 + 1$. Since for any

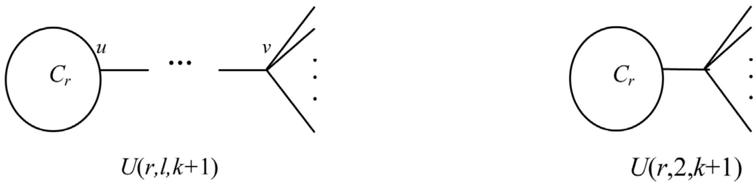

Figure 3. The unicyclic graphs $U(r, l, k)$ and $U(r, 2, k)$.

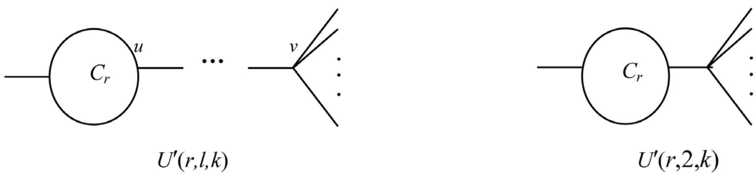

Figure 4. The unicyclic graphs $U'(r, l, k)$ and $U'(r, 2, k)$.

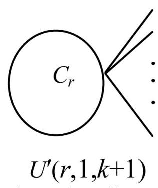

Figure 5. The unicyclic graph $U(r, 1, k + 1)$.

$M \in \mathcal{M}(U(r,2,k+1))$, there exist $r/2$ edges in M, such that they all belong to $E(C_r)$, by Theorem 2.1 we have $\eta(U(r,2,k+1)) = n - 2m(U(r,2,k+1)) + 2 = n - r$. □

Let $l = 1$ in $U(r, l, k+1)$ (**Figure 3**), we get the following graph $U(r, 1, k+1)$ (**Figure 5**).

Theorem 3.3 The nullity set of $U_{0,1}(n, r)$ is $\{0, 1, 2, \ldots, n-r-1\}$.

Proof. By Corollary 2.3, we only need to show that for each $k \in \{0,1,2,\cdots,n-r-1\}$, there exist a unicyclic graph $U \in \mathcal{U}_{0,1}(n,r)$ such that $\eta(U) = k$, where r is odd.

Case 1. $k = 0$. Let $U = U(r, n-r)$ (see **Figure 1**). If $n \equiv 1(\mathrm{mod}\ 2)$, using Lemma 2.6, after $(n - r)/2$ steps, we get C_r, by Lemma 2.6 and 2.5 we have $\eta(U(r,n-r)) = \eta(C_r) = 0$. If $n \equiv 0(\mathrm{mod}\ 2)$, using Lemma 2.6, after $(n - 2)/2$ steps, we get a P_2, by Lemmas 2.6 we have $\eta(U(n,r)) = \eta(P_2) = 0$.

Case 2. $k = n - r - 1$. Let $U = U(r,1,k+1)$ (see **Figure 5**), where $r + k + 1 = n$, using Lemma 2.5, after $(r+1)/2$ steps, we get kK_1, by Lemmas 2.3 we have $\eta(U(r,1,k+1)) = \eta(kK_1) = k = n - r - 1$.

Case 3. $1 \le k \le n - r - 2$. Let $U = U(r,l,k+1)$ (see **Figure 3**), where $r + l + k = n$. If $n \ne k(\mathrm{mod}\ 2)$, Using Lemma 2.6, after $l/2$ steps, we get $C_r \cup kK_1$, by Lemmas 2.3 and 2.5 we have $\eta(U) = \eta(C_r \cup kK_1) = \eta(C_r) + \eta(kK_1) = k$. Similarly, If $n \equiv k(\mathrm{mod}\ 2)$, we have $\eta(U(r,l,k+1)) = \eta(kK_1) = k$.

Theorem 3.4 The nullity set of $U_{0,2}(n, r)$ is $\{0,1,2,\cdots,n-r\}$.

Proof. Similar to Theorem 2.3, if $l \equiv 1(\mathrm{mod}\ 2)$, we consider the graph $U(r, l, k)$ with k pendants (see **Figure 3**), where $r + l + k - 1 = n$. If $l \equiv 0(\mathrm{mod}\ 2)$, we consider the graph $U'(r, l, k)$ with k pendants (see **Figure 4**), where $r + l + k = n$.

If we take $r = 3$ in Theorem 2.3 and $r = 4$ in Theorem 2.4, then we have the following Corollary:

Corollary 3.9 [18] The nullity set of U_n is $\{0,1,2,\cdots,n-4\}$.

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