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A Note on the Almost Sure Central Limit Theorem in the Joint Version for the Maxima and Partial Sums of Certain Stationary Gaussian Sequences*

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Abstract

Considering a sequence of standardized stationary Gaussian random variables, a universal result in the almost sure central limit theorem for maxima and partial sum is established. Our result generalizes and improves that on the almost sure central limit theory previously obtained by Marcin Dudzinski [1]. Our result reaches the optimal form.

Keywords

Almost Sure Central Limit Theorem, Stationary Gaussian Sequence, Slowly Varying Functions at Infinity

1. Introduction

In this paper, we let $\left(X, X_n\right)_{n\in\mathbb{N}}$ be a standardized stationary Gaussian sequence, also let

$$M_n := \max_{1\le i\le n} X_i, \quad M_{m,n} := \max_{m+1\le i\le n} X_i, \quad S_n := \sum_{i=1}^{n} X_i, \quad \sigma_n := \sqrt{\mathrm{Var}\left(S_n\right)}, \quad D_n = \sum_{k=1}^{n} d_k, \quad \left\{d_k\right\}$$

be some sequence of weights, $\mathrm{I}(.)$ and $\Phi(.)$ denote the indicator function and the standard normal distribution function, respectively.

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The ASCLT has been first introduced independently by Brosamler [2] and Schatte [3] for partial sum, since then the concept has already started to receive applications in many fields. For example, Fahrner and Sadtmuller [4] and Cheng *et al.* [5] extended this almost sure central limit theorem for partial sums to the case of maxima of i.i.d. random variables. Under some conditions, they proved as follows:

$$\lim_{n\to\infty}\frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k}\mathrm{I}\left(a_k\left(S_k-b_k\right)\le x\right)=G\left(x\right)\ \text{ a.s.},$$

for all $x\in R$, where $a_k>0$ and $b_k\in R$ satisfy $a_k\left(S_k-b_k\right)\xrightarrow{d}G$, where $G(:)$ is some non-degenerate distribution function. Afterwards, Marcin Dudzinski [5] showed the ASCLT in its two-dimensional version, *i.e.*

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k I\left(a_k\left(M_k-b_k\right)\le x,\frac{S_k}{\sigma_k}\le y\right)=\mathrm{e}^{-\tau}\Phi\left(y\right)\ \text{ a.s. for }\ \forall x,y\in\mathbb{R}. \tag{1.1}$$

In this paper, we extend the weight $d_k=\frac{1}{k}$ to the weight $d_k=\frac{\exp\left(\left(\ln k\right)^{\beta}\right)}{k}$, $0\le\beta<\frac{1}{2}$.

Our purpose is to prove that if $\left(X,X_n\right)_{n\in N}$ is a standardized stationary Gaussian sequence, the covariance function $r\left(t\right)$ fulfills

$$r\left(t\right)=\frac{L\left(t\right)}{t^{\alpha}},\ \ \alpha>0\ \text{ and }\ t=1,2,\cdots, \tag{1.2}$$

where $L\left(t\right)$ is a positive slowly varying function at infinity. Moreover if the numerical sequence $\left\{u_n\right\}$ satisfies the following relation

$$\lim_{n\to\infty}n\left(1-\Phi\left(u_n\right)\right)=\tau\ \text{ for some }\ 0\le\tau<\infty, \tag{1.3}$$

then we have

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k I\left(a_k\left(M_k-b_k\right)\le x,\frac{S_k}{\sigma_k}\le y\right)=\mathrm{e}^{-\tau}\Phi\left(y\right)\ \text{ a.s. for any }\ x,y\in R, \tag{1.4}$$

where $D_n=\sum_{k=1}^{n}d_k$ and $d_k=\frac{\exp\left(\left(\ln k\right)^{\beta}\right)}{k}$, $0\le\beta<\frac{1}{2}$.

In the following, denote by $a_n\sim b_n$ if $\frac{a_n}{b_n}\to 1$ as $n\to\infty$, by $a_n\ll b_n$ if there exists a constant $c>0$ such that $a_n\le cb_n$ for sufficiently large n. The c stands for a constant, which may vary from one line to another.

2. Main Results

Theorem 2.1. Assume that $\left(X,X_n\right)_{n\in\mathbb{N}}$ be a standardized stationary Gaussian sequence, the covariance function $r\left(t\right)$ satisfies (1.2) for some $\alpha>0$. If the numerical sequence $\left\{u_n\right\}$ satisfies (1.3), then (1.4) holds with $d_k=\frac{\exp\left(\left(\ln k\right)^{\beta}\right)}{k}$, $0\le\beta<\frac{1}{2}$.

Let $D=\left\{D_n\right\}_{n\in\mathbb{N}}$ be a positive non-decreasing sequence with $\lim_{n\to\infty}D_n=\infty$. We say that $\left\{x_k\right\}_{k\in\mathbb{N}}$ is D-summable to a finite limit x if

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k x_k=x, \tag{2.1}$$

where $d_k=D_k-D_{k-1}$.

Remark 2.1. By a classical theorem of Harly (see Chandrasekharan and Minakshisundaram [6]), if two sequences D and D^* satisfy $D_n^* = O(D_n)$, then D-summability implies the D^*-summability, *i.e.*, if a sequence $\{x_n\}_{n\in N}$ is D-summable to x, then it is also D^*-summable to x. These results show that if (2.1) holds with the weight $\{d_k\}_{k\in\mathbb{N}}$, then for $0 \le d_k^* \le d_k$, $D_n^* \to \infty$, (2.1) also holds with the weight $\{d_k^*\}_{k\in\mathbb{N}}$. So, if ASCLT holds with $\{d_k\}_{k\in\mathbb{N}}$, then for $0 \le d_k^* \le d_k$, $D_n^* \to \infty$, ASCLT also holds with $\{d_k^*\}_{k\in\mathbb{N}}$. So, by this, if we use larger weights, we should expect to get stronger results. Theorem 2.1 remains valid if we extend the weights from $d_k = \frac{1}{k}$ to $d_k = \frac{\exp\left((\ln k)^\beta\right)}{k}$, $0 < \beta < \frac{1}{2}$. When $\beta = 0$, the weights $d_k = \frac{1}{k}$.

Lemma 2.2. Under the same assumptions as in Theorem 2.1, if the numerical sequence $\{u_n\}$ fulfills (3), then there exists some $\gamma > 0$ and for any $x, y \in \mathbb{R}$ and $m < n$,

$$E\left|I\left(M_n \le u_n, \frac{S_n}{\sigma_n} \le y\right) - I\left(M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y\right)\right| \ll \left(\frac{m}{n}\right)^\gamma, \tag{2.2}$$

$$\left|Cov\left(I\left(M_m \le u_m, \frac{S_m}{\sigma_m} \le y\right), I\left(M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y\right)\right)\right| \ll \left(\frac{m}{n}\right)^\gamma, \tag{2.3}$$

where $u_m = \frac{x}{a_m} + b_m, u_n = \frac{x}{a_n} + b_n$.

3. Proofs

3.1. Proof of Theorem 2.1

Under the assumptions of Theorem 1 on $X_1, X_2, \cdots$, $r(t)$ and $\{u_n\}$, by Theorem 4.3.3 in Leadbetter *et al.* [7], we have $\lim_{n\to\infty} P(M_n \le u_n) = \mathrm{e}^{-\tau}$ for τ which is defined in (1.3). Let y be a real number, for each $n \ge 1$, Y_n denotes a standard normal variable, which is independent of $(X_1, \cdots, X_n)$ and has the same distribution as $\frac{S_n}{\sigma_n}$. From the proof of Lemma 2.2, we get that $\left|P\left(M_n \le u_n, \frac{S_n}{\sigma_n} \le y\right) - P(M_n \le u_n)P(Y_n \le y)\right| \ll \frac{1}{n^\gamma} \to 0$, as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} P\left(M_n \le u_n, \frac{S_n}{\sigma_n} \le y\right) = \lim_{n\to\infty} P(M_n \le u_n)P(Y_n \le y) = \mathrm{e}^{-\tau}\Phi(y),$$

by Toeplitz Lemma, we obtain

$$\lim_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k P\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right) = \mathrm{e}^{-\tau}\Phi(y) \tag{3.1}$$

where $D_n = \sum_{k=1}^n d_k$ and $d_k = \frac{\exp\left((\ln k)^\beta\right)}{k}$, $0 < \beta < \frac{1}{2}$.

To prove Theorem 2.1 for $0 < \beta < \frac{1}{2}$. We should prove the following:

$$\frac{1}{D_n}\sum_{k=1}^n d_k\left[I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right) - P\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right)\right] \to 0 \quad \text{a.s.} \tag{3.2}$$

In order to prove (3.2), it suffices to show the following holds (see Lemma 3.1 in Csaki and Gonchigdanzan

[8]) for some $\varepsilon > 0$

$$\mathrm{Var}\left(\sum_{k=1}^{n} d_k I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right)\right) \ll D_n^2 \left(\ln D_n\right)^{-(1+\varepsilon)} \tag{3.3}$$

Set $\xi_k = I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right) - P\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right)$, by Lemma 2.2, we have

$$\begin{aligned}
\left|E\left(\xi_k \xi_l\right)\right| &= \left|Cov\left(I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right), I\left(M_l \le u_l, \frac{S_l}{\sigma_l} \le y\right)\right)\right| \\
&\le \left|Cov\left(I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right), I\left(M_l \le u_l, \frac{S_l}{\sigma_l} \le y\right)\right) - I\left(M_{k,l} \le u_l, \frac{S_l}{\sigma_l} \le y\right)\right| \\
&\quad + \left|Cov\left(I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right), I\left(M_{k,l} \le u_l, \frac{S_l}{\sigma_l} \le y\right)\right)\right| \\
&\ll E\left|I\left(M_l \le u_l, \frac{S_l}{\sigma_l} \le y\right) - I\left(M_{k,l} \le u_l, \frac{S_l}{\sigma_l} \le y\right)\right| \\
&\quad + \left|Cov\left(I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right), I\left(M_{k,l} \le u_l, \frac{S_l}{\sigma_l} \le y\right)\right)\right| \\
&\ll \left(\frac{k}{l}\right)^{\gamma} \qquad \text{for } 1 \le k < l.
\end{aligned}$$

We know that

$$\mathrm{Var}\left(\sum_{k=1}^{n} d_k I\left(M_k \le u_k, \frac{S_k}{\sigma_k} \le y\right)\right) \le E\left(\sum_{k=1}^{n} d_k \xi_k\right)^2 \le 2 \sum_{1 \le k \le l \le n} d_k d_l \left|E\left(\xi_k \xi_l\right)\right|$$

$$\ll \sum_{1 \le k \le l \le n, \frac{k}{l} \le (\ln D_n)^{-\frac{2}{\gamma}}} d_k d_l \left(\frac{k}{l}\right)^{\gamma} + \sum_{1 \le k \le l \le n, \frac{k}{l} > (\ln D_n)^{-\frac{2}{\gamma}}} d_k d_l =: T_{n1} + T_{n2}$$

For T_{n1}, we have the following estimate:

$$T_{n1} \le \sum_{1 \le k \le l \le n, \frac{k}{l} \le (\ln D_n)^{-\frac{2}{\gamma}}} d_k d_l \left(\frac{k}{l}\right)^{\gamma} \ll \sum_{1 \le k \le l \le n, \frac{k}{l} \le (\ln D_n)^{-\frac{2}{\gamma}}} d_k d_l \frac{1}{\left(\ln D_n\right)^2} \ll \frac{D_n^2}{\left(\ln D_n\right)^2} \ll \frac{D_n^2}{\left(\ln D_n\right)^{1+\varepsilon}}.$$

By the elementary calculation, it is easy to see that $D_n \sim \frac{1}{\beta}(\ln n)^{1-\beta} \exp\left((\ln n)\right)^{\beta}$, $\ln D_n \sim (\ln n)^{\beta}$, $\ln \ln D_n \sim \ln \ln n$.

For $0 < \beta < \frac{1}{2}$, we have $\varepsilon := \frac{1-2\beta}{2\beta} > 0$ and $\frac{1}{2\beta} = 1 + \varepsilon$. Then

$$\begin{aligned}
T_{n2} &\ll \sum_{1 \le k \le l \le n, \frac{k}{l} > (\ln D_n)^{-\frac{2}{\gamma}}} d_k d_l \ll \sum_{1 \le k \le l \le n} d_k \sum_{k \le l < (\ln D_n)^{-\frac{2}{\lambda}}} \frac{\exp\left((\ln n)^{\beta}\right)}{l} \ll \frac{D_n}{\left(\ln D_n\right)^{\frac{1-\beta}{\beta}}} \sum_{k=1}^{n} d_k \ln \ln D_n \ll \frac{D_n^2 \ln \ln D_n}{\left(\ln D_n\right)^{\frac{1-\beta}{\beta}}} \\
&= \frac{D_n^2}{\left(\ln D_n\right)^{\frac{1}{2\beta}}} \frac{\ln \ln D_n}{\left(\ln D_n\right)^{\frac{1-2\beta}{2\beta}}} \ll \frac{D_n^2}{\left(\ln D_n\right)^{\frac{1}{2\beta}}} = \frac{D_n^2}{\left(\ln D_n\right)^{1+\varepsilon}}
\end{aligned}$$

Then (9) and thus (8) follows from above estimates. By using (7), we complete the proof of **Theorem 2.1**.

3.2. Proof of Lemma 2.2

We first consider that (1.2) holds with some $0<\alpha<1$. Let $1\le i\le n$, we have

$$\begin{aligned}0<Cov\left(X_i,\frac{S_n}{\sigma_n}\right)&=\frac{1}{\sigma_n}Cov\left(X_i,\sum_{j=1}^{n}X_j\right)=\frac{1}{\sigma_n}\left(1+\sum_{j=1}^{i-1}r(i-j)+\sum_{j=i+1}^{n}r(j-i)\right)\\&\ll\frac{2}{\sigma_n}+\frac{2}{\sigma_n}\sum_{t=1}^{n-1}\frac{L(t)}{t^{\alpha}}\quad\text{for some }\ 0<\alpha<1\end{aligned}. \tag{4.1}$$

By the application of the Karamata's theorem (see Mielniczuk [9]), we obtain

$$\sigma_n\sim\left(\frac{2}{(1-\alpha)(2-\alpha)}\right)^{\frac{1}{2}}L(n)^{\frac{1}{2}}n^{1-\frac{\alpha}{2}}. \tag{4.2}$$

From (4.1) and (4.2), for some $0<\alpha<1$, we have

$$Cov\left(X_i,\frac{S_n}{\sigma_n}\right)\ll\frac{1}{L(n)^{\frac{1}{2}}n^{1-\frac{\alpha}{2}}}\sum_{t=1}^{n-1}\frac{L(t)}{t^{\alpha}}\ll\frac{L(n)n^{1-\alpha}}{(L(n))^{\frac{1}{2}}n^{1-\frac{\alpha}{2}}}=\frac{(L(n))^{\frac{1}{2}}}{n^{\frac{\alpha}{2}}}.$$

Since $L(n)$ is a slowly varying function at infinity, for any $\varepsilon>0$, $L(n)\ll n^{\varepsilon}$. So $Cov\left(X_i,\frac{S_n}{\sigma_n}\right)\ll\frac{n^{\varepsilon}}{n^{\frac{\alpha}{2}}}=\frac{1}{n^{\frac{\alpha}{2}-\varepsilon}}$ for any $\varepsilon>0$ and some $0<\alpha<1$. Hence,

$$0<\sup_{1\le i\le n}Cov\left(X_i,\frac{S_n}{\sigma_n}\right)\ll\frac{1}{n^{\beta}}\to 0,\ \text{as}\ n\to\infty,\ \text{for some}\ 0<\beta<\frac{1}{2}, \tag{4.3}$$

$$\text{so there exist}\ \lambda,\ n_0\ \text{such that}\ 0<\sup_{1\le i\le n}Cov\left(X_i,\frac{S_n}{\sigma_n}\right)<\lambda<1,\ \text{for any}\ n>n_0. \tag{4.4}$$

By (1.2), $\lim_{t\to\infty}r(t)=\lim_{t\to\infty}\frac{L(t)}{t^{\alpha}}=0$, thus there exist δ, n_1 such that

$$0<\sup_{t>n_1}r(t)=\delta<1. \tag{4.5}$$

Let y be an arbitrary real number and $m<n$. Suppose that Y_n is a random variable, which has the same distribution as $\frac{S_n}{\sigma_n}$, but Y_n is independent of $(X_1,\cdots,X_n)$, then we have

$$\begin{aligned}&E\left|I\left(M_n\le u_n,\frac{S_n}{\sigma_n}\le y\right)-I\left(M_{m,n}\le u_n,\frac{S_n}{\sigma_n}\le y\right)\right|\\&=P\left(M_{m,n}\le u_n,\frac{S_n}{\sigma_n}\le y\right)-P\left(M_n\le u_n,\frac{S_n}{\sigma_n}\le y\right)\\&\le\left|P\left(M_n\le u_n,\frac{S_n}{\sigma_n}\le y\right)-P(M_n\le u_n)P(Y_n\le y)\right|\\&\quad+\left|P\left(M_{m,n}\le u_n,\frac{S_n}{\sigma_n}\le y\right)-P(M_{m,n}\le u_n)P(Y_n\le y)\right|\\&\quad+\left|P(M_{m,n}\le u_n)-P(M_n\le u_n)\right|\\&=:A_1+A_2+A_3\end{aligned}.$$

By Theorem 4.2.1 in Leadbetter *et al.* (1983) and (4.3), (4.4) and (4.5), we get that

$$A_1 + A_2 \ll \sum_{i=1}^{n} Cov\left(X_i, \frac{S_n}{\sigma_n}\right) \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \ll n\frac{1}{n^{\beta}} \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right). \tag{4.6}$$

As we know $\{u_n\}$ fulfills (1.3), which implies that

$$\exp\left(-\frac{u_n^2}{2}\right) \sim \frac{c\sqrt{2\pi}u_n}{n}, \quad u_n \sim (2\ln n)^{\frac{1}{2}}, \quad \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \ll \frac{(\ln n)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}}},$$

combine this and (4.6), we have

$$A_1 + A_2 \ll n\frac{1}{n^{\beta}} \frac{(\ln n)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}}} = \frac{(\ln n)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}+\beta-1}}.$$

We set $\beta + \frac{1}{1+\lambda} - 1 > 0$, thus

$$A_1 + A_2 \ll \frac{1}{n^{\gamma}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for some} \quad 0 < \gamma < \beta + \frac{1}{1+\lambda} - 1. \tag{4.7}$$

Next, we estimate A_3.

$$\begin{aligned} A_3 &= \left|P\left(M_{m,n} \le u_n\right) - P\left(M_n \le u_n\right)\right| \\ &\le \left|P\left(M_{m,n} \le u_n\right) - \prod_{i=m+1}^{n} \Phi\left(u_n\right)\right| + \left|P\left(M_n \le u_n\right) - \prod_{i=1}^{n} \Phi\left(u_n\right)\right| \\ &\quad + \left|\prod_{i=m+1}^{n} \Phi\left(u_n\right) - \prod_{i=1}^{n} \Phi\left(u_n\right)\right| \\ &=: B_1 + B_2 + B_3 \end{aligned} \tag{4.8}$$

Let δ is defined as in (4.5), set $\delta < \frac{\alpha}{2-\alpha}$ such that $\frac{2}{1+\delta} + \alpha - 2 > 0$. By Theorem 4.2.1 in Leadbetter *et al.* and (1.3), we have

$$B_1 + B_2 \ll n\exp\left(-\frac{u_n^2}{1+\delta}\right)\sum_{t=1}^{n} r(t) \ll \frac{(\ln n)^{\frac{1}{1+\delta}}}{n^{\frac{2}{1+\delta}-1}} \sum_{t=1}^{n} \frac{L(t)}{t^{\alpha}} \ll \frac{(\ln n)^{\frac{1}{1+\delta}} L(n)}{n^{\frac{2}{1+\delta}+\alpha-2}},$$

since $L(n)$ and $\ln n$ are slowly varying functions at infinity, we have $(\ln n)^{\frac{1}{1+\delta}} L(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$, then

$$B_1 + B_2 \ll \frac{1}{n^{\gamma}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for some } \gamma > 0. \tag{4.9}$$

Meanwhile, following from the elementary inequality $x^{n-m} - x^n \le \frac{m}{n}$, $0 \le x \le 1$. We get

$$B_3 \le \frac{m}{n} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for } m < n. \tag{4.10}$$

By (4.8), (4.9) and (4.10), if (1.2) holds with $0 < \alpha < 1$, then (2.2) holds.

Provided (1.2) also holds with some $\alpha \ge 1$, since $r(.)$ is positive, we get $\sigma_n = \sqrt{\mathrm{Var}(S_n)} \ge n^{\frac{1}{2}}$, this imply

$0 < Cov\left(X_i, \frac{S_n}{\sigma_n}\right) \ll \frac{2}{n^{\frac{1}{2}}} \sum_{t=1}^{n-1} \frac{L(t)}{t^{\alpha}}$. As $\sum_{t=1}^{n} \frac{L(t)}{t^{\alpha}}$ is a slowly varying function infinity, we get $\sum_{t=1}^{n} \frac{L(t)}{t^{\alpha}} \ll n^{\varepsilon}$ for any $\varepsilon > 0$. We obtain that (4.3) and (4.4). By using Theorem 4.2.1 in Leadbetter *et al.* and by (4.6)-(4.10), let A_1 - A_3 be replaced by C_1 - C_3 in (4.3), we also get $C_1 + C_2 \ll \left(\frac{m}{n}\right)^{\gamma}$, let D_1 - D_3 be replaced by B_1 - B_3 in (4.8), we also obtain that $D_1 + D_2 \ll n \exp\left(-\frac{u_n^2}{1+\delta}\right) \sum_{t=1}^{n} r(t) \ll \frac{(\ln n)^{\frac{1}{1+\delta}}}{n^{\frac{2}{1+\delta}-1}} n^{\varepsilon} \ll \left(\frac{m}{n}\right)^{\gamma}$ for some $\gamma > 0$, like (4.11), $D_3 \le \frac{m}{n} \ll \left(\frac{m}{n}\right)^{\gamma}$, we get $C_3 \ll \left(\frac{m}{n}\right)^{\gamma}$, at last , we obtain when (1.2) holds with $\alpha \ge 1$, then (2.2) also holds.

Next, we prove (2.3). First we consider the situation of some $0 < \alpha < 1$. Let $i \ge m+1$. Since $L(t)$ is a positive slowly varying function, then we have $(L(m))^{\frac{1}{2}} \ll m^{\varepsilon}$ for any $\varepsilon > 0$, $\frac{L(t)}{t^{\alpha}}$ is monotonic decreasing, so $\sum_{t=i-m}^{i-1} \frac{L(t)}{t^{\alpha}} \ll \sum_{t=1}^{m} \frac{L(t)}{t^{\alpha}}$, by (1.2) and (4.2), we obtain

$$
\begin{aligned}
0 < Cov\left(X_i, \frac{S_m}{\sigma_m}\right) &\ll \frac{1}{\sigma_m} \sum_{t=i-m}^{i-1} \frac{L(t)}{t^{\alpha}} \ll \frac{1}{L(m)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}} \sum_{t=i-m}^{i-1} \frac{L(t)}{t^{\alpha}} \\
&\ll \frac{1}{L(m)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}} \sum_{t=1}^{m} \frac{L(t)}{t^{\alpha}} \ll \frac{L(m) m^{1-\alpha}}{L(m)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}} = \frac{L(m)^{\frac{1}{2}}}{m^{\frac{\alpha}{2}}}, \\
&\ll \frac{m^{\varepsilon}}{m^{\frac{\alpha}{2}}} \ll \frac{1}{m^{\eta}} \qquad \text{for some } 0 < \eta < \frac{1}{2}
\end{aligned}
$$

then there exist numbers μ, m_0, such that $0 < \sup_{i \ge m+1} Cov\left(X_i, \frac{S_m}{\sigma_m}\right) < \mu < 1$ for all $m > m_0$. By the Normal Comparison Lemma, we obtain

$$
\begin{aligned}
&\left| Cov\left(I\left(M_m \le u_m, \frac{S_m}{\sigma_m} \le y\right), I\left(M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y\right)\right)\right| \\
&= \left| P\left(M_m \le u_m, \frac{S_m}{\sigma_m} \le y, M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y\right) - P\left(M_m \le u_m, \frac{S_m}{\sigma_m} \le y\right) P\left(M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y\right)\right| \\
&\ll \sum_{i=1}^{m} \sum_{j=m+1}^{n} r(j-i) \exp\left(-\frac{u_m^2 + u_n^2}{2(1+r(j-i))}\right) + \sum_{i=1}^{m} Cov\left(X_i, \frac{S_n}{\sigma_n}\right) \exp\left(-\frac{u_m^2}{2\left(1+Cov\left(X_i, \frac{S_n}{\sigma_n}\right)\right)}\right) \qquad (4.11) \\
&\quad + \sum_{i=m+1}^{n} Cov\left(X_i, \frac{S_m}{\sigma_m}\right) \exp\left(-\frac{u_n^2}{2\left(1+Cov\left(X_i, \frac{S_m}{\sigma_m}\right)\right)}\right) + Cov\left(\frac{S_m}{\sigma_m}, \frac{S_n}{\sigma_n}\right) \\
&=: E_1 + E_2 + E_3 + E_4
\end{aligned}
$$

By (4.5), set $\delta < \frac{\alpha}{2-\alpha} < 1$, then $\frac{2}{1+\delta} + \alpha - 2 > 0$, we have

$$E_1 = \sum_{i=1}^{m}\sum_{j=m+1}^{n} r(j-i)\exp\left(-\frac{u_m^2+u_n^2}{2(1+r(j-i))}\right) \ll \exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\sum_{i=1}^{m}\sum_{t=m+1-i}^{n-i} r(t)$$
$$\ll m\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\sum_{t=m+1-i}^{n-i}\frac{L(t)}{t^{\alpha}} \ll m\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\sum_{t=1}^{n}\frac{L(t)}{t^{\alpha}} .$$
$$\ll \left(\frac{m}{n}\right)^{1-\frac{1}{1+\delta}} \frac{1}{n^{\frac{2}{1+\delta}-2+\alpha}} L(n) \ll \left(\frac{m}{n}\right)^{1-\frac{1}{1+\delta}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for some } 0<\gamma<1$$

By (4.4), set $\lambda < \frac{\beta}{1-\beta} < 1$, then $-\beta + 1 - \frac{1}{1+\lambda} < 0$, we have

$$E_2 = \sum_{i=1}^{m} Cov\left(X_i, \frac{S_n}{\sigma_n}\right)\exp\left(-\frac{u_m^2}{2\left(1+Cov\left(X_i, \frac{S_n}{\sigma_n}\right)\right)}\right)$$
$$\ll \exp\left(-\frac{u_m^2}{2(1+\lambda)}\right)\sum_{i=1}^{m} Cov\left(X_i, \frac{S_n}{\sigma_n}\right) \ll \frac{m^{1-\frac{1}{1+\lambda}}}{n^{\beta}}(\ln m)^{\frac{1}{2(1+\lambda)}} .$$
$$\ll \frac{m^{\beta}}{n^{\beta}} m^{-\beta+1-\frac{1}{1+\lambda}} (\ln m)^{\frac{1}{2(1+\lambda)}} \ll \left(\frac{m}{n}\right)^{\beta} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for } 0<\gamma<\beta<\frac{1}{2}$$

and

$$\sum_{i=m+1}^{n} Cov\left(X_i, \frac{S_m}{\sigma_m}\right) = \frac{1}{\sigma_m}\sum_{i=m+1}^{n}\sum_{j=1}^{m} r(i-j) = \frac{1}{\sigma_m}\sum_{i=1}^{m}\sum_{j=m+1}^{n} r(j-i)$$
$$= \frac{1}{\sigma_m}\sum_{i=1}^{m}\sum_{k=m+1-i}^{n-i} r(k) \ll \frac{1}{\sigma_m}\sum_{i=1}^{m}\sum_{k=1}^{n} r(k) \ll \frac{m}{\sigma_m}\sum_{k=1}^{n}\frac{L(k)}{k^{\alpha}}$$

$$E_3 \ll \exp\left(-\frac{u_n^2}{2(1+\mu)}\right)\sum_{i=m+1}^{n} Cov\left(X_i, \frac{S_m}{\sigma_m}\right)$$
$$\ll \exp\left(-\frac{u_n^2}{2(1+\mu)}\right)\frac{m}{\sigma_m}\sum_{k=1}^{n}\frac{L(k)}{k^{\alpha}}$$
$$\ll \left(\frac{m}{n}\right)^{\frac{\alpha}{2}} \frac{1}{n^{\frac{1}{1+\mu}+\frac{\alpha}{2}-1}} L(n)\left(L(m)\right)^{\frac{1}{2}} (\ln n)^{\frac{1}{2(1+\mu)}} .$$
$$\ll \left(\frac{m}{n}\right)^{\frac{\alpha}{2}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for } 0<\gamma<\frac{\alpha}{2}$$

$$E_4 \ll Cov\left(\frac{S_m}{\sigma_m}, \frac{S_{n+2m}-S_{2m}}{\sigma_n}\right) + E\left(\frac{S_m}{\sigma_m}\right)E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m}-S_{2m}}{\sigma_n}\right)$$
$$- E\left(\frac{S_m}{\sigma_m}\right)E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m}-S_{2m}}{\sigma_n}\right) . \qquad (4.12)$$
$$=: F_1 + F_2 + F_3$$

Since $\frac{L(t)}{t^{\alpha}}$ is monotonic decreasing, so for $1 \le i \le m$, $r(j-i) \le r(j-m)$, by (4.2), Karamata's theorem and the property of the slowly varying function, we obtain

$$\begin{aligned} F_1 &\ll \frac{1}{\sigma_m \sigma_n} Cov\left(\sum_{i=1}^{m} X_i, \sum_{j=2m+1}^{2m+n} X_j \right) = \frac{1}{\sigma_m \sigma_n} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(j-i) \\ &\le \frac{1}{\sigma_m \sigma_n} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(j-m) \ll \frac{m}{\sigma_m \sigma_n} \sum_{k=1}^{n} r(k) \\ &\ll \frac{m}{L(m)^{\frac{1}{2}} L(n)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}} n^{1-\frac{\alpha}{2}}} L(n) n^{1-\alpha} \ll \left(\frac{m}{n}\right)^{\frac{\alpha}{2}} \left(\frac{L(n)}{L(m)}\right)^{\frac{1}{2}} \cdot \\ &\ll \left(\frac{m}{n}\right)^{\frac{\alpha}{2}} \left(\frac{n}{m}\right)^{\frac{\alpha}{4}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for } 0 < \gamma < \frac{\alpha}{4} < \frac{1}{4} \end{aligned}$$

By $\sigma_n = \sqrt{\mathrm{Var}(S_n)}$, then we get $E\left(\frac{S_m}{\sigma_m}\right)^2 = 1$, by the stationary of the sequence $(X, X_n)_{n \in \mathbb{N}}$, we get $S_{n+2m} - S_n \overset{d}{=} S_{2m}$, thus $E(S_{n+2m} - S_n)^2 = E(S_{2m})^2$ and by the fact that $EX_i X_j > 0$ for all i, j and (4.2), we deduce that

$$\begin{aligned} F_2 &\le E\left|\frac{S_m}{\sigma_m}\right| E\left|\frac{S_{n+2m} - S_n}{\sigma_n} - \frac{S_{2m}}{\sigma_n}\right| \le \sqrt{E\left(\frac{S_m}{\sigma_m}\right)^2 E\left(\frac{S_{n+2m} - S_{2m}}{\sigma_n}\right)^2} \\ &\ll \frac{\sigma_{2m}}{\sigma_n} \ll \left(\frac{m}{n}\right)^{1-\frac{\alpha}{2}} \left(\frac{L(m)}{L(n)}\right)^{1/2} \ll \left(\frac{m}{n}\right)^{1-\frac{\alpha}{2}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for some } \gamma < 1 \end{aligned},$$

where the second inequality follows from Jensen inequality.

For F_3, we have

$F_3 = -E\left(\frac{S_m}{\sigma_m}\right) E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) = 0$, so we prove (2.3) for some $0 < \alpha < 1$.

Next, we prove (2.3) for some $\alpha > 1$.

$$\left| Cov\left(I\left(M_m \le u_m, \frac{S_m}{\sigma_m} \le y \right), I\left(M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y \right) \right) \right| =: G_1 + G_2 + G_3 + G_4,$$

where G_1 - G_4 are defined as E_1 - E_4 in (4.11), but for (1.2) holds for $\alpha > 1$. Similarly as the proof of E_1 - E_4, it is easy to check that $G_1 + G_2 + G_3 \ll \left(\frac{m}{n}\right)^{\gamma}$ for some $\gamma > 0$.

Define $G_4 =: H_1 + H_2 + H_3$ as in (4.12), by Karamata's theorem, we get

$$\begin{aligned} H_1 &\le \frac{1}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(j-i) \ll \frac{1}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \sum_{i=1}^{m} \sum_{t=m+1}^{2m+n-1} r(t) \\ &\ll \frac{m}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \sum_{t=m+1}^{2m+n} r(t) \ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \frac{1}{m^{\alpha-1}} L(m) \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for } 0 < \gamma < \frac{1}{2} \end{aligned}.$$

By the definition of σ_n and Karamata's theorem, it is easy to obtain that

$$n \le \sigma_n^2 = n + 2\sum_{t=1}^{n-1} (n-t) r(t) = n + 2n \sum_{t=1}^{n-1} \frac{L(t)}{t^{\alpha}} - 2\sum_{t=1}^{n-1} t \times r(t) \ll n + 2n \sum_{t=1}^{n-1} \frac{L(t)}{t^{\alpha}} \ll n + 2n \sum_{t=1}^{\infty} \frac{L(t)}{t^{\alpha}} \ll n + cn,$$

then we have $\sigma_n = O\left(n^{1/2}\right)$. Similarly as the proof of F_2,

$$H_2 \ll \frac{\sigma_{2m}}{\sigma_n} \ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for} \quad 0 < \gamma < \frac{1}{2},$$

by the stationary of the sequence $\left(X, X_n\right)_{n\in\mathbb{N}}$, it is easy to see that $H_3 = -E\left(\frac{S_m}{\sigma_m}\right)E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) = 0$.

So we have proved (2.3) for the case of $\alpha > 1$.

Finally we should prove (2.3) for $\alpha = 1$.

$$\left|Cov\left(I\left(M_m \le u_m, \frac{S_m}{\sigma_m} \le y\right), I\left(M_{m,n} \le u_n, \frac{S_n}{\sigma_n} \le y\right)\right)\right| =: I_1 + I_2 + I_3 + I_4,$$

where we replaced E_1 - E_4 by I_1 - I_4 as in (4.11), but for (1.2) holds for some $\alpha = 1$.

Similarly, it is easy to check that $I_1 + I_2 + I_3 \ll \left(\frac{m}{n}\right)^{\gamma}$ for some $\gamma > 0$. Define $I_4 =: J_1 + J_2 + J_3$. Since $r(t)$ is monotonic decreasing, define $\hat{L}(n) = 1 + 2\sum_{t=1}^{n-1} r(t)$, by the application of the proposition (Potter's TH) of the slowly varying function, for any $0 < \delta_1 < \frac{1}{2}$, we have

$$J_1 \ll \frac{m}{\sigma_m\sigma_n}\sum_{t=m+1}^{2m+n+1} r(t) \ll \frac{m}{\sigma_m\sigma_n}\sum_{t=1}^{m+n+1} r(t) \ll \frac{m\hat{L}(n+m)}{\left(\hat{L}(m)\right)^{\frac{1}{2}}\left(\hat{L}(n)\right)^{\frac{1}{2}} m^{\frac{1}{2}} n^{\frac{1}{2}}}$$

$$= \left(\frac{m}{n}\right)^{\frac{1}{2}} \frac{\left(\hat{L}(n+m)\right)^{\frac{1}{2}}}{\left(\hat{L}(m)\right)^{\frac{1}{2}}} \frac{\left(\hat{L}(n+m)\right)^{\frac{1}{2}}}{\left(\hat{L}(n)\right)^{\frac{1}{2}}} \ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \left(\frac{n+m}{m}\right)^{\delta_1} \left(\frac{n+m}{n}\right)^{\delta_2} .$$

$$\ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \left(\frac{n+m}{m}\right)^{\delta_1} \ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \left(\frac{2n}{m}\right)^{\delta_1} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for some } 0 < \gamma < \frac{1}{2}$$

$$J_2 \ll \frac{\hat{\sigma}_{2m}}{\hat{\sigma}_n} \ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \frac{\hat{L}(m)^{\frac{1}{2}}}{\hat{L}(n)^{\frac{1}{2}}} \ll \left(\frac{m}{n}\right)^{\frac{1}{2}} \left(\frac{m}{n}\right)^{-\delta_1} \ll \left(\frac{m}{n}\right)^{\gamma} \quad \text{for some } 0 < \gamma < \frac{1}{2}.$$

Similarly as $F_3 = 0$ when $0 < \alpha < 1$, we have $J_3 = 0$, So we have $I_4 \ll \left(\frac{m}{n}\right)^{\gamma}$. Finally, we get that

$I_1 + I_2 + I_3 + I_4 \ll \left(\frac{m}{n}\right)^{\gamma}$ for some $\gamma > 0$, when $\alpha = 1$.

References

[1] Dudzinski, M. (2008) The Almost Sure Central Limit Theorems in The Joint Version for The Maxima and Sums of Certain Stationary Gaussian Sequence. *Statistics & Probability Letters*, **78**, 347-357. http://dx.doi.org/10.1016/j.spl.2007.07.007

[2] Brosamler, G.A. (1988) An Almost Everywhere Central Limit Theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, **104**, 561-574. http://dx.doi.org/10.1017/S0305004100065750

[3] Schatte, P. (1988) On Strong Versions of the Central Limit Theorem. *Mathematische Nachrichten*, **137**, 249-256. http://dx.doi.org/10.1002/mana.19881370117

[4] Fahrner, I. and Stadtmuller, U. (1998) On Almost Sure Central Max-Limit Theorems. *Statistics & Probability Letters*, **37**, 229-236. http://dx.doi.org/10.1016/S0167-7152(97)00121-1

[5] Cheng, H., Peng, L. and Qi, Y.C. (1998) Almost Sure Convergence in Extreme Value Theory. *Mathematische Nachrichten*, **190**, 43-50. http://dx.doi.org/10.1002/mana.19981900104

[6] Chandrasekharan, K. and Minakshisundaram, S. (1952) Typical Means. Oxford University Press, Oxford.

[7] Leadbetter, M.R., Lindgren, G. and Rootzen, H. (1983) Extremes and Related Properties of Random Sequences and Processes. Springer, New York. http://dx.doi.org/10.1007/978-1-4612-5449-2

[8] Csaki, E. and Gonchigdanzan, K. (2002) Almost Sure Limit Theorems for The Maximum of Stationary Gaussian Sequences. *Statistics & Probability Letters*, **58**, 195-203. http://dx.doi.org/10.1016/S0167-7152(02)00128-1

[9] Mielniczuk, J. (2002) Some Remarks on the Almost Sure Central Limit Theorem for Dependent Sequences. *Limit Theorems in Probability and Statistics*, **2**, 391-403.