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On the Solutions of the Equation $x^3 + Ax = B$ in $\mathbb{Z}_3^*$ with Coefficients from $\mathbb{Q}_3$

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ABSTRACT

Recall that in [1] it is obtained the criteria solvability of the Equation $x^3 + ax = b$ in $\mathbb{Z}_p^*$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for $p > 3$. Since any *p*-adic number *x* has a unique form $x = p^k x^*$, where $x^* \in \mathbb{Z}_p^*$ and $k \in \mathbb{Z}$, in [1] it is also shown that from the criteria in $\mathbb{Z}_p^*$ it follows the criteria in $\mathbb{Z}_p$ and $\mathbb{Q}_p$. In this paper we provide the algorithm of finding the solutions of the Equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$.

KEYWORDS

***p*-Adic Numbers; Solvability of Equation; Congruence**

1. Introduction

In the present time description of different structures in mathematics are studying over field of p-adic numbers. In particular, p-adic analysis is one of the intensive developing directions of modern mathematics. Numerous applications of p-adic numbers have found their own reflection in the theory of p-adic differential equations, p-adic theory of probabilities, p-adic mathematical physics, algebras over p-adic numbers and others.

The field of p-adic numbers were introduced by German mathematician K. Hensel at the end of the 19th century [2]. The investigation of p-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of the power series into number theory. Their canonical representation is similar to expansion of analytical functions in power series, which is analogy between algebraic numbers and algebraic functions. There are several books devoted to study p-adic numbers and p-adic analysis [3-6].

Classification of algebras in small dimensions plays important role for the studying of properties of varieties of algebras. It is known that the problem of classification of finite dimensional algebras involves a study on equations for structural constants, *i.e.* to the decision of some systems of the Equations in the corresponding field. Classifications of complex Leibniz algebras have been investigated in [7-10] and many other works. In similar complex case, the problem of classification in p-adic case is reduced to the solution of the Equations in the field. The classifications of Leibniz algebras over the field of p-adic numbers have been obtained in [11-13].

In the field of complex numbers the fundamental Abel's theorem about insolvability in radicals of general Equation of n-th degree $(n > 5)$ is well known. In this field square equation is solved by discriminant, for cubic Equation Cardano's formulas were widely applied. In the field of p-adic numbers square equation does not always has a solution. Note that the criteria of solvability of the Equation $x^2 = a$ is given in [6,14,15] we can find the solvability criteria for the Equation $x^q = a$, where q is an arbitrary natural number.

In this paper we consider $p-$ adic cubic equation $y^3 + ry^2 + sy + t = 0$. By replacing $y = x - \frac{r}{3}$, this equation become the so-called depressed cubic equation

$$x^3 + ax = b. \tag{1}$$

The solvability criterion for the cubic equation $x^3 + ax = b.$ over 3-adic numbers is different from the case $p > 3$. Note that solvability criteria for $p > 3$ is obtained in [1]. The problem of finding a solvability criteria of the cubic equation for the case $p = 3$ is complicated. This problem was partially solved in [16], namely, it is obtained solvability criteria of cubic equation with condition $|a|_3 \neq \frac{1}{3}$.

In this paper we obtain solvability criteria of cubic equation for $p = 3$ without any conditions. Moreover, the algorithm of finding the solutions of the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$ is provided.

2. Preliminaries

Let $\mathbb{Q}$ be a field of rational numbers. Every rational number $x \neq 0$ can be represented by the form $x = p^{\gamma(x)} \frac{n}{m}$, where $n, \gamma(x) \in \mathbb{Z}$, m is a positive integer, $(p,n) = 1, (p,m) = 1$ and p is a fixed prime number. In $\mathbb{Q}$ a norm has been defined as follows:

$$|x|_p = \begin{cases} p^{-\gamma(x)}, x \neq 0, \\ 0, \quad\quad x = 0. \end{cases}$$

The norm $|x|_p$ is called a p-*adic norm* of x and it satisfies so called the strong triangle inequality. The completion of $\mathbb{Q}$ with respect to p-adic norm defines the p-adic field which is denoted by $\mathbb{Q}_p$ ([4,6]). It is well known that any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)} \left(x_0 + x_1 p + x_2 p^2 + \cdots \right),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p-1$, $x_0 \neq 0$, $(j = 0,1,\cdots)$. p-Adic number x, for which $|x|_p \leq 1$, is called *integer* p-adic number, and the set of such numbers is denoted by $\mathbb{Z}_p$. Integer $x \in \mathbb{Z}_p$, for which $|x|_p = 1$, is called *unit* of $\mathbb{Z}_p$, and their set is denoted by $\mathbb{Z}_p^*$.

For any numbers a and m it is known the following result.

Theorem 2.1 [3]. *If* $(a,m) = 1$, *then a congruence* $ax \equiv b(mod m)$ *has one and only one solution.*

We also need the following Lemma.

Lemma 2.1 [14]. *The following is true*:

$$\left(\sum_{i=0}^{\infty} x_i p^i \right)^q = x_0^q + \sum_{k=1}^{\infty} \left(q x_0^{q-1} x_k + N_k \left(x_0, x_1, \cdots, x_{k-1} \right) \right) p^k,$$

where $x_0 \neq 0$, $0 \leq x_j \leq p-1$, $N_1 = 0$ and for $k \geq 2$

$$N_k = N_k \left(x_0, \cdots, x_{k-1} \right) = \sum_{\substack{m_0, m_1, \cdots, m_{k-1}: \\ \sum_{i=0}^{k-1} m_i = q, \sum_{i=0}^{k-1} i m_i = k}} \frac{q!}{m_0! m_1! \cdots m_{k-1}!} x_0^{m_0} x_1^{m_1} \cdots x_{k-1}^{m_{k-1}}$$

From Lemma 2.1 by $q = 3$ we have

$$\left(\sum_{i=0}^{\infty} x_i p^i \right)^3 = x_0^3 + \sum_{k=1}^{\infty} \left(3 x_0^2 x_k + N_k \left(x_0, x_1, \cdots, x_{k-1} \right) \right) p^k.$$

For $j \leq k$ we put

$$P_k^j = P_k^j\left(x_0, x_1, \cdots, x_{j-1}\right) = \sum_{\substack{m_0, m_1, \cdots, m_{j-1}: \\ \sum_{i=0}^{j-1} m_i = 3, \sum_{i=1}^{j-1} i m_i = k,}} \frac{6}{m_0! m_1! \cdots m_{j-1}!} 6 x_0^{m_0} x_1^{m_1} \cdots x_{j-1}^{m_{j-1}}$$

Also the following identity is true:

$$\left(\sum_{i=0}^{\infty} a_i p^i\right)\left(\sum_{j=0}^{\infty} x_j p^j\right) = \sum_{k=0}^{\infty}\left(\sum_{s=0}^{k} x_s a_{k-s}\right) p^k. \tag{2}$$

3. The Main Result

In this paper we study the cubic Equation (1) over the field 3-adic numbers, *i.e.* $a, b, x \in \mathbb{Q}_3$.

Put

$$x = 3^{\gamma(x)}\left(x_0 + x_1 3 + \cdots\right), a = 3^{\gamma(a)}\left(a_0 + a_1 3 + \cdots\right), b = 3^{\gamma(b)}\left(b_0 + b_1 3 + \cdots\right),$$

where $x_j, a_j, b_j \in \{0,1,2\}, x_0, a_0, b_0 \neq 0, (j = 0, 1, \cdots)$.

Since any 3-adic number x has a unique form $x = 3^k x^*$, where $x^* \in \mathbb{Z}_3^*$ and $k \in \mathbb{Z}$, we will be limited to search a decision from $\mathbb{Z}_3^*$, *i.e.* $\gamma(x) = 0$.

Putting the canonical form of a, b and x in (1), we get

$$\left(\sum_{k=0}^{\infty} x_k 3^k\right)^3 + 3^{\gamma(a)}\left(\sum_{k=0}^{\infty} a_k 3^k\right)\left(\sum_{k=0}^{\infty} x_k 3^k\right) = 3^{\gamma(b)} \sum_{k=0}^{\infty} b_k 3^k.$$

By Lemma 2.1 and Equality (2), the Equation (1) becomes to the following form:

$$\begin{aligned} &x_0^3 + \sum_{k=1}^{\infty}\left(3x_0^2 x_k + N_k\left(x_0, x_1, \cdots, x_{k-1}\right)\right)3^k + 3^{\gamma(a)}\left(a_0 x_0 + \sum_{k=1}^{\infty}\left(\sum_{s=0}^{k} x_s a_{k-s}\right)3^k\right) \\ &= 3^{\gamma(b)}\left(b_0 + \sum_{k=1}^{\infty} b_k 3^k\right). \end{aligned} \tag{3}$$

Proposition 3.1 *If one of the following conditions*:

$$\begin{aligned} &1)\quad \gamma(a) = 0 \quad\text{and}\quad \gamma(b) < 0;\quad 2)\quad \gamma(a) > 0 \quad\text{and}\quad \gamma(b) > 0; \\ &3)\quad \gamma(a) > 0 \quad\text{and}\quad \gamma(b) < 0;\quad 4)\quad \gamma(a) < 0 \quad\text{and}\quad \gamma(b) = 0; \\ &5)\quad \gamma(a) < 0 \quad\text{and}\quad \gamma(b) > 0, \end{aligned}$$

is fulfilled, then the Equation (1) has not a solution in $\mathbb{Z}_3^*$.

Proof. 1) Let $\gamma(a) = 0$ and $\gamma(b) < 0$. Multiplying Equation (3) by $3^{-\gamma(b)}$, we get the following congruence $b_0 \equiv 0(\mathrm{mod}\,3)$, which is not correct. Consequently, Equation (1) has no solution in $\mathbb{Z}_3^*$.

2) Let $\gamma(a) > 0$ and $\gamma(b) > 0$. Then from (3) it follows a congruence $x_0^3 \equiv 0(\mathrm{mod}\,3)$, which has no a nonzero solution.Therefore, in $\mathbb{Z}_3^*$ Equation (1) does not have a solution.

In other cases, we analogously get the congruences

$$b_0 \equiv 0(\mathrm{mod}\,3) \quad\text{or}\quad a_0 x_0 \equiv 0(\mathrm{mod}\,3),$$

which are not hold. Therefore, in $\mathbb{Z}_3^*$ there is no solution.■

From the Proposition 3.1 we have that the cubic equation may have a solution if one of the following four cases

$$\begin{aligned} &1)\quad \gamma(a) = 0,\quad \gamma(b) = 0,\quad 2)\quad \gamma(a) = 0,\quad \gamma(b) > 0, \\ &3)\quad \gamma(a) < 0,\quad \gamma(b) < 0,\quad 4)\quad \gamma(a) > 0,\quad \gamma(b) = 0 \end{aligned}$$

is hold.

In the following theorem we present an algorithm of finding of the Equation $x^3 + ax = b$ for the first case.

Theorem 3.1 *Let* $\gamma(a) = \gamma(b) = 0$ *and* $a_0 = 1$. *Then* x *to be a solution of the Equation* (1) *in* $\mathbb{Z}_3^*$ *if and* *only if the congruences*

$$x_0^3 + a_0x_0 \equiv b_0 \pmod 3,$$

$$x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv b_1 \pmod 3,$$

$$x_ka_0 + \cdots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \cdots, x_{k-1}) + M_k(x_0, x_1, \cdots, x_{k-1}) \equiv b_k \pmod 3, k \ge 2$$

are fulfilled, where integers $M_k(x_0, \cdots, x_{k-1})$ are defined consequently from the following correlations

$$x_0^3 + a_0x_0 = b_0 + M_1(x_0)\cdot 3,$$

$$x_1a_0 + x_0a_1 + N_1(x_0) = b_1 - M_1(x_0) + M_2(x_0, x_1)\cdot 3,$$

$$x_{k-1}a_0 + x_{k-2}a_1 + \cdots + x_0a_{k-1} + x_0^2x_{k-2} + N_{k-1}(x_0, x_1, \cdots, x_{k-2})$$
$$= b_{k-1} - M_{k-1}(x_0, x_1, \cdots, x_{k-2}) + M_k(x_0, x_1, \cdots, x_{k-1})\cdot 3, k \ge 3.$$

Proof. Let

$$x_0 + x_1\cdot 3 + x_2\cdot 3^2 + \cdots, 0 \le x_j \le 2, x_0 \ne 0, (j = 0,1,\cdots)$$

is a solution of Equation (1), then Equality (3) becomes

$$x_0^3 + \sum_{k=1}^{\infty}\left(3x_0^2x_k + N_k(x_0, x_1, \cdots, x_{k-1})\right)3^k + a_0x_0 + \sum_{k=1}^{\infty}\left(\sum_{s=0}^{k} x_sa_{k-s}\right)3^k = b_0 + \sum_{k=1}^{\infty} b_k3^k.$$

So we have

$$x_0^3 + a_0x_0 + \left(x_1a_0 + x_0a_1 + N_1(x_0)\right)\cdot 3 + \sum_{k=2}^{\infty}\left(x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \cdots, x_{k-1})\right)3^k$$
$$= b_0 + \sum_{k=1}^{\infty} b_k3^k,$$

from which it follows the necessity in fulfilling the congruences of the theorem.

Now let x is satisfied the congruences of the theorem. Since $(a_0, 3) = 1$, then by Theorem 2.1 it implies that these congruences have the solutions x_k.

Then

$$x_0^3 + \sum_{k=1}^{\infty}\left(3x_0^2x_k + N_k(x_0, x_1, \cdots, x_{k-1})\right)3^k + a_0x_0 + \sum_{k=1}^{\infty}\left(\sum_{s=0}^{k} x_sa_{k-s}\right)3^k$$

$$= x_0^3 + a_0x_0 + \left(N_1 + x_0a_1 + a_0x_1\right)3 + \sum_{k=2}^{\infty}\left(x_0^2x_{k-1} + N_k(x_0, x_1, \cdots, x_{k-1}) + x_0a_k + x_1a_{k-1} + \cdots + x_{k-1}a_1 + x_ka_0\right)3^k$$

$$= b_0 + M_1(x_0)\cdot 3 + \left(b_1 - M_1(x_0) + M_2(x_0, x_1)\cdot 3\right)\cdot 3 + \sum_{k=2}^{\infty}\left(b_k - M_k(x_0, x_1, \cdots, x_{k-1}) + M_{k+1}(x_0, x_1, \cdots, x_k)\cdot 3\right)\cdot 3^k$$

$$= b_0 + \sum_{k=1}^{\infty} b_k3^k.$$

Therefore, we show that $x = \sum_{k=0}^{\infty} x_k3^k$ is a solution of the Equation (1).■

Let us examine a case $\gamma(a) = 0, \gamma(b) > 0$ and get necessary and sufficient conditions for a solution of Equation (1).

Theorem 3.2 *Let* $\gamma(a) = 0$, $\gamma(b) = m > 0$ *and* $a_0 = 2$. *Then* x *to be a solution of Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if the congruences*

$$x_0^3 + a_0 x_0 \equiv 0(\text{mod}3),$$
$$x_1 a_0 + x_0 a_1 + N_1(x_0) + M_1(x_0) \equiv 0(\text{mod}3),$$
$$x_k a_0 + \cdots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0,\cdots,x_{k-1}) + M_k(x_0,\cdots,x_{k-1}) \equiv 0(\text{mod}3), 2 \le k \le m-1,$$
$$x_k a_0 + \cdots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0,\cdots,x_{k-1}) + M_k(x_0,\cdots,x_{k-1}) \equiv b_{k-m}(\text{mod}3), k \ge m$$

are fulfilled, where integers $M_k(x_0, x_1, \cdots, x_{k-1})$ are defined consequently from the following correlations

$$x_0^3 + 2x_0 = M_1(x_0)\cdot 3,$$
$$x_1 a_0 + x_0 a_1 + N_1(x_0) = -M_1(x_0) + 3M_2(x_0, x_1),$$
$$x_k a_0 + x_{k-1} a_1 + \cdots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \cdots, x_{k-1})$$
$$= -M_k(x_0, x_1, \cdots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \cdots, x_k), 2 \le k \le m-1,$$
$$x_k a_0 + x_{k-1} a_1 + \cdots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \cdots, x_{k-1})$$
$$= b_{k-m} - M_k(x_0, x_1, \cdots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \cdots, x_k), k \ge m.$$

Proof. Let x is a solution of the Equation (1), then Equality (3) becomes

$$x_0^3 + \sum_{k=1}^{\infty}\left(3x_0^2 x_k + N_k(x_0, x_1, \cdots, x_{k-1})\right)3^k + a_0 x_0 + \sum_{k=1}^{\infty}\left(\sum_{s=0}^{k} x_s a_{k-s}\right)3^k = 3^m\left(b_0 + \sum_{k=1}^{\infty} b_k 3^k\right).$$

Therefore, we have

$$x_0^3 + a_0 x_0 + \left(x_1 a_0 + x_0 a_1 + N_1(x_0)\right)\cdot 3 + \sum_{k=2}^{\infty}\left(x_k a_0 + x_{k-1} a_1 + \cdots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, \cdots, x_{k-1})\right)3^k$$
$$= 3^m\left(b_0 + \sum_{k=1}^{\infty} b_k 3^k\right),$$

from which it follows the necessity in fulfilling the congruences of the theorem.

Now let x is satisfied the congruences of the theorem. Since $(a_0, 3) = 1,$ then by Theorem 2.1 there are solutions x_k of the congruences.

Putting element x to Equality (3), we have

$$x_0^3 + \sum_{k=1}^{\infty}\left(3x_0^2 x_k + N_k(x_0, x_1, \cdots, x_{k-1})\right)3^k + a_0 x_0 + \sum_{k=1}^{\infty}\left(\sum_{s=0}^{k} x_s a_{k-s}\right)3^k$$
$$= x_0^3 + a_0 x_0 + (N_1 + x_0 a_1 + a_0 x_1)3 + \sum_{k=2}^{\infty}\left(x_0^2 x_{k-1} + N_k(x_0, x_1, \cdots, x_{k-1}) + x_0 a_k + x_1 a_{k-1} + \cdots + x_{k-1} a_1 + x_k a_0\right)3^k$$
$$= M_1(x_0)\cdot 3 + \left(-M_1(x_0) + M_2(x_0, x_1)\cdot 3\right)\cdot 3 + \sum_{k=2}^{m-1}\left(-M_k(x_0, x_1, \cdots, x_{k-1}) + M_{k+1}(x_0, x_1, \cdots, x_k)\cdot 3\right)\cdot 3^k$$
$$+ \sum_{k=m}^{\infty}\left(b_{k-m} - M_k(x_0, x_1, \cdots, x_{k-1}) + M_{k+1}(x_0, x_1, \cdots, x_k)\cdot 3\right)\cdot 3^k$$
$$= 3^m\left(b_0 + \sum_{k=1}^{\infty} b_k 3^k\right).$$

Therefore, we show that x is a solution of Equation (1).■

The following theorem gives necessary and sufficient conditions for a solution of Equation (1) for the case $\gamma(a) < 0$ and $\gamma(b) < 0$.

Theorem 3.3 *Let*

$$\gamma(a) = \gamma(b) = -m < 0, (m > 0).$$

Then x to be a solution of Equation (1) in $\mathbb{Z}_3^*$ if and only if the next congruences

$$
\begin{aligned}
&a_0x_0 \equiv b_0 \pmod 3,\\
&x_k a_0 + x_{k-1}a_1 + \cdots + x_0 a_k + M_k\left(x_0, x_1, \cdots, x_{k-1}\right) \equiv b_k \pmod 3, 1 \le k \le m-1,\\
&x_m a_0 + x_{m-1}a_1 + \cdots + x_0 a_m + x_0^3 + M_m\left(x_0, x_1, \cdots, x_{m-1}\right) \equiv b_m \pmod 3,\\
&x_{m+1} a_0 + x_m a_1 + \cdots + x_0 a_{m+1} + M_{m+1}\left(x_0, x_1, \cdots, x_m\right) \equiv b_{m+1} \pmod 3,\\
&x_k a_0 + x_{k-1}a_1 + \cdots + x_0 a_k + x_0^2 x_{k-m-1} + N_{k-m}\left(x_0, x_1, \cdots, x_{k-m-1}\right) + M_k\left(x_0, x_1, \cdots, x_{k-1}\right) \equiv b_k \pmod 3, k \ge m+2
\end{aligned}
$$

are fulfilled, where integers $M_k\left(x_0, x_1, \cdots, x_{k-1}\right)$ are defined consequently from the equalities

$$
\begin{aligned}
&a_0x_0 = b_0 + M_1\left(x_0\right)\cdot 3,\\
&x_k a_0 + x_{k-1}a_1 + \cdots + x_0 a_k = b_k - M_k\left(x_0, \cdots, x_{k-1}\right) + 3M_{k+1}\left(x_0, \cdots, x_k\right), 1 \le k \le m-1,\\
&x_m a_0 + x_{m-1}a_1 + \cdots + x_0 a_m + x_0^3 = b_m - M_m\left(x_0, x_1, \cdots, x_{m-1}\right) + 3M_{m+1}\left(x_0, x_1, \cdots, x_m\right),\\
&x_{m+1} a_0 + x_m a_1 + \cdots + x_0 a_{m+1} = b_{m+1} - M_{m+1}\left(x_0, \cdots, x_m\right) + 3M_{m+2}\left(x_0, \cdots, x_{m+1}\right),\\
&x_k a_0 + x_{k-1}a_1 + \cdots + x_0 a_k + x_0^2 x_{k-m-1} + N_{k-m}\left(x_0, x_1, \cdots, x_{k-m-1}\right)\\
&= b_k - M_k\left(x_0, x_1, \cdots, x_{k-1}\right) + 3M_{k+1}\left(x_0, x_1, \cdots, x_k\right), k \ge m+2.
\end{aligned}
$$

Proof. The proof of the Theorem can be obtained by similar way to the proofs of Theorems 3.1 and 3.2.■

Examining various cases of $\gamma(a)$ and $\gamma(b)$ we need to study only the case $\gamma(a) > 0$ and $\gamma(b) = 0$. Because of appearance of uncertainty of a solution, we divide this case to $\gamma(a) > 1$ and $\gamma(a) = 1$.

Theorem 3.4 *Let* $\gamma(a) = 2$, $\gamma(b) = 0$ *and* $(b_0, b_1) = (1, 0)$ *or* $(2, 2)$. *Then* x *to be a solution of Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if he next congruences*

$$
\begin{aligned}
&x_0^3 \equiv b_0 \pmod 3,\\
&x_0^3 \equiv b_0 + b_1\cdot 3 \pmod 9,\\
&x_0^2 x_1 + x_0 a_0 + M_1\left(x_0\right) \equiv b_2 \pmod 3,\\
&x_0^2 x_2 + P_3^2\left(x_0, x_1\right) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 + M_2\left(x_0, x_1\right) \equiv b_3 \pmod 3,\\
&x_0^2 x_{k-1} + P_k^{k-1}\left(x_0, x_1, \cdots, x_{k-2}\right) + 2x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \cdots + x_0 a_{k-2} + M_{k-1}\left(x_0, x_1, \cdots, x_{k-2}\right)\\
&\equiv b_k \pmod 3, k \ge 4
\end{aligned}
$$

are fulfilled, where integers $M_k\left(x_0, x_1, \cdots, x_{k-1}\right)$ are defined from the equalities

$$
\begin{aligned}
&x_0^3 = b_0 + b_1\cdot 3 + M_1\left(x_0\right)\cdot 9,\\
&x_0^2 x_1 + x_0 a_0 = b_2 - M_1\left(x_0\right) + 3M_2\left(x_0, x_1\right),\\
&x_0^2 x_2 + P_3^2\left(x_0, x_1\right) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 = b_3 - M_2\left(x_0, x_1\right) + 3M_3\left(x_0, x_1, x_2\right),\\
&x_0^2 x_{k-1} + P_k^{k-1}\left(x_0, x_1, \cdots, x_{k-2}\right) + 2x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \cdots + x_0 a_{k-2}\\
&= b_k - M_{k-1}\left(x_0, x_1, \cdots, x_{k-2}\right) + 3M_k\left(x_0, x_1, \cdots, x_{k-1}\right), k \ge 4.
\end{aligned}
$$

Proof. Analogously to the proof of Theorem 3.1.■

Theorem 3.5 *Let* $\gamma(a) = 3$, $\gamma(b) = 0$ *and* $(b_0, b_1) = (1, 0)$ *or* $(2, 2)$. *Then* x *to be a solution of the Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if he next congruences*

$$
\begin{aligned}
&x_0^3 \equiv b_0 \pmod 3,\\
&x_0^3 \equiv b_0 + b_1\cdot 3 \pmod 9,\\
&x_0^2 x_1 + M_1\left(x_0\right) \equiv b_2 \pmod 3,\\
&x_0^2 x_2 + P_3^2\left(x_0, x_1\right) + x_0 a_0 + x_0 x_1^2 + M_2\left(x_0, x_1\right) \equiv b_3 \pmod 3,\\
&x_0^2 x_{k-1} + P_k^{k-1}\left(x_0, x_1, \cdots, x_{k-2}\right) + 2x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \cdots + x_0 a_{k-3} + M_{k-1}\left(x_0, x_1, \cdots, x_{k-2}\right)\\
&\equiv b_k \pmod 3, k \ge 4
\end{aligned}
$$

are fulfilled, where integers $M_k\left(x_0,x_1,\cdots,x_{k-1}\right)$ are defined from the equalities

$$
\begin{aligned}
&x_0^3=b_0+b_1\cdot 3+M_1\left(x_0\right)\cdot 9,\\
&x_0^2x_1=b_2-M_1\left(x_0\right)+3M_2\left(x_0,x_1\right),\\
&x_0^2x_2+P_3^2\left(x_0,x_1\right)+x_0a_0+x_0x_1^2=b_3-M_2\left(x_0,x_1\right)+3M_3\left(x_0,x_1,x_2\right),\\
&x_0^2x_{k-1}+P_k^{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+2x_0x_1x_{k-2}+x_{k-3}a_0+x_{k-4}a_1+\cdots+x_0a_{k-3}\\
&=b_k-M_{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+3M_k\left(x_0,x_1,\cdots,x_{k-1}\right),k\ge 4.
\end{aligned}
$$

Proof. Analogously to the proof of Theorem 3.1.■

Similarly to Theorem 3.4, it is proved the following

Theorem 3.6 *Let* $\gamma(a)=m\ge 4$, $\gamma(b)=0$ *and* $\left(b_0,b_1\right)=(1,0)$ *or* $(2,2)$. *Then* x *to be a solution of the Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if he next congruences*

$$
\begin{aligned}
&x_0^3\equiv b_0\left(\operatorname{mod}3\right),\\
&x_0^3\equiv b_0+b_1\cdot 3\left(\operatorname{mod}9\right),\\
&x_0^2x_1+M_1\left(x_0\right)\equiv b_2\left(\operatorname{mod}3\right),\\
&x_0^2x_2+P_3^2\left(x_0,x_1\right)+x_0x_1^2+M_2\left(x_0,x_1\right)\equiv b_3\left(\operatorname{mod}3\right),\\
&x_0^2x_{k-1}+P_k^{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+2x_0x_1x_{k-2}\equiv b_k\left(\operatorname{mod}3\right),4\le k\le m-1,\\
&x_0^2x_{m-1}+P_m^{m-1}\left(x_0,x_1,\cdots,x_{m-2}\right)+2x_0x_1x_{m-2}+x_0a_0\equiv b_m\left(\operatorname{mod}3\right),\\
&x_0^2x_{k-1}+P_k^{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+2x_0x_1x_{k-2}+x_{k-m}a_0+\cdots+x_0a_{k-m}\equiv b_k\left(\operatorname{mod}3\right),k\ge m+1
\end{aligned}
$$

are fulfilled, where integers $M_k\left(x_0,x_1,\cdots,x_{k-1}\right)$ are defined from the equalities

$$
\begin{aligned}
&x_0^3=b_0+b_1\cdot 3+M_1\left(x_0\right)\cdot 9,\\
&x_0^2x_1=b_2-M_1\left(x_0\right)+3M_2\left(x_0,x_1\right),\\
&x_0^2x_2+P_3^2\left(x_0,x_1\right)+x_0x_1^2=b_3-M_2\left(x_0,x_1\right)+3M_3\left(x_0,x_1,x_2\right),\\
&x_0^2x_{k-1}+P_k^{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+2x_0x_1x_{k-2}\\
&=b_k-M_{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+3M_k\left(x_0,x_1,\cdots,x_{k-1}\right),4\le k\le m-1,\\
&x_0^2x_{m-1}+P_m^{m-1}\left(x_0,\cdots,x_{m-2}\right)+2x_0x_1x_{m-2}+x_0a_0=b_m-M_{m-1}\left(x_0,\cdots,x_{m-2}\right)+3M_m\left(x_0,\cdots,x_{m-1}\right),\\
&x_0^2x_{k-1}+P_k^{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+2x_0x_1x_{k-2}+x_{k-m}a_0+\cdots+x_0a_{k-m}\\
&=b_k-M_{k-1}\left(x_0,x_1,\cdots,x_{k-2}\right)+3M_k\left(x_0,x_1,\cdots,x_{k-1}\right),k\ge m+1.
\end{aligned}
$$

Now we consider Equality (3) with $\gamma(a)=1$, $\gamma(b)=0$. Put

$$A_0=x_0^2+a_0,A_k=\frac{A_{k-1}}{3}+a_k+R_k,\text{where } R_k=\sum_{j=0}^{k}x_jx_{k-j},k\ge 1,$$

$$
N_j'=\begin{cases}\dfrac{N_{j-1}}{3},j=3s-1,\\[2mm] \dfrac{N_{j-1}}{3}+x_{\frac{j}{3}}^3,j=3s,\\[2mm] \dfrac{N_{j-1}-x_{\frac{j-1}{3}}^3}{3},j=3s+1,\end{cases}\qquad
S_j^i=\begin{cases}\dfrac{P_{j-1}^i}{3},j=3s-1,\\[2mm] \dfrac{P_{j-1}^i}{3}+x_{\frac{j}{3}}^3,j=3s,\\[2mm] \dfrac{P_{j-1}^i-x_{\frac{j-1}{3}}^3}{3},j=3s+1.\end{cases}
$$

Theorem 3.7 *Let* $\gamma(a)=1$, $\gamma(b)=0$ *and* $x\in\mathbb{Z}_3^*$ *to be so that* $A_0=x_0^2+a_0\not\equiv 0\left(\operatorname{mod}3\right)$. *Then* x *to be a solution of Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if the congruences*

$$x_0^3 \equiv b_0 \pmod 3,$$
$$x_0 a_0 + M_1(x_0) \equiv b_1 \pmod 3,$$
$$\left(x_0^2 + a_0\right) x_{k-1} + x_{k-2} a_1 + \cdots + x_0 a_{k-1} + N_k'(x_0, \cdots, x_{k-2}) + M_k(x_0, \cdots, x_{k-2})$$
$$\equiv b_k \pmod 3, k \ge 2$$

are faithfully, where $M_1(x_0) = \frac{x_0^3 - b_0}{3}$ and integers $M_k(x_0, \cdots, x_{k-2}), (k \ge 2)$ are defined from the equalities

$$x_0 a_0 + M_1(x_0) = b_1 + M_2(x_0) \cdot 3,$$
$$\left(x_0^2 + a_0\right) x_{k-1} + N_k'(x_0, x_1, \cdots, x_{k-2}) + x_{k-2} a_1 + \cdots + x_0 a_{k-1} + M_k(x_0, \cdots, x_{k-2})$$
$$= b_k + M_{k+1}(x_0, \cdots, x_{k-1}) \cdot 3, k \ge 2.$$

Proof. Let the congruences $x_0^3 \equiv b_0 \pmod 3, x_0 a_0 + \frac{x_0^3 - b_0}{3} \equiv b_1 \pmod 3$, has a solution x_0. Then denote by $M_2(x_0)$ the number satisfying the equality $3M_2(x_0) = x_0 a_0 + M_1(x_0) - b_1$.

Using Theorem 2.1, we have existence of solutions x_k of the congruences

$$\left(x_0^2 + a_0\right) x_{k-1} + x_{k-2} a_1 + \cdots + x_0 a_{k-1} + N_k'(x_0, \cdots, x_{k-2}) + M_k(x_0, \cdots, x_{k-2})$$
$$\equiv b_k \pmod 3, k \ge 2.$$

The next chain of equalities

$$x_0^3 + a_0 x_0 \cdot 3 + \sum_{k=2}^{\infty} \left(\left(x_0^2 + a_0\right) x_{k-1} + N_k'(x_0, x_1, \cdots, x_{k-2}) + x_{k-2} a_1 + \cdots + x_0 a_{k-1} \right) 3^k$$
$$= b_0 + M_1(x_0) \cdot 3 + \left(b_1 - M_1(x_0) + M_2(x_0) \cdot 3\right) \cdot 3 + \sum_{k=2}^{\infty} \left(b_k - M_k(x_0, x_1, \cdots, x_{k-2}) + M_{k+1}(x_0, x_1, \cdots, x_{k-1}) \cdot 3\right) \cdot 3^k$$
$$= b_0 + \sum_{k=1}^{\infty} b_k 3^k,$$

shows that x is a solution of Equation (1).■

From the proof of Theorem 3.7, it is easy to see that if $A_0 = x_0^2 + a_0 \equiv 0 \pmod 3$, then we have the following congruences and appropriate equalities

a) $x_0^3 \equiv b_0 \pmod 3$, *i.e.* $x_0^3 = b_0 + M_1(x_0) \cdot 3$;

b) $x_0 a_0 + M_1(x_0) \equiv b_1 \pmod 3$, then $x_0 a_0 + M_1(x_0) = b_1 + M_2(x_0) \cdot 3$;

c) $x_0 a_1 + M_2(x_0) \equiv b_2 \pmod 3$, then

$$x_0 a_1 + M_2(x_0) = b_2 + M_3(x_0) \cdot 3; \tag{4}$$

d) $\frac{A_0}{3} x_1 + x_1 a_1 + x_0 a_2 + x_0 x_1^2 + x_1^3 + M_3(x_0) \equiv b_3 \pmod 3$, it follows that

$$\frac{A_0}{3} x_1 + x_1 a_1 + x_0 a_2 + x_0 x_1^2 + x_1^3 + M_3(x_0) = b_3 + M_4(x_0, x_1) \cdot 3.$$

Since $A_1 = \frac{A_0}{3} + a_1 + 2x_0 x_1$, then the congruence d) can be written in the form

$$\left(A_1 - x_0 x_1\right) x_1 + x_0 a_2 + x_1^3 + M_3(x_0) \equiv b_3 \pmod 3,$$

and so we have

$$\left(A_1 - x_0 x_1\right) x_1 + x_0 a_2 + x_1^3 + M_3(x_0) = b_3 + M_4(x_0, x_1) \cdot 3.$$

If for any natural number k we have $A_k \equiv 0(\bmod 3)$, then we could establish the criteria of solvability for Equation (1). However, if there exists k, such that $A_k \not\equiv 0(\bmod 3)$, then the criteria of solvability can be found, and therefore, we need the following

Lemma 3.1 *Let* $\gamma(a)=1$, $\gamma(b)=0$ *and* $x \in \mathbb{Z}_3^*$ *to be so that* $A_{k-j} \equiv 0(\bmod 3), 1 \le j \le k$, $A_k \not\equiv 0(\bmod 3)$ *for some fixed* k. *If* x *be a solution of Equation* (1), *then it is true the following system of the congruences*

$$
\begin{aligned}
&x_0^3 \equiv b_0 (\bmod 3),\\
&x_0 a_0 + M_1(x_0) \equiv b_1 (\bmod 3),\\
&x_{j-1}a_j + x_{j-2}a_{j+1} + \cdots + x_0 a_{2j-1} + S_{2j}^j + M_{2j}(x_0, x_1, \cdots, x_{j-1}) \equiv b_{2j} (\bmod 3),\\
&(A_j - x_0 x_j) x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \cdots + x_0 a_{2j} + S_{2j+1}^j + M_{2j+1}(x_0, x_1, \cdots, x_{j-1})\\
&\equiv b_{2j+1} (\bmod 3),\\
&A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \cdots + x_0 a_{2k+i} + S_{2k+1+i}^{k+i} + M_{2k+1+i}(x_0, x_1, \cdots, x_{k+i-1})\\
&\equiv b_{2k+1+i} (\bmod 3),
\end{aligned}
\tag{5}
$$

where $1 \le j \le k$ and integers $M_k(x_0, \cdots, x_{k-2})$ are defined from the equalities

$$
\begin{aligned}
&3 \cdot M_1(x_0) = x_0^3 - b_0,\\
&3 \cdot M_2(x_0) = x_0 a_0 + M_1(x_0) - b_1,\\
&3 \cdot M_{2j+1}(x_0, \cdots, x_{j-1}) = x_{j-1}a_j + \cdots + x_0 a_{2j-1} + S_{2j}^j + M_{2j}(x_0, \cdots, x_{j-1}) - b_{2j},\\
&3 \cdot M_{2j+2}(x_0, x_1, \cdots, x_j) = (A_j - x_0 x_j) x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \cdots + x_0 a_{2j} +\\
&\qquad + S_{2j+1}^j + M_{2j+1}(x_0, x_1, \cdots, x_{j-1}) - b_{2j+1},\\
&3 \cdot M_{2k+2+i}(x_0, x_1, \cdots, x_{k+i}) = A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0 a_{2k+i} +\\
&\qquad + S_{2k+1+i}^{k+i} + M_{2k+1+i}(x_0, x_1, \cdots, x_{k+i-1}) - b_{2k+1+i}.
\end{aligned}
\tag{6}
$$

Proof. We will prove Theorem by induction. Let $k=1$, *i.e.*

$$A_0 = x_0^2 + a_0 \equiv 0(\bmod 3), A_1 = \frac{A_0}{3} + a_1 + 2x_0 x_1 \not\equiv 0(\bmod 3),$$

then the system of the congruences (4) are true. Note that $S_2^1 = 0$, $S_3^1 = x_1^3$.

From (3) it is easy to get

$$\frac{A_0}{3} x_{t-2} + x_{t-2}a_1 + x_{t-3}a_2 + \cdots + x_0 a_{t-1} + S_t^{t-2} + 2x_0 x_1 x_{t-2} + M_t(x_0, \cdots, x_{t-3}) \equiv b_t (\bmod p), t \ge 4.$$

Therefore,

$$A_1 x_{t-2} + x_{t-3}a_2 + \cdots + x_0 a_{t-1} + S_t^{t-2} + M_t(x_0, \cdots, x_{t-3}) \equiv b_t (\bmod 3), t \ge 4, \tag{7}$$

where

$$3 \cdot M_{t+1}(x_0, \cdots, x_{t-2}) = A_1 x_{t-2} + x_{t-3}a_2 + \cdots + x_0 a_{t-1} + S_t^{t-2} + M_t(x_0, \cdots, x_{t-3}) - b_t, t \ge 4.$$

Obviously, the statement of Lemma is true for $k=1$, *i.e.* for $i = t-3$.

Let $k=2$, *i.e.* $A_0 \equiv 0(\bmod 3)$, $A_1 \equiv 0(\bmod 3)$, $A_2 = \frac{A_1}{3} + a_2 + x_1^2 + 2x_0 x_2 \not\equiv 0(\bmod 3)$, then from the equalities (7) it follows that the following congruences are be added to the system (4):

e) $x_1 a_2 + x_0 a_3 + S_4^2 + M_4(x_0, x_1) \equiv b_4 (\bmod 3)$, it follows

$$3\cdot M_5\left(x_0,x_1\right)=x_1a_2+x_0a_3+S_4^2+M_4\left(x_0,x_1\right)-b_4;$$

f) $\frac{A_1}{3}x_2+x_2a_2+x_1a_3+x_0a_4+S_5^3+M_5\left(x_0,x_1\right)\equiv b_5\left(\operatorname{mod}3\right)$, it follows

$$3\cdot M_6\left(x_0,x_1,x_2\right)=\frac{A_1}{3}x_2+x_2a_2+x_1a_3+x_0a_4+S_5^3+M_5\left(x_0,x_1\right)-b_5;$$

h) $\frac{A_1}{3}x_{t-2}+x_{t-2}a_2+x_{t-3}a_3+\cdots+x_0a_t+S_{t+1}^{t-1}+M_{t+1}\left(x_0,x_1,\cdots,x_{t-3}\right)\equiv b_{t+1}\left(\operatorname{mod}3\right)$,

where $t\ge5$ and $M_{t+2}\left(x_0,x_1,\cdots,x_{t-2}\right)$ are defined by equalities

$$3\cdot M_{t+2}\left(x_0,x_1,\cdots,x_{t-2}\right)=\frac{A_1}{3}x_{t-2}+x_{t-2}a_2+x_{t-3}a_3+\cdots+x_0a_t+S_{t+1}^{t-1}+M_{t+1}\left(x_0,x_1,\cdots,x_{t-3}\right)-b_{t+1}.$$

Since $S_4^2=0$, $S_2^2=0$, $S_5^3=x_0x_2^2+x_1^2x_2$, $S_{t+1}^{t-1}=S_{t+1}^{t-2}+x_1^2x_{t-2}+2x_0x_2x_{t-2}$, we denote by $i=t-4$ and have

e) $x_1a_2+x_0a_3+M_4\left(x_0,x_1\right)\equiv b_4\left(\operatorname{mod}3\right)$,

f) $\left(A_2-x_0x_2\right)x_2+x_1a_3+x_0a_4+M_5\left(x_0,x_1\right)\equiv b_5\left(\operatorname{mod}3\right)$,

h) $A_2x_{i+2}+x_{i+1}a_3+x_ia_4+\ldots+x_0a_{i+4}+S_{i+5}^{i+2}+M_{i+5}\left(x_0,x_1,\cdots,x_{i+1}\right)\equiv b_{i+5}\left(\operatorname{mod}3\right)$, where

$$3\cdot M_5\left(x_0,x_1\right)=x_1a_2+x_0a_3+M_4\left(x_0,x_1\right)-b_4,$$
$$3\cdot M_6\left(x_0,x_1,x_2\right)=\left(A_2-x_0x_2\right)x_2+x_1a_3+x_0a_4+M_5\left(x_0,x_1\right)-b_5.$$
$$3\cdot M_{6+i}\left(x_0,\cdots,x_{i+2}\right)=A_2x_{i+2}+x_{i+1}a_3+\cdots+x_0a_{i+4}+S_{i+5}^{i+2}+M_{i+5}\left(x_0,\cdots,x_{i+1}\right)-b_{i+5}.$$

So we showed that the statement of Lemma is true for $k=2$.

Let the system of congruences (5) and (6) is true for k. Since $A_k\equiv0\left(\operatorname{mod}3\right)$, then from the congruences

$$A_kx_{k+i}+x_{k+i-1}a_{k+1}+x_{k+i-2}a_{k+2}+\cdots+x_0a_{2k+i}+S_{2k+1+i}^{k+i}+M_{2k+1+i}\left(x_0,x_1,\cdots,x_{k+i-1}\right)\equiv b_{2k+1+i}\left(\operatorname{mod}3\right),i\ge1$$

we derive

$$x_ka_{k+1}+x_{k-1}a_{k+2}+\cdots+x_0a_{2k+1}+S_{2k+2}^{k+1}+M_{2k+2}\left(x_0,\cdots,x_k\right)\equiv b_{2k+2}\left(\operatorname{mod}3\right),$$
$$\frac{A_k}{3}x_{k+1}+x_{k+1}a_{k+1}+x_ka_{k+2}+\cdots+x_0a_{2k+2}+S_{2k+3}^{k+2}+M_{2k+3}\left(x_0,\cdots,x_k\right)\equiv b_{2k+3}\left(\operatorname{mod}3\right),$$
$$\frac{A_k}{3}x_{k+1+i}+x_{k+1+i}a_{k+1}+x_{k+i}a_{k+2}+\cdots+x_0a_{2k+i+2}+S_{2k+i+3}^{k+i+2}+M_{2k+i+3}\left(x_0,x_1,\cdots,x_{k+i}\right)$$
$$\equiv b_{2k+i+3}\left(\operatorname{mod}3\right),i\ge1.$$

It is easy to check that

$$S_{2k+3}^{k+2}=S_{2k+3}^{k+1}+R_{k+1}x_{k+1}-x_0x_{k+1}^2,S_{2k+i+3}^{k+i+2}=S_{2k+i+3}^{k+i+1}+R_{k+1}x_{k+1+i},i\ge1.$$

By these correlations we deduce

$$\frac{A_k}{3}x_{k+1}+x_{k+1}a_{k+1}+x_ka_{k+2}+\cdots+x_0a_{2k+2}+S_{2k+3}^{k+2}+M_{2k+3}\left(x_0,x_1,\cdots,x_k\right)$$
$$=\left(A_{k+1}-x_0x_{k+1}\right)x_{k+1}+x_ka_{k+2}+\cdots+x_0a_{2k+2}+S_{2k+3}^{k+1}+M_{2k+3}\left(x_0,x_1,\cdots,x_k\right).$$

For $i\ge1$ we get

$$\frac{A_k}{3}x_{k+1+i}+x_{k+1+i}a_{k+1}+\cdots+x_0a_{2k+i+2}+S_{2k+i+3}^{k+i+2}+M_{2k+i+3}\left(x_0,x_1,\cdots,x_{k+i}\right)$$
$$=A_{k+1}x_{k+1+i}+x_{k+i}a_{k+2}+\cdots+x_0a_{2k+i+2}+S_{2k+i+3}^{k+i+1}+M_{2k+i+3}\left(x_0,x_1,\cdots,x_{k+i}\right).$$

Consequently, we have

$$x_k a_{k+1} + x_{k-1}a_{k+2} + \cdots + x_0 a_{2k+1} + S_{2(k+1)}^{k+1} + M_{2(k+1)}(x_0,\cdots,x_k) \equiv b_{2(k+1)} (\operatorname{mod} 3),$$

$$(A_{k+1} - x_0 x_{k+1})x_{k+1} + x_k a_{k+2} + \cdots + x_0 a_{2k+2} + S_{2k+3}^{k+1} + M_{2k+3}(x_0,\cdots,x_k) \equiv b_{2k+3} (\operatorname{mod} 3),$$
$$A_{k+1}x_{k+1+i} + x_{k+i}a_{k+2} + \cdots + x_0 a_{2k+i+2} + S_{2k+i+3}^{k+i+1} + M_{2k+i+3}(x_0,x_1,\cdots,x_{k+i}) \equiv b_{2k+i+3} (\operatorname{mod} 3), i \ge 1.$$

So we established that the system of congruences (5)-(6) is true for $k+1$. ■

Using the Lemma 3.1 we obtain the following Theorems.

Theorem 3.8 *Let* $\gamma(a)=1$, $\gamma(b)=0$ *and* $x \in \mathbb{Z}_3^*$ *to be such that* $A_{k-j} \equiv 0(\operatorname{mod} 3), 1 \le j \le k$, $A_k \not\equiv 0(\operatorname{mod} 3)$ *for some fixed* $k(k \ge 1)$. *Then* x *to be a solution of the Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if the system of the congruences*

$$x_0^3 \equiv b_0 (\operatorname{mod} 3),$$
$$x_0 a_0 + M_1(x_0) \equiv b_1 (\operatorname{mod} 3),$$
$$x_{j-1}a_j + x_{j-2}a_{j+1} + \cdots + x_0 a_{2j-1} + S_{2j}^j + M_{2j}(x_0,x_1,\cdots,x_{j-1}) \equiv b_{2j} (\operatorname{mod} 3),$$
$$(A_j - x_0 x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \cdots + x_0 a_{2j} + S_{2j+1}^j + M_{2j+1}(x_0,x_1,\cdots,x_{j-1}) \equiv b_{2j+1} (\operatorname{mod} 3),$$

has a solution, where $1 \le j \le k$ and integers $M_k(x_0,x_1,\cdots,x_{k-1})$ are defined from the equalities

$$3 \cdot M_1(x_0) = x_0^3 - b_0,$$
$$3 \cdot M_2(x_0) = x_0 a_0 + M_1(x_0) - b_1,$$
$$3 \cdot M_{2j+1}(x_0,\cdots,x_{j-1}) = x_{j-1}a_j + x_{j-2}a_{j+1} + \cdots + x_0 a_{2j-1} + S_{2j}^j + M_{2j}(x_0,\cdots,x_{j-1}) - b_{2j},$$
$$3 \cdot M_{2j+2}(x_0,x_1,\cdots,x_j)$$
$$= (A_j - x_0 x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \cdots + x_0 a_{2j} + S_{2j+1}^j + M_{2j+1}(x_0,x_1,\cdots,x_{j-1}) - b_{2j+1}.$$

Theorem 3.9 *Let* $\gamma(a)=1$, $\gamma(b)=0$ *and* $x \in \mathbb{Z}_3^*$ *to be so that* $A_k \equiv 0(\operatorname{mod} 3)$ *for all* $k \in \mathbb{N}$. *Then* x *to be a solution of the Equation* (1) *in* $\mathbb{Z}_3^*$ *if and only if the system of the congruences*

$$x_0^3 \equiv b_0 (\operatorname{mod} 3),$$
$$x_0 a_0 + M_1(x_0) \equiv b_1 (\operatorname{mod} 3),$$
$$x_{j-1}a_j + x_{j-2}a_{j+1} + \cdots + x_0 a_{2j-1} + S_{2j}^j + M_{2j}(x_0,x_1,\cdots,x_{j-1}) \equiv b_{2j} (\operatorname{mod} 3),$$
$$(A_j - x_0 x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \cdots + x_0 a_{2j} + S_{2j+1}^j + M_{2j+1}(x_0,x_1,\cdots,x_{j-1})$$
$$\equiv b_{2j+1} (\operatorname{mod} 3),$$

has a solution, where $j \ge 1$ and integers $M_k(x_0,x_1,\cdots,x_{k-1})$ are defined from the equalities

$$3 \cdot M_1(x_0) = x_0^3 - b_0,$$
$$3 \cdot M_2(x_0) = x_0 a_0 + M_1(x_0) - b_1,$$
$$3 \cdot M_{2j+1}(x_0,\cdots,x_{j-1}) = x_{j-1}a_j + x_{j-2}a_{j+1} + \cdots + x_0 a_{2j-1} + S_{2j}^j + M_{2j}(x_0,\cdots,x_{j-1}) - b_{2j},$$
$$3 \cdot M_{2j+2}(x_0,x_1,\cdots,x_j) = (A_j - x_0 x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0 a_{2j} + S_{2j+1}^j + M_{2j+1}(x_0,x_1,\cdots,x_{j-1}) - b_{2j+1}.$$

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