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Solution of Some Integral Equations Involving Confluent *k*-Hypergeometric Functions

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ABSTRACT

The principle aim of this research article is to investigate the properties of *k*-fractional integration introduced and defined by Mubeen and Habibullah [1], and secondly to solve the integral equation of the form $g(x)=\int_0^x \frac{(x-t)^{\gamma/k}}{\Gamma_k(\gamma)}\,{}_1F_1\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-x\right] f(t)\mathrm{d}t$, for $k>0,\beta>0,\gamma>0,0<x<t<\infty$, where ${}_1F_1\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ x\right]$ is the confluent *k*-hypergeometric functions, by using *k*-fractional integration.

Keywords: Linear Integral Equations; Fractional Integrals; Confluent Hypergeometric Functions

1. Introduction

Erdélyi [2] investigated the solutions of integral equations whose kernels contain Legendre functions. Love [3] solved the integral equations involving hypergeometric functions using fractional derivatives. Using variance of fractional integration, Habibullah [4] investigated the solution of the integral equations involving confluent hypergeometric functions and Srivastava [5] discussed the equations with polynomial kernels.

Diaz *et al*. [6-8] have introduced *k*-gamma and *k*-beta functions and proved a number of their properties that we are interested in. They have also studied *k*-zeta function and *k*-hypergeometric function based on Pochhammer *k*-symbols for factorial functions. These studies were then followed by works of Mansour [9], Kokologiannaki [10], Krasniqi [11,12] and Merovci [13] elaborating and strengthening the scope of *k*-gamma and *k*-beta functions. Very recently, Mubeen and Habibullah [14] gave a simple and useful integral representations of generalized *k*-hypergeometric and confluent *k*-hypergeometric functions that could helpful in completing the present research paper.

2. Fractional Integration

Mubeen and Habibullah [1] defined a *k*-fractional integration as a variant of Riemann-Liouville fractional integral as

$$I_k^{\alpha}(f)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{(\alpha/k)-1} f(t)\mathrm{d}t,$$

for $k>0,0<x<t<\infty$. It reduces to the classical Riemann-Liouville fractional integral by taking $k\to 1$ as

$$I^{\alpha}(f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, 0<x<t<\infty.$$

3. *k*-Hypergeometric and Confluent *k*-Hypergeometric Differential Equations

The following *k*-hypergeometric function defined by Mubeen and Habibullah [14]

$$\omega={}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k); \\ (\gamma,k);\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{z^n}{n!}, k>0$$

is the solution of the linear second order differential equation of the form

$$kz(1-kz)\omega''+\left[\gamma-k(\alpha+\beta+k)z\right]\omega'-\alpha\beta\omega=0.$$

In this article, we call it *k*-hypergeometric differential equation. It reduces to ordinary hypergeometric differential equation by taking $k\to 1$.

And also the following confluent k-hypergeometric function defined by Mubeen and Habibullah [14]

$$\omega = {}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{z^n}{n!}, k>0,$$

is the solution of the linear second order differential equation of the form

$$kz\omega'' + (\gamma - kz)\omega' - \beta\omega = 0.$$

In this article, we call it confluent k-hypergeometric differential equation. It reduces to ordinary hypergeometric differential equation by taking $k \to 1$.

4. Main Results

Theorem 4.1. If $\lambda > 0, \gamma > 0$ and $x > t$, then

$$\int_t^x \frac{(x-s)^{(\lambda/k)-1}}{\Gamma_k(\lambda)}\frac{(s-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-s\right]\mathrm{d}s = \frac{k(x-t)^{((\gamma+\lambda)/k)-1}}{\Gamma_k(\gamma+\lambda)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma+\lambda,k);\end{matrix}\ t-x\right]$$

Proof. Consider

$$\begin{aligned}
&\int_0^1 \frac{(1-u)^{(\lambda/k)-1}}{\Gamma_k(\lambda)}\frac{u^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ zu\right]\mathrm{d}u \\
&= \int_0^1 \frac{(1-u)^{(\lambda/k)-1}}{\Gamma_k(\lambda)}\frac{u^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{z^n u^n}{n!}\mathrm{d}u \\
&= k\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{z^n}{n!}\left\{\frac{1}{k}\int_0^1\frac{(1-u)^{(\lambda/k)-1}}{\Gamma_k(\lambda)}\frac{u^{(\gamma/k)+n-1}}{\Gamma_k(\gamma)}\mathrm{d}u\right\} \\
&= k\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{z^n}{n!}\frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)\Gamma_k(\gamma+\lambda+nk)} \\
&= \frac{k}{\Gamma_k(\gamma+\lambda)}\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{(\gamma+\lambda)_{n,k}}\frac{z^n}{n!} \\
&= \frac{k}{\Gamma_k(\gamma+\lambda)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma+\lambda,k);\end{matrix}\ z\right]
\end{aligned}$$

Put $u = \frac{s-t}{x-t}$ and $z = t-x$ in the above equation, then we get the desired result.

Theorem 4.2. Let

$$\int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-x\right]f(t)\mathrm{d}t = g(x)$$

for $k > 0, \beta > 0, \gamma > 0$ and $0 < x < t < \infty$.

If $g(x)$ is a given function, then

$$f(x) = \mathrm{e}^{-x}I_k^{-\beta}\mathrm{e}^{x}I_k^{-\gamma}I_k^{\beta}g(x).$$

Proof. Set

$$H_k(\beta,\gamma)f(x) = \int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-x\right]f(t)\mathrm{d}t,$$

where $k > 0, \beta > 0, \gamma > 0$ and $0 < x < t < \infty$.

Apply I_k^{λ} on both sides, we get the following

$$I_k^{\lambda}H_k(\beta,\gamma)f(x) = \int_0^x \frac{(x-s)^{(\lambda/k)-1}}{\Gamma_k(\lambda)} \times\left\{\int_0^s \frac{(s-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-s\right]f(t)\mathrm{d}t\right\}\mathrm{d}s.$$

Changing the order of integration by using Fubini's theorem.

$$I_k^{\lambda}H_k(\beta,\gamma)f(x) = \int_0^x f(t) \times\left\{\int_t^x \frac{(x-s)^{(\lambda/k)-1}}{\Gamma_k(\lambda)}\frac{(s-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-s\right]\mathrm{d}s\right\}\mathrm{d}t.$$

By Theorem 4.1, we have

$$I_k^{\lambda}H_k(\beta,\gamma)f(x) = k\int_0^x \frac{(x-t)^{((\gamma+\lambda)/k)-1}}{\Gamma_k(\gamma+\lambda)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma+\lambda,k);\end{matrix}\ t-x\right]f(t)\mathrm{d}t.$$

This implies that

$$I_k^{\lambda}H_k(\beta,\gamma)f(x) = kH_k(\beta,\gamma+\lambda)f(x).$$

Since

$$\begin{aligned}
&\int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ t-x\right]f(t)\mathrm{d}t = g(x), \\
&H_k(\beta,\gamma)f(x) = g(x), \\
&I_k^{\beta}H_k(\beta,\gamma)f(x) = I_k^{\beta}g(x), \\
&kH_k(\beta,\beta+\gamma)f(x) = I_k^{\beta}g(x), \\
&I_k^{\gamma}H_k(\beta,\beta)f(x) = I_k^{\beta}g(x), \\
&H_k(\beta,\beta)f(x) = I_k^{-\gamma}I_k^{\beta}g(x).
\end{aligned}$$

This may be written as

$$\int_0^x \frac{(x-t)^{(\beta/k)-1}}{\Gamma_k(\beta)}\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\beta,k);\end{matrix}\ t-x\right]f(t)\mathrm{d}t = I_k^{-\gamma}I_k^{\beta}g(x).$$

Since ${}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\beta,k);\end{matrix}\ t-x\right] = \mathrm{e}^{t-x}$, we obtain

$$\int_0^x \frac{(x-t)^{(\beta/k)-1}}{\Gamma_k(\beta)} e^{t-x} f(t)\,dt = I_k^{-\gamma} I_k^{\beta} g(x),$$

$$\int_0^x \frac{(x-t)^{(\beta/k)-1}}{\Gamma_k(\beta)} e^{t} f(t)\,dt = e^x I_k^{-\gamma} I_k^{\beta} g(x),$$

$$I_k^{\beta} e^x f(x) = e^x I_k^{-\gamma} I_k^{\beta} g(x),$$

$$e^x f(x) = I_k^{-\beta} e^x I_k^{-\gamma} I_k^{\beta} g(x),$$

$$f(x) = e^{-x} I_k^{-\beta} e^x I_k^{-\gamma} I_k^{\beta} g(x).$$

This is the solution of the integral equation, if it exists. This integral equation implies that

$$g(x) = I_k^{-\beta} I_k^{\gamma} e^{-x} I_k^{\beta} e^x f(x).$$

Now, we find a solution of another integral equation

$$\int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)} {}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ x-t\right] f(t)\,dt = g(x),$$

for $k>0, \beta>0, \gamma>0$ and $0<x<t<\infty$.

Theorem 4.3. Let

$$\int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)} {}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ x-t\right] f(t)\,dt = g(x),$$

for $k>0, \beta>0, \gamma>0$ and $0<x<t<\infty$.

If $g(x)$ is a given function, then

$$f(x) = I_k^{\beta-\gamma} e^x I_k^{-\gamma} I_k^{\gamma-\beta} e^{-x} g(x).$$

Proof. Consider

$$\int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)} {}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ x-t\right] f(t)\,dt = g(x).$$

Using the Mubeen's relation [15]

$${}_1F_{1,k}\left[\begin{matrix}(\beta,k); \\ (\gamma,k);\end{matrix}\ x\right] = e^x\, {}_1F_{1,k}\left[\begin{matrix}(\gamma-\beta,k); \\ (\beta,k);\end{matrix}\ -x\right],$$

we obtain the following

$$\int_0^x \frac{(x-t)^{(\gamma/k)-1}}{\Gamma_k(\gamma)} e^x\, {}_1F_{1,k}\left[\begin{matrix}(\gamma-\beta,k); \\ (\gamma,k);\end{matrix}\ t-x\right] f(t)\,dt = g(x).$$

Thus, if $k>0, \beta>0, \gamma>0$ and $0<x<t<\infty$, then

$$f(x) = I_k^{\beta-\gamma} e^x I_k^{-\gamma} I_k^{\gamma-\beta} e^{-x} g(x).$$

Also, we have the following result

$$g(x) = e^x I_k^{\beta-\gamma} I_k^{\gamma} e^{-x} I_k^{\gamma-\beta} f(x).$$

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