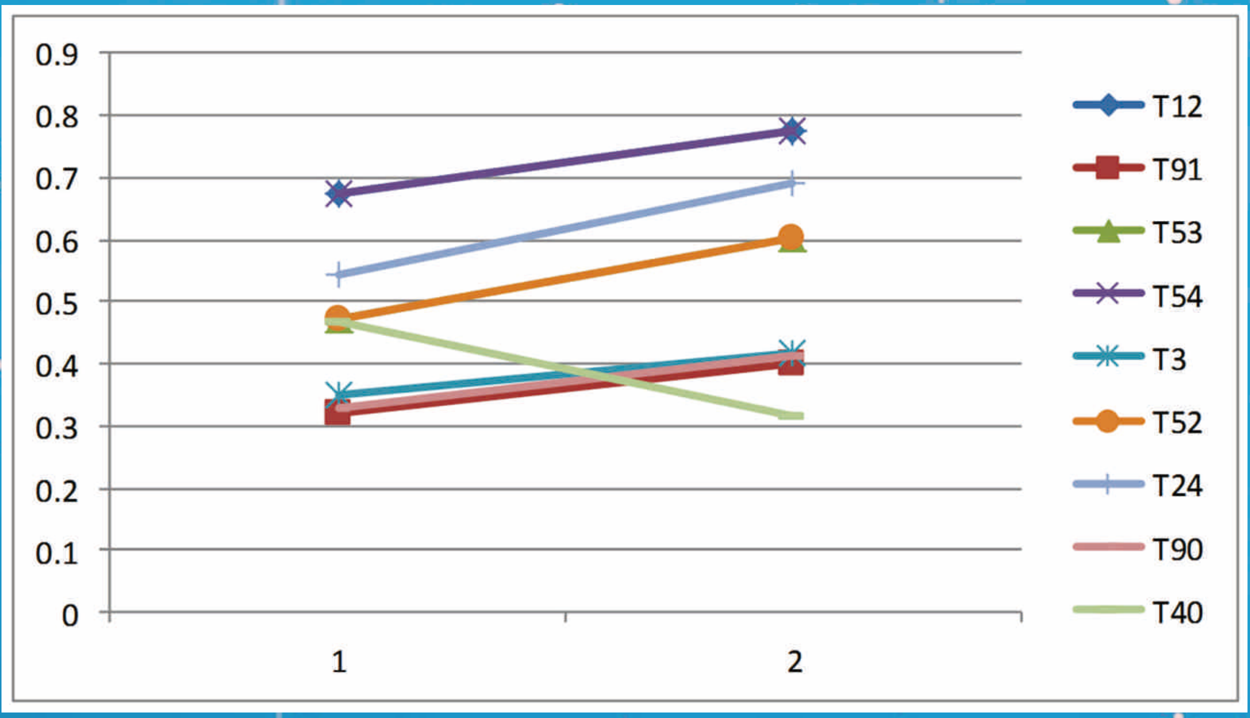
ISSN: 2160-8830 Vol. 4, No. 6, November 2014

Scientific Research

American Journal of Operations Research



ISSN: 2160-8830
9 772160 883007 06

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ISSN 2160-8830 (Print) ISSN 2160-8849 (Online)
http://www.scirp.org/journal/ajor

American Journal of Operations Research, 2014, 4, 331-374
Published Online November 2014 in SciRes. http://www.scirp.org/journal/ajor

Scientific Research

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Volume 4 Number 6 November 2014

The figure on the front cover is from the article published in American Journal of Operations Research, 2014, Vol. 4, No. 6, pp. 365-374 by William P. Fox.

American Journal of Operations Research (AJOR)

Journal Information

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American Journal of Operations Research, 2014, 4, 331-339
Published Online November 2014 in SciRes. http://www.scirp.org/journal/ajor
http://dx.doi.org/10.4236/ajor.2014.46032

Scientific Research

A Penalty Function Algorithm with Objective Parameters and Constraint Penalty Parameter for Multi-Objective Programming

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Received 28 August 2014; revised 22 September 2014; accepted 6 October 2014

Abstract

In this paper, we present an algorithm to solve the inequality constrained multi-objective programming (MP) by using a penalty function with objective parameters and constraint penalty parameter. First, the penalty function with objective parameters and constraint penalty parameter for MP and the corresponding unconstraint penalty optimization problem (UPOP) is defined. Under some conditions, a Pareto efficient solution (or a weakly-efficient solution) to UPOP is proved to be a Pareto efficient solution (or a weakly-efficient solution) to MP. The penalty function is proved to be exact under a stable condition. Then, we design an algorithm to solve MP and prove its convergence. Finally, numerical examples show that the algorithm may help decision makers to find a satisfactory solution to MP.

Keywords

Multi-Objective Programming, Penalty Function, Objective Parameters, Constraint Penalty Parameter, Pareto Weakly-Efficient Solution

1. Introduction

Multi-objective programming is an important model in solving vector optimization problems. Many methods had been given to find solutions to multiobjective programming [1]. It is well-known that the penalty function is one of efficient methods in studying multiobjective programming. For example, in 1984, White [2] presented an exact penalty function for multiobjective programming. Sunaga, Mazeed and Kondo [3] applied penalty function

How to cite this paper: Meng, Z.Q., Shen, R. and Jiang, M. (2014) A Penalty Function Algorithm with Objective Parameters and Constraint Penalty Parameter for Multi-Objective Programming. *American Journal of Operations Research*, **4**, 331-339.
http://dx.doi.org/10.4236/ajor.2014.46032

formulation to interactive multiobjective programming problems. Ruan and Huang [4] studied weak calmness and weak stability of exact penalty functions for multiobjective programming. By penalty function, Liu [5] derived necessary and sufficient conditions without a constraint qualification for e-Pareto optimality of multi-objective programming, and the generalized e-saddle point for Pareto optimality of the vector Lagrangian. Huang and Yang [6] gave nonlinear Lagrangian for multiobjective optimization to duality and exact penalization. Chang and Lin [7] solved interval goal programming by using S-shaped penalty function. Antczak [8] studied the vector exact l_1 penalty method for nondifferentiable convex multiobjective programming problems. Huang, Teo and Yang [9] discussed calmness of exact penalization in vector optimization with cone constraints. Huang [10] proved calmness of exact penalization in constrained scalar set-valued optimization. Meng, Shen and Jiang [11] defined an objective penalty function based on objective weight for multiobjective optimization problem and presented an interactive algorithm. This paper defines a penalty function with objective parameters and constraint penalty parameter which differs from an objective penalty function in [11].

Because it is almost not possible for decision makers (DMs) to obtain all efficient solutions to MP, it is significant to present an efficient algorithm of MP so that DMs finds an easy and satisfactory solution to the MP. Luque, Ruiz and Steuer pointed out that an efficient algorithms not only help decision makers learn more about efficient solutions, but also navigate to a final solution as quickly as possible [12]. This paper presents an algorithm by modifying every objective parameter of penalty function so that a final solution is easily and quickly obtained. In Section 2, we introduce a penalty function with the objective parameters and constraint penalty parameter, and its algorithm. In Section 3, we give numerical results to show that the proposed algorithm is efficient.

2. Penalty Function with Objective Parameters and Constraint Penalty Parameter

In this paper we consider the following inequality constrained multi-objective programming:

$$\begin{aligned} (\mathrm{MP}) \qquad & \min \quad f(x)=\left(f_1(x), f_2(x),\cdots, f_q(x)\right) \\ & \text{s.t.} \quad g_i(x)\le 0,\ i=1,2,\cdots,m, \end{aligned} \tag{1}$$

where $f_j : R^n \to R^1 \cup \{+\infty\}$, $g_i : R^n \to R^1 \cup \{+\infty\}$, for $j \in J = \{1,2,\cdots,q\}$, $i \in I = \{1,2,\cdots,m\}$.

We denote the feasible set of MP (1) by $X = \left\{x \in R^n \middle| g_i(x) \le 0, i \in I\right\}$. As usual, $\bar{x} \in X$ is called a Pareto weakly-efficient solution if there is no $x \in X$ such that $f_j(x) < f_j(\bar{x})$ for all $j \in J$, *i.e.* $f(x) < f(\bar{x})$. $\bar{x} \in X$ is called a Pareto efficient solution if there is no $x \in X$ such that $f_j(x) \le f_j(\bar{x})$ for all $j \in J$ and $f_j(x) < f_j(\bar{x})$ for at least one $j \in J$, *i.e.* $f(x) \preceq f(\bar{x})$.

Let functions $Q : R \to R \cup \{+\infty\}$ and $P : R \to R \cup \{+\infty\}$ satisfy

$$\begin{cases} Q(t) = 0 & \text{if and only if } t \le 0 \\ Q(t) > 0 & \text{if and only if } t > 0 \\ Q(t_2) > Q(t_1) & \text{if and only if } t_2 > t_1 > 0 \end{cases}$$

where $\lim_{t \to -\infty} Q(t) = 0$ and

$$\begin{cases} P(t) = 0 & \text{if and only if } t \le 0, \\ P(t) > 0 & \text{if and only if } t > 0 \\ P(t_2) > P(t_1) & \text{if and only if } t_2 > t_1 > 0. \end{cases}$$

Let

$$F_j(x, M_j, \rho) = Q\left(f_j(x) - M_j\right) + \rho \sum_{i \in I} P\left(g_i(x)\right), \quad j = 1,2,\cdots,q,$$

where $M_j\ (j = 1,2,\cdots,q)$ is an objective parameter and $\rho > 0$ is the constraint penalty parameter. Let $\boldsymbol{M} = (M_1, M_2, \cdots, M_q)$ and the penalty function of (1) be defined as:

$$\boldsymbol{F}(x,\boldsymbol{M},\rho)=\left(F_1(x,M_1,\rho),F_2(x,M_2,\rho),\cdots,F_q(x,M_q,\rho)\right).$$

Consider the following unconstraint penalty optimization problem:

$$\mathrm{MP}(\boldsymbol{M},\rho) \qquad \min \boldsymbol{F}(x,\boldsymbol{M},\rho), \qquad s.t. \quad x\in R^n.$$

For $x\in R^n$, let index set

$$J^0(x,\boldsymbol{M},\rho)=\left\{j\in J \,\middle|\, F_j(x,M_j,\rho)=0, \text{ for } j\in J\right\},$$

$$J^+(x,\boldsymbol{M},\rho)=\left\{j\in J \,\middle|\, F_j(x,M_j,\rho)>0, \text{ for } j\in J\right\}.$$

We have $J=J^0(x,\boldsymbol{M},\rho)\cup J^+(x,\boldsymbol{M},\rho)$.

Theorem 1. *Suppose that for given* $(\boldsymbol{M},\rho)$, x_M^* *is a Pareto weakly-efficient solution to* $\mathrm{MP}(\boldsymbol{M},\rho)$. *Then the following three assertions hold:*

1) If $J^0(x_M^*,\boldsymbol{M},\rho)\neq\varnothing$, then x_M^* is a feasible solution to (MP), $f_j(x_M^*)\le M_j$ for all $j\in J^0(x_M^*,\boldsymbol{M},\rho)$ and $f_j(x_M^*)>M_j$ for all $j\in J^+(x_M^*,\boldsymbol{M},\rho)$.

2) If $J^0(x_M^*,\boldsymbol{M},\rho)=\varnothing$ (*i.e.* $\boldsymbol{F}(x_M^*,\boldsymbol{M},\rho)>0$), then there is no $x\in X$ such that $f(x)<f(x_M^*)$.

3) If $\boldsymbol{F}(x_M^*,\boldsymbol{M},\rho)>0$ and x_M^* is a feasible solution to (MP), then x_M^* is a Pareto weakly-efficient solution to (MP).

Proof. 1) The conclusion is obvious from the definitions of P and Q.

2) Suppose that there be an $x\in X$ such that $f(x)<f(x_M^*)$. When $f_j(x_M^*)\le M_j$ for some $j\in J$, we have

$$Q\left(f_j(x)-M_j\right)=Q\left(f_j(x_M^*)-M_j\right)<Q\left(f_j(x_M^*)-M_j\right)+\rho\sum_{i=1}^m P\left(g_i(x_M^*)\right).$$

When $f_j(x_M^*)>M_j$ for some $j\in J$, we have

$$Q\left(f_j(x)-M_j\right)<Q\left(f_j(x_M^*)-M_j\right)\le Q\left(f_j(x_M^*)-M_j\right)+\rho\sum_{i=1}^m P\left(g_i(x_M^*)\right).$$

Hence, $\boldsymbol{F}(x,\boldsymbol{M},\rho)<\boldsymbol{F}(x_M^*,\boldsymbol{M},\rho)$, then x_M^* is not a Pareto weakly-efficient solution to $\mathrm{MP}(\boldsymbol{M},\rho)$.

3) According to 2), the conclusion holds.

Theorem 2. *Suppose that for a given* $(\boldsymbol{M},\rho)$, x_M^* *is a Pareto efficient solution to* $\mathrm{MP}(\boldsymbol{M},\rho)$. *Then the following three assertions hold:*

1) If $J^0(x_M^*,\boldsymbol{M},\rho)\neq\varnothing$, then x_M^* is a feasible solution to (MP), $f_j(x_M^*)\le M_j$ for all $j\in J^0(x_M^*,\boldsymbol{M},\rho)$ and $f_j(x_M^*)>M_j$ for all $j\in J^+(x_M^*,\boldsymbol{M},\rho)$.

2) If $J^0(x_M^*,\boldsymbol{M},\rho)\neq\varnothing$ (*i.e.* $\boldsymbol{F}(x_M^*,\boldsymbol{M},\rho)>0$), then there is no $x\in X$ such that $f(x)\preceq f(x_M^*)$.

3) If $\boldsymbol{F}(x_M^*,\boldsymbol{M},\rho)>0$ and x_M^* is a feasible solution to (MP), then x_M^* is a Pareto efficient solution to (MP).

Proof. 1) The conclusion is obvious from the definitions of P and Q.

2) Suppose that there be an $x\in X$ such that $f(x)\preceq f(x_M^*)$. When $f_j(x_M^*)\le M_j$ for some $j\in J$, we have

$$Q\left(f_j(x)-M_j\right)=Q\left(f_j(x_M^*)-M_j\right)<Q\left(f_j(x_M^*)-M_j\right)+\rho\sum_{i=1}^m P\left(g_i(x_M^*)\right).$$

When $f_j(x_M^*)>M_j$ for some $j\in J$, we have

$$Q\left(f_j(x)-M_j\right)\preceq Q\left(f_j\left(x_M^*\right)-M_j\right)\preceq Q\left(f_j\left(x_M^*\right)-M_j\right)+\rho\sum_{i=1}^{m}P\left(g_i\left(x_M^*\right)\right).$$

Hence, $\boldsymbol{F}(x,\boldsymbol{M},\rho)\preceq\boldsymbol{F}\left(x_M^*,\boldsymbol{M},\rho\right)$, then x_M^* is not a Pareto efficient solution to $\mathrm{MP}(\boldsymbol{M},\rho)$.

3) According to 2), the conclusion holds.

Based on Theorem 1, we develop an algorithm to compute an efficient solution to (MP). The algorithm solves the problem $\mathrm{MP}(\boldsymbol{M},\rho)$ sequentially, and is called Multiobjective Penalty Function Algorithm (MPFA for short).

MPFA Algorithm:

Step 1: Choose $x^0\in X$, $\rho_1>0$, $N>1$ and $M_j^*<\min\limits_{x\in X}f_j(x)$ for each $j\in J$. Let $k=1$, and $M_j^1=\dfrac{M_j^*+f_j\left(x^0\right)}{2}\,(j\in J)$.

Step 2: Solve $\min\limits_{x\in R^n}\boldsymbol{F}\left(x,\boldsymbol{M}^k,\rho_k\right)$, where $\boldsymbol{M}^k=\left(M_1^k,M_2^k,\cdots,M_q^k\right)$. Let x^k be a Pareto weakly-efficient solution.

Step 3: If $J^0\left(x^k,\boldsymbol{M}^k,\rho_k\right)\neq\varnothing$, for each $j\in J$, let $M_j^{k+1}=\dfrac{M_j^*+M_j^k}{2}$, $\rho_{k+1}=N\rho_k, k+1:=k$ and go to Step 2. Otherwise, $\boldsymbol{F}\left(x^k,\boldsymbol{M}_k,\rho_k\right)>0$, go to Step 4.

Step 4: If x^k is not feasible to (MP), for each $j\in J$, let $M_j^{k+1}=\dfrac{M_j^*+M_j^k}{2}$, $\rho_{k+1}=N\rho_k, k+1:=k$ and go to Step 2. Otherwise, stop and x^k is a Pareto weakly-efficient solution to (MP).

In the MPFA algorithm, it is assumed that for each $j\in J$ $M_j^*<\min\limits_{x\in X}f_j(x)$ can always be obtained .

The convergence of the MPFA algorithm is proved in the following theorem. For some $j\in J$, let

$$S\left(L,f_j\right)=\left\{x^k\,\middle|\,L\geq Q\left(f_j\left(x^k\right)-M_j^k\right),k=1,2,\cdots\right\},$$

which is called a Q-level set. $S\left(L,f_j\right)$ is bounded if, for any given $L>0$ and a convergent sequence $M_j^k\to M_j^*$, $S\left(L,f_j\right)$ is bounded.

Theorem 3. *Suppose that* Q, $f_j\,(j\in J)$ *and* $g_i\,(i\in I)$ *are continuous on* R^n*, and the Q-level set* $S\left(L,f_j\right)$ *is bounded for all* $j\in J$*. Let* $\left\{x^k\right\}$ *be the sequence generated by the MPFA algorithm.*

1) If $\left\{x^k\right\}\left(k=1,2,\cdots,\bar{k}\right)$ is a finite sequence (*i.e.*, the MPFA algorithm stops at the $\bar{k}$ -th iteration), then $x^{\bar{k}}$ is a Pareto weakly-efficient solution to (MP).

2) If $\left\{x^k\right\}$ is an infinite sequence, then $\left\{x^k\right\}$ is bounded and any limit point of it is a Pareto weakly-efficient solution to (MP).

Proof. For all $j\in J$, it is clear that the sequence $\left\{M_j^k\right\}$ decreases with

$$M_j^{k+1}-M_j^*=\frac{M_j^k-M_j^*}{2},\quad k=1,2,\cdots. \tag{2}$$

Therefore, $\left\{M_j^k\right\}$ converges to M_j^* for all $j\in J$.

1) If the MPFA algorithm terminates at the $\bar{k}$th iteration and the second situation of Step 4 occurs, by Theorem 1, $x^{\bar{k}}$ is a Pareto weakly-efficient solution to (MP).

2) We first show that the sequence $\left\{x^k\right\}$ is bounded. From the MPFA algorithm, we have $M_j^*<f_j(x)$ for all $x\in X$. Since $\left\{M_j^k\right\}$ converges to M_j^* for all $j\in J$, there is a k' such that $M_j^k<f_j(x)$ for all $x\in X$ and all $k>k'$. If $x^k\in X$ for each $k>k'$, we have $Q\left(f_j\left(x^k\right)-M_j^k\right)>0$ for all $j\in J$. Hence, we

have $\boldsymbol{F}\left(x^k, \boldsymbol{M}_k, \rho_k\right) > 0$ for all $k > k'$. By Theorem 1, there is a $j \in J$ such that

$$f_j\left(x^k\right) \le f_j\left(x^0\right),\ k = k'+1, k'+2, \cdots.$$

So,

$$Q\left(f_j\left(x^k\right) - M_j^k\right) \le Q\left(f_j\left(x^0\right) - M_j^k\right),\ k = k'+1, k'+2, \cdots.$$

Therefore, there is an $L > 0$ such that

$$Q\left(f_j\left(x^k\right) - M_j^k\right) \le Q\left(f_j\left(x^0\right) - M_j^k\right) < L,\ k = 1, 2, \cdots.$$

Since $S\left(L, f_j\right)$ is bounded, the sequence $\left\{x^k\right\}$ is bounded. Without loss of generality, we assume $x^k \to x^*$. Since x^k is a Pareto weakly-efficient solution to $\mathrm{MP}\left(\boldsymbol{M}^k, \rho_k\right)$, for some j, there are infinite $k > k'$ such that

$$Q\left(f_j\left(x^k\right) - M_j^k\right) + \rho_k \sum_{i=1}^{m} P\left(g_i\left(x^k\right)\right) \le Q\left(f_j\left(x^0\right) - M_j^k\right).$$

We have

$$\sum_{i=1}^{m} P\left(g_i\left(x^k\right)\right) \le \frac{1}{\rho_k}\left[Q\left(f_j\left(x^0\right) - M_j^k\right) - Q\left(f_j\left(x^k\right) - M_j^k\right)\right].$$

When $\rho_k \to +\infty$, we have $\sum_{i=1}^{m} P\left(g_i\left(x^*\right)\right) = 0$. Hence, $x^* \in X$. If x^* is not a Pareto weakly-efficient solution to (MP), there is an $x \in X$ such that $f(x) < f\left(x^*\right)$. Let $\delta = \min\left\{f_j\left(x^*\right) - f_j(x) \mid j = 1, 2, \cdots, q\right\}$. From $x^k \to x^*$, there is some k such that

$$f_j\left(x^*\right) - f_j\left(x^k\right) < \delta \le f_j\left(x^*\right) - f_j(x),\ j = 1, 2, \cdots, q.$$

So, we have $f(x) < f\left(x^k\right)$, which by Theorem 1 is a contradiction. Hence, x^* is a Pareto weakly-efficient solution to (MP).

Theorem 3 means that the MPFA algorithm is convergent in theory. Now, we discuss the exactness of the penalty function for (MP). If there are an $\boldsymbol{M}' \in R^q$ and ρ' such that a Pareto weakly-efficient solution x^* to (MP) is also a Pareto weakly-efficient solution to $\left(\mathrm{P}\left(\boldsymbol{M}, \rho\right)\right)$ for $\forall \boldsymbol{M} < \boldsymbol{M}'$ and $\forall \rho > \rho'$, then $\boldsymbol{F}\left(x, \boldsymbol{M}, \rho\right)$ is called an exact penalty function.

Let (MP(s)) be a perturbed problem of (MP) given by

$$\begin{array}{lll}(\mathrm{MP}(s)) & \min & f(x) = \left(f_1(x), f_2(x), \cdots, f_q(x)\right) \\ & \text{s.t.} & g_i(x) \le s_i,\ i = 1, 2, \cdots, m,\end{array} \tag{3}$$

where $s = \left(s_1, s_2, \cdots, s_m\right)$. Similar to that for a constrained penalty function in [12], we define stability.

Definition 1. Let x be any feasible solution to (MP) and x_s any feasible solution to (MP(s)) for each $s \in R^m$. If there is an $\boldsymbol{M}'$ such that for $\forall j \in J$

$$\frac{Q\left(f_j(x) - M_j\right) - Q\left(f_j\left(x_s\right) - M_j\right)}{\rho} \le |s|_p,\quad \forall \boldsymbol{M} < \boldsymbol{M}' \text{ and } \forall \rho > \rho'$$

where $|s|_p = \sum_{i=1}^{m} P\left(s_i\right)$, then (MP) is stable.

We have an exact result of the penalty function.

Theorem 4. *Let* x^* *be an optimal solution to (MP). If (MP) is stable,* $\boldsymbol{F}\left(x, \boldsymbol{M}, \rho\right)$ *is an exact penalty function.*

Proof. Suppose that $\boldsymbol{F}(x,\boldsymbol{M},\rho)$ is not an exact penalty function. Let x_s^* a Pareto weakly-efficient solution to (MP(*s*)). According to the definition of stability, we obtain that there is an $\boldsymbol{M}_1'$ satisfying

$$\frac{Q\left(f_j(x)-M_j\right)-Q\left(f_j(x_s)-M_j\right)}{\rho}\le|s|_P,\quad \forall \boldsymbol{M}<\boldsymbol{M}_1' \text{ and } \forall\rho>\rho' \tag{4}$$

This implies that there is some $\boldsymbol{M}'<\boldsymbol{M}_1'$ such that $f_j(x^*)>M_j'$ for $\forall j\in J$. Then, there always exists some $\boldsymbol{M}<\boldsymbol{M}'$ such that x^* is not a Pareto weakly-efficient solution to (MP(*M*)), *i.e.* there is some x' such that

$$F_j(x',M_j,\rho)<F_j(x^*,M_j,\rho)=Q\left(f_j(x^*)-M_j\right),\ \forall j\in J.$$

Thus,

$$Q\left(f_j(x')-M_j\right)+\rho\sum_{i\in I}P\left(g_i(x')\right)<Q\left(f_j(x^*)-M_j\right),\ \forall j\in J.$$

Suppose that x' is a feasible solution to (MP). If $f_j(x^*)<M_j$ for $j\in J$, we have $f_j(x^*)<M_j'<f_j(x^*)$. Otherwise if $f_j(x^*)\ge M_j$ for $j\in J$, from $Q\left(f_j(x')-M_j\right)<Q\left(f_j(x^*)-M_j\right)$, $f_j(x')<f_j(x^*)$, which shows that x^* is not a Pareto weakly-efficient solution to (MP). A contradiction occurs. Hence, x' is not a feasible solution to (MP) and $\sum_{i\in I}P\left(f_i(x')\right)>0$.

Let $s'=(s_1',s_2',\cdots,s_m')^{\mathrm{T}}$ with $s_i'=g_i(x')$, $i=1,2,\cdots,m$, and x_s^* be a Pareto weakly-efficient solution to (P(*s*′)). Then, there is some j such that $f_j(x_s^*)\le f_j(x')$ and $f_j(x_s^*)-M_j\le f_j(x')-M_j$. Thus,

$$Q\left(f_j(x_s^*)-M_j\right)\le Q\left(f_j(x')-M_j\right).$$

Therefore,

$$\begin{aligned}Q\left(f_j(x_s^*)-M_j\right)+\rho\sum_{i\in I}P(s_i')&\le Q\left(f_j(x')-M_j\right)+\rho\sum_{i\in I}P(s_i')\\&=F_j(x',M_j,\rho)<Q\left(f_j(x^*)-M_j\right),\end{aligned}$$

which shows that

$$Q\left(f_j(x^*)-M_j\right)-Q\left(f_j(x_s^*)-M_j\right)>\rho|s'|_P,$$

where $|s'|_P=\sum_{i\in I}P(s_i')$. This inequality contradicts to (4). Hence, (MP) is stable which yields a contradiction with the assumption and proves that $\boldsymbol{F}(x,\boldsymbol{M},\rho)$ is an exact penalty function.

3. Numerical Examples

In the MPFA algorithm, it is not easy to solve multiobjective problem $\min_{x\in R^n}\boldsymbol{F}\left(x,\boldsymbol{M}^k,\rho_k\right)$. Let

$$\bar{\boldsymbol{F}}(x,\boldsymbol{M},\rho)=F_1(x,M_1,\rho)+F_2(x,M_2,\rho)+\cdots+F_q(x,M_q,\rho).$$

It is easily known that an optimal solution to the problem $\min_{x\in R^n}\bar{\boldsymbol{F}}\left(x,\boldsymbol{M}^k,\rho_k\right)$ is a Pareto weakly-efficient solutions to the problem $\min_{x\in R^n}\boldsymbol{F}\left(x,\boldsymbol{M}^k,\rho_k\right)$. Hence, we replace the problem $\min_{x\in R^n}\boldsymbol{F}\left(x,\boldsymbol{M}^k,\rho_k\right)$ in the Step 2 of the MPFA algorithm with the problem $\min_{x\in R^n}\bar{\boldsymbol{F}}\left(x,\boldsymbol{M}^k,\rho_k\right)$. Let $Q'(t)>0$ for $t>0$. When $M_j<f_j(x)$, we have

$$\frac{\partial \bar{\boldsymbol{F}}(x,\boldsymbol{M},\rho)}{\partial M_j}=\frac{\partial F_j(x,M_j,\rho)}{\partial M_j}=-Q'\left(f_j(x)-M_j\right)<0.$$

Hence, when M_j decreases, the *j*-th objective $F_j(x,M_j,\rho)$ will decrease too. For fixed $(x,M_i,\rho)(\text{each } i\in J)(i\neq j)$,

$$\lim_{M_j\to-\infty}\frac{F_i(x,M_i,\rho)}{F_j(x,M_j,\rho)}=0.$$

So, we may obtain different Pareto weakly-efficient solutions at given different $(M_1,M_2,\cdots,M_q)$. By controlling M_j, we can control the *j*-th objective value $F_j(x,M_j,\rho)$.

We have applied the MPFA algorithm to several examples programmed by Matlab 6.5. The aim of numerical examples is to check the convergence of the algorithm and to control changes in objectives.

Example 1. Consider the following problem:

$$\begin{aligned}(P1)\qquad &\min\quad f(x_1,x_2)=\left\{-2x_1^4-x_2^4,x_1^4+4x_2^4\right\}\\ &\text{s.t.}\quad 2x_1+3x_2\le 6,-x_1\le 0,-x_2\le 0.\end{aligned}$$

Let penalty function

$$\begin{aligned}\bar{\boldsymbol{F}}(x,\boldsymbol{M},\rho)=&\max\left\{-2x_1^4-x_2^4-M_1,0\right\}^2+\max\left\{x_1^4+4x_2^4-M_2,0\right\}^2\\ &+\rho\max\left\{2x_1+3x_2-6,0\right\}^2+\rho\max\left\{-x_1,0\right\}^2+\rho\max\left\{-x_2,0\right\}^2.\end{aligned}$$

Let the starting point $(x_1^0,x_1^0)=(0,0)$, $\rho=1000$, $N=100$ and constraint error

$$e(x)=\max\{2x_1+3x_2-6,0\}+\max\{-x_1,0\}+\max\{-x_2,0\}.$$

Clearly, if $e(x)=0$, x is a feasible solution. We take different parameters (M_1^*,M_2^*) in the MPFA algorithm, the results are shown in **Table 1**.

In **Table 1**, when M_1 or M_2 decreases, the first objective value $f_1(x_1,x_2)$ or $f_2(x_1,x_2)$ decrease too. Objective parameter can control change of each objective function. It helps decision makers learn about the change of each objective function and choose a satisfactory solution as quickly as possible.

Example 2. Consider the problem:

$$\begin{aligned}(P2)\qquad &\min\quad f(x_1,x_2)=\{x_1-2x_2,-2x_1+x_2,-x_1-x_2\}\\ &\text{s.t.}\quad x_2\le 2x_1^4-8x_1^3+8x_1^2+2\\ &\qquad\ \ x_2\le 4x_1^4-32x_1^3+88x_1^2-96x_1+36\\ &\qquad\ \ 0\le x_1\le 3\\ &\qquad\ \ 0\le x_2\le 4\end{aligned}$$

We want to find a solution that three objectives are as small as possible with the first and second objective value less than −2 and the third objective value less than −5.

Let penalty function

$$\begin{aligned}\bar{\boldsymbol{F}}(x,\boldsymbol{M},\rho)=&\max\{x_1-2x_2-M_1,0\}^2+\max\{-2x_1+x_2-M_2,0\}^2+\max\{-x_1-x_2-M_3,0\}^2\\ &+\rho\max\left\{x_2-2x_1^4+8x_1^3-8x_1^2-2,0\right\}^2+\rho\max\{x_1-3,0\}^2\\ &+\rho\max\{x_2-4,0\}^2+\rho\max\{-x_1,0\}^2+\rho\max\{-x_2,0\}^2.\end{aligned}$$

Let the starting point $(x_1^0,x_1^0)=(0,0)$, $\rho=1000$, $N=100$ and constraint error

Table 1. Numerical results with different objective parameters.

k	(M_1^*, M_2^*)	$e(x^k)$	(x_1^k, x_2^k)	$(f_1(x_1^k, x_2^k), f_2(x_1^k, x_2^k))$
5	(−4000.000000, −40.000000)	0.000000	(2.955488, 0.027052)	(−152.597224, 76.298614)
3	(−40.000000, −4000.000000)	0.000000	(0.004439, 0.004726)	(−0.000000, 0.000000)
2	(−400.000000, −400.000000)	0.000000	(2.514867, 0.000009)	(−80.000006, 40.000003)

Table 2. Numerical results with different objective parameters.

k	(M_1^*, M_2^*, M_3^*)	$e(x^k)$	(x_1^k, x_2^k)	$(f_1(x_1^k, x_2^k), f_2(x_1^k, x_2^k), f_3(x_1^k, x_2^k))$
5	(−10.000000, −10.000000, −10.000000)	0.000000	(2.329518, 3.178479)	(−4.027439, −1.480558, −5.507997)
5	(−10.000000, −20.000000, −10.000000)	0.000000	(2.534721, 2.039602)	(−1.544482, −3.029841, −4.574323)
5	(−11.000000, −20.000000, −10.000000)	0.000000	(2.489790, 2.311039)	(−2.132288, −2.668541, −4.800829)
5	(−12.000000, −20.000000, −10.000000)	0.000000	(2.444095, 2.577823)	(−2.711552, −2.310366, −5.021918)

$e(x) = \max\{x_2 - 2x_1^4 + 8x_1^3 - 8x_1^2 - 2, 0\} + \max\{x_1 - 3, 0\} + \max\{x_2 - 4, 0\} + \max\{-x_1, 0\} + \max\{-x_2, 0\}$.

We take different parameters (M_1^*, M_2^*, M_3^*) in the MPFA algorithm and get the results shown in **Table 2**.

In **Table 2**, we find a satisfactory solution $(x_1, x_2) = (2.444095, 2.577823)$ when taking different (M_1^*, M_2^*, M_3^*).

4. Conclusion

In this paper, we define a penalty function with objective parameters and constraint penalty parameter for MP and the corresponding unconstraint penalty optimization problem. Under some conditions, we prove that a Pareto efficient solution (or a weakly-efficient solution) to UPOP is a Pareto efficient solution (or a weakly-efficient solution) to MP, and the penalty function is exact under a stable condition. We present the MPFA algorithm to solve the multi-objective programming with inequality constraints by using the nonlinear penalty function with objective parameters. With this algorithm, we may find a satisfactory solution.

Acknowledgments

We thank the Editor and the referee for their comments. The research is supported by the National Natural Science Foundation of China under grunt 11271329 and 10971193.

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American Journal of Operations Research, 2014, 4, 340-350
Published Online November 2014 in SciRes. http://www.scirp.org/journal/ajor
http://dx.doi.org/10.4236/ajor.2014.46033

Mathematical Modeling in Heavy Traffic Queuing Systems

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Received 6 September 2014; revised 2 October 2014; accepted 18 October 2014

Academic Editor: P. G. Khot, RTM Nagpur University, India

Abstract

In this article, modeling in queuing systems with heavy traffic customer flows is reviewed. Key areas include their limiting distributions, asymptotic behaviors, modeling issues and applications. Heavy traffic flows are features of queuing in modern communications, transportation and computer systems today. Initially, we reviewed the onset of asymptotic modeling for heavy traffic single server queuing systems and then proceeded to multi server models supporting diffusion approximations developed recently. Our survey shows that queues with heavy traffic customer flows have limiting distributions and extreme value maximum. In addition, the diffusion approximation can conveniently model performance characters such as the queue length or the waiting time distributions in these systems.

Keywords

Queuing Process, Brownian Process, Martingale, Regularly Varying Functions

1. Introduction

There are times when queueing systems behave like fluid. A good scenario is when customers of a busy bus station experience rush hour. Therein, the scenery looks highly saturated and stable or completely unstable. Either way, the system dynamics resembles a continuous fluid flow rather than discrete. Medhi's analogy in [1] of fluid flow of people coming out of a subway or an electric train during rush hour is similar to the example above. The wide sense approximate continuity in such traffic flows is created by the heaviness of queueing traffic into the system. Broadly speaking, a heavy traffic queueing system can be defined as a queueing system

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How to cite this paper: Sani, S. and Daman, O.A. (2014) Mathematical Modeling in Heavy Traffic Queuing Systems. *American Journal of Operations Research*, **4**, 340-350. http://dx.doi.org/10.4236/ajor.2014.46033

whose server occupation rate is barely less than unity and this phenomenon as Boxma *et al.* [2] indicates is a feature in modern communications and computer systems today. Researches have shown that early investigation in this area was carried out by Kingman [3] on a general queue called the G/G/1 and the result is referred to as central limit theorem for[1] queueing theory, see Medhi [1].

Our objective in this paper is to survey works on heavy traffic queueing systems generally in the light of both mathematical and statistical realities with emphasis on those queues supporting the diffusion approximation. This includes their distributions, analyses, modeling and application. It is anticipated that a survey of this kind will provide an excellent background in heavy traffic studies especially in packets and internet traffic prevalent in computers, communications and telecommunications systems. To achieve an optimum survey process as in this case, it is essential that one bears in mind the Poisson traffic controversy now prevalent in telecommunications and computer traffic modeling not because the controversy is relevant or not, but because there are diverse opinions worthy of sharing especially as regards the new traffic models such as the self-similar model. Not only that, recent studies have shown that the Poisson based models are equally relevant and could be used to describe the internet traffic central in this controversy. For instance, Lee and Kim [4] of late have shown that at small time scales, inter-arrival times of computer protocols such as the Simple Mail Transfer Protocol (SMTP) sessions traffic for the internet are exponentially distributed and independent of each other which makes it possible to model this kind of traffic session arrivals as a Poisson process. Before then, Karagiannis *et al.* [5] have indicated that the observed long-range dependence in the internet traffic does not make the Poisson based models obsolete. That at a sub-second time scales, backbone traffic appears to be well described by Poisson packet arrivals and went further to provide evidence that the ongoing pattern of the internet evolution may potentially affect the future characteristics of its traffic. Similarly, Kos and Bester [6] wrote that for a traffic model to be suitable, it should be able to represent traffic with few parameters devoid of complications and intricacies. This is the point the Poisson based models exceed all other models.

Thus, in agreement with this line of research opined that in addition to self similar models, the Poisson traffic models and their assumptions will conveniently capture the internet traffic at a given time scaling and are equally relevant. The survey is organized as follows: in section 2, we survey the onset of mathematical modeling in stable heavy traffic queues. Here, stability is in the context of Whitt [7] to mean highly saturated stable systems. In section 3, an overview of the diffusion approximation is provided with focus on unstable queues, their relevant distributions and asymptotic behaviors. We also deal with characteristics such as self similarity, long-range behavior etc. that define burstiness in computer and internet traffic giving rise to recent traffic models. Sections 4 and 5 deal with issues on modeling heavy traffic queueing systems as data traffic. Here, we studied the Poisson traffic controversy and provide a summary that best fits all lines of research relative to modern day traffic modeling. In section 6, we study applications of heavy traffic systems and contributions of data traffic science to computer and telecommunication systems development and advancement. The review is concluded in chapter 7 and 8 with open problems and summaries.

2. Heavy Traffic Approximation

Heavy traffic approximation started with the work of Kingman [3] on a one-server model with a general arrival and service time distributions called the G/G/1. Kingman [3] proved that for the G/G/1 queue under heavy traffic, the waiting time distribution could be approximated by an exponential distribution. The result is called the central limit theorem for heavy traffic queuing systems given below:

Theorem 2.1

Suppose the traffic intensity $\rho \le 1$. Let $W(t)$ denotes the steady state waiting time distribution in a G/G/1 queue. Then $W(t)$ could be approximated by the exponential distribution

$$W(t) = 1 - \exp\left(\frac{-2(1-\rho)}{\lambda\left(\sigma_u^2 + \sigma_v^2\right)}\right)t \tag{1}$$

where λ is the arrival rate, σ_u^2 and σ_v^2 are the variances of the inter-arrival and service time distributions of customers in the system.

Medhi [1] noted that in 1964, Kingman made a conjecture for the seemingly more significant G/G/c queue

[1]This expository paper is a component of a Doctoral research Literature Review process in Queuing Systems with Heterogeneous Servers.

under heavy traffic arising from developments and insights on the G/M/c model. He conjectured that the waiting time distribution in a G/G/c queue equally could be approximated similar to the G/G/1 model. Ten years later in 1974, Kollerstrom [8] proved the conjecture[2] affirming that the heavy traffic waiting time distribution is of the form

$$W(t)=1-\exp\left(\frac{-2\left(\frac{1}{\lambda}\right)(1-\rho)}{\sigma_u^2+\frac{\sigma_v^2}{c^2}}\right)t \tag{2}$$

Here, ρ is the occupation rate and c is the number of servers.

The two equations above represent remarkable achievements in heavy traffic analysis of modern systems with light tail or short range dependencies. Moreover, they signify outstanding developments and breakthroughs in heavy traffic approximation via classical analysis of Laplace transforms of relevant distributions as far as operations research is concerned. Apart from operational significance, the works leading to the referenced equations boasted similar works notably on the subject of convergence and behaviors of similar systems under different conditions and assumptions. Convergence here refers to convergence in distribution of sequence of stochastic queuing processes such as the arrival or service process, the waiting time or queue length process, etc, see Whitt [7]. On convergence over the years in this area, a lot has been written on queues under heavy traffic. The bulk of these works applies the diffusion approximation technique which will be discussed later in this survey on heavy traffic queuing systems. For saturated stable systems for instance, Whitt [7] noted that Borovkov [9] investigated the asymptotic behavior of a single phase case with Poisson arrival and service time distributions working independently of the service process in heavy traffic. Weak convergence limits and asymptotics for heavy traffic queues are well presented also in Borovkov [9]. Similarly, Abate and Whitt [10] studied a multi-channel queuing system and approximated the asymptotic decay rates of the queue length and the customer service distribution in form of tail probabilities under heavy traffic. The result shows that both the queue length and the service time distributions depend on the first 3 moments of their distributions. The M/G/1 queue with priority classes is an essential model giving its numerous applications. Priorities normally aroused sequel to the realistic nature of services in systems. Abate and Whitt [11] derived limit theorems for the case when the priorities are preemptive and non-preemptive with resumption tendency. They proved that in the low-priority case, the limiting waiting-time distribution is a geometric random sum of independent and identically distributed random variables similar to the M/G/1 first come first served (fcfs) waiting-time distribution. On the asymptotic behavior of tail probabilities for this model, Abate and Whitt [11] added that there is routinely a region such that the tail probabilities have non-exponential asymptotics even if the service time distribution is exponential. In addition, the asymptotics formed tend to be determined by the non-exponential asymptotics for the high-priority busy-period distribution.

Essentially, heavy traffic approximations in queues under the classical procedures are difficult especially for multi-channel queues that functions as integrated systems. These queues formed the bulk found in present day computers, communications and telecommunication systems. As Whitt [7] posited, it is not easy to work with triple or quadruple transforms and this makes it hard to obtain knitted results. Consequently, we observed two implications out of this limitation. First, attention was like shifted to obtaining other forms of approximations among others; the diffusion approximation which describes a queuing process as a Brownian motion and appears suitable for describing heavy traffic systems, see Medhi [1]. Secondly, asymptotics of similar models for instance the M/G/c and the M/M/c that seemed more realistic were attempted and derived. On the latter for instance, Boxma *et al.* [2] derived the asymptotics for the heterogeneous server M/G/2 with an exponential and a general server of regular variation and a cumulative service time distribution denoted by B(t). By regular variation, we mean a distribution whose compliment can be approximated by a slowly varying process at infinity index. The M/G/2 model of Boxma *et al.* [2] is simply the trivial prototype of a discrete channel system with two distinct service processes arising from two servers such that the regularly varying component keeps changing, making its complimentary distribution fatter as time grows large. Boxma *et al.* [2] have shown that such a model under light traffic is asymptotically similar to the Kingman's solution distribution (waiting time distribution is

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semi-exponential distribution). However, under heavy traffic the regularly varying nature of the service time distribution of the general server will have a long tail effect on the complementary waiting time distribution of customers in the system. The Boxma *et al.* [2] asymptotic result for the model in question under heavy traffic is summarized in the theorem below:

Theorem 2.2

Suppose $\lambda > \mu$ and $B(t)$ is the service distribution of customers served by the regularly varying general server at index $-\nu$ with mean β. If $L(t)$ is a slowly varying function on some neighborhood such that

$1-B(t) \sim t^{-\nu}L(t), \quad t \to \infty, \nu \in (m, m+1), \forall m \in \mathbb{N}$

Then the complementary waiting time distribution denoted by $1 - W(t)$ is given by;

$$1-W(t) \sim \frac{1-Q_0-Q_1}{(\nu-1)\beta(1-\lambda\beta+\mu\beta)}\left(\frac{(\lambda-\mu)}{\lambda}\right)^{\nu-1} t^{1-\nu}L(t), \; t \to \infty \tag{3}$$

where Q_i is the probability that there are exactly i-customers in the system at a steady time η. From the above result, a sufficient condition for defining heavy tail phenomena in the M/G/c model generally under heavy traffic is evident. It is summarized in the lemma below

Lemma 2.1

A sufficient condition for heavy tail phenomena in a heavy traffic M/G/2 model is that, either/both the arrival or service process of customers in the system is/are significantly regularly varying at a known index as time grows large.

Remark 2.1

It is trivial since regularly varying distributions are subclass of heavy tail or more precisely, sub-exponential distributions. The light proof (by simple argument) of the lemma follows.

Proof

Given that server-1 is exponential and server 2 is general with a regularly varying distribution $B(t)$. Then $1-B(t) \sim t^{-\nu}L(t), t \to \infty,$ where $L(t)$ is a slowly varying function such that

$$\lim_{t\to\infty}\frac{L(tx)}{L(t)} = 1, \quad x \to \infty. \tag{4}$$

If $L(t)$ is undominatedly non-decreasing, we have $\limsup_{t\to\infty} L(t) = \infty.$ So that the open set of service times on server 2 is trivially sub exponential. Similarly if $L(t)$ is dominatedly non-decreasing and $\nu \in (-\infty, 0]$, then $1 - B(t)$ is open in $\mathbb{R}$. Though, finite for a given supreme point, may be long-tailed if the supreme value is closely to infinity. This suffices. The other case follows with similar argument.

Significance here is statistical and implies relativeness with the other component. On the consequence of the former limitation, Medhi [1] indicated that researchers were motivated to seek other approximation techniques. This quest gave birth to the diffusion approximation prevalent in queuing system modeling today. Essentially, the motivation leading to the above results and many others came from the works of Kingman and others on the asymptotic behaviors of queuing systems under heavy traffic.

3. The Diffusion Approximation

The diffusion approximation in heavy traffic queuing systems came to light in the works of Iglehart [12], Gaver [13] and Newell [14]. It consists basically two conceptually different kinds of approximations; the diffusion limits for queues justified by heavy traffic limit theorems for unstable queues and the diffusion models as continuous approximations for stable queues, see Kimura [15]. The technique involves approximating the limit of a sequence of stochastic queuing variables as a Brownian motion (diffusion). By a Brownian motion we mean a continuous time stochastic process satisfying the Markov and Gaussian properties respectively. Iglehart [12] established the first limit theorem for the palm model via this technique and Gaver [13] considered it on certain congestion problems in 1968; see Guadong *et al.* [16] and Medhi [1]. Similarly, Newell [14] applied the diffusion approximation on queue length distributions of certain queuing systems under heavy traffic. The idea covering the diffusion approximation is approximating the already randomized and discrete-natured queuing arrivals and departures as continuously non randomized processes analogous to fluid flow in and out of a reservoir.

Then the asymptotic behavior of a queue will only involve deriving a functional law of large numbers or a functional central limit theorem.

3.1. Suitability and Robustness

Early works involved providing justifications on the suitability of the diffusion approximation in the light of analytical and numerical considerations of various queuing processes. For instance in 1970, Iglehart and Whitt [17] justified the suitability of the diffusion approximation by establishing a limit theorem for the G/G/c queue. It was proved that both the queue length and the waiting time distributions could be approximated by a Brownian motion. In 1974, Reiser and Kobayashi [18] studied the accuracy of the diffusion approximation on some networks of queuing systems. Network analysis appeared challenging given its complexities especially through classical approaches. The accuracy was considered for wide classes of distributional form of inter arrival and service times for various models. Reiser and Kobayashi [18] concluded similar to Iglehart and Whitt [17] that the diffusion approximation is quite adequate in most cases, more adequate than the exponential server model prevalent in computer system modeling.

The subject of control especially in multi-server queuing systems is a tool for describing suitability on a model resulting from smoothness in behavior either in transient or limiting case. Rami *et al.* [19] studied a controlled multi server queuing system restricted in the Halfin-Whitt regime, a heavy traffic regime where quality and efficiency are assured. They derived a diffusion model with a singular control term that describes the scaling limit of the queuing model. The singular term constrains the diffusion to adapt to certain subsets for any time t in the neighborhood of $(0,\infty)$. In addition to providing null-controllability conditions for this model, they have shown that an analogous asymptotic result holds for such multi-server systems via the diffusion approximation. Similarly, Lee and Werasinghe [20] analyzed a sequence of single-server queuing systems with impatient customers under heavy traffic. Customer impatience in form of abandonments or reneging is a significant feature of queuing systems arising from long queues upon entry. They proved that the drift coefficient of the limiting diffusion (the mean) is influenced non-linearly by the sequence of patience-time distribution. In addition, both the queue length and the waiting time processes have stable limiting distributions. The relationship between the drift coefficient and the diffusion parameter of a diffusion process is approximately linear. Wagenmakers *et al.* [21] studies on the mean and the variance of a diffusion model for sojourn times has shown that within the range of plausible values, the relationship between the two is linear.

In the 1980's, with the advent of the facsimile (fax) machine and the internet in the years preceding the facsimile, the nature of system traffic changed tremendously. Data traffic of the internet age replaces the voice traffic of the telephone age. This shift creates most research interest in heavy traffic queuing studies via the diffusion approximation and since then, several models have been developed to approximate performance of systems such as tail probabilities, moments and distributions.

3.2. Limiting Distributions

As indicated earlier, recent research focus in heavy traffic queuing systems is in the study of tail behaviors of queues in form of limiting and extreme value distributions for various models. For instance, Glynn and Whitt [22] proved the extreme value limit theorem for heavy traffic queues with general arrival and service time distributions called the G1/G/1. Using strong approximations under regularity conditions, the extreme waiting time among n-sized customers was derived. The number of customers in the system is assumed increasing as time grows large. It was shown that, when the traffic intensity ρ approaches 1 from the left and n approaches infinity at a suitable rate, the normalized maximum wait among n-customers converges to the Gumbel extreme-value distribution. Also, Glynn and Whitt [22] added that the normalization depends only on the means and the variances of the inter arrival and service time distributions. On the contrary, if ρ is a fixed point, then the maximum waiting time fails to converge to the Gumbel distribution. The General Gumbel probability distribution for a continuous random variable $W(t)$ is given by

$$P(W \le w) = \exp(-\exp(w)), \quad -\infty < w < \infty. \tag{5}$$

In addition, the Gumbel extreme value distribution holds for the queue length distribution. The lemma below summarizes the ρ- region of convenience for the Gumbel distribution in the G1/G/1 model.

Lemma 3.1

In a heavy traffic G1/G/1 queue under regularity conditions, if ρ *approaches one from the left origin and n steadily increases to* ∞ *then, the Gumbel distribution sufficiently model the extreme value limit of the waiting time or the queue length distribution of customers in the system.*

Similarly, Szczotka and Woyczynski [23] studied the G/G/1 queues with service and/or the inter arrival times of heavy tailed probability distributions. Szczotka and Woyczynski [23] obtained that the waiting time distribution is exponential if the tail of the distribution of inter arrival times is heavier than that of the service times and is non-exponential in the opposite case. In other words, if the service times have a heavy-tailed distribution heavier than that of the inter arrival times in the domain of attraction of a Levy process then, the limiting distribution is a Mittag-Leffler distribution. In addition, Szczotka and Woyczynski [23] emphasized that under these modeling conditions, the queue length distribution can be analyzed. Limic [24] studied the heavy traffic behavior of a G/G/1 Last-in-First-Out (LIFO) preemptive resume queue and derived a diffusion approximation for the model. Limic [24] showed that the queue length process exhibits perhaps an unexpected heavy traffic behavior. In addition, the diffusion limit depends on the type of arrivals and services in a fairly intricate way which is related to the Wiener-Hopf factorization for random walks. Earlier in 1999, in their heavy traffic analysis of the G1/G/1 queue with heavy tailed service or arrival distribution of the regularly varying type, Boxma and Cohen [25] have shown that if the traffic intensity of the G/G/1 system approaches unity and the tail of the service distribution is heavier than that of the arrival distribution, the stationary actual waiting time distribution together with a contraption factor is a function of the traffic load and converges to the Kovalenko distribution. In contrast, if the reversed is the case and all other factors kept constant, the stationary actual waiting time distribution will still depend on the traffic load but converge to the negative exponential distribution. Hence, limiting distributions of queuing systems to a large extend depend on the model constructed. Moreover, a slight variation of significant parameters may shift distribution of systems.

Whitt [26] has provided a summary of functional limit theorems for both noisy and non-noisy single server queues. By a noisy queue we mean a queuing system with a measurable diffusion component. Using the open mapping theorem, Whitt [26] indicated that similarly to the convergence of stochastic functions to reflected Brownian motion as captivated by Donsker theorem, a discrete-time queuing model with cumulative net-input process of stationary increments and jumps of infinitesimal variance or mean will converge to a reflected stable process such as the Gaussian or the Lévy process. More explicitly, for a sequence of queuing models (multi-server systems), the limit is strictly a reflected Lévy process. Finally, Whitt [26] established that the functional central limit theorem for the customers in the queue when the input process is a superposition of many independent processes with complex dependence is a Gaussian process. Kruk *et al.* [27] presented a heavy-traffic analysis of a single-server queue under a scheduling policy called Earliest-Deadline-First (EDF). In this queuing discipline, customers have deadlines and are served until their deadlines elapse. The system performance is measured by the fraction of reneged work shown to be minimized by the service policy. The evolution of the lead time distribution of customers in the queue is described by a measure-valued process. It was shown that, in the heavy traffic regime, the limit of this (properly scaled) process is a deterministic function of the limit of the scaled workload process. In addition, the limit is a doubly reflected Brownian motion. The polling system queue[3] where a single server revisits the queuing system in a cyclic order has also been studied and limiting distributions analyzed especially relative to unfinished jobs. For instance, the work of Coffman *et al.* [28] on polling queues under the exhaustive-service discipline is worthy of mentioning. Coffman *et al.* [28] showed that under the standard heavy traffic scaling, the total unfinished work in the system tends to Bessel-type diffusion in the heavy-traffic limit. What all these results signify in essence is that, limiting distributions of unstable (random) queuing systems are themselves stable.

4. Modeling in Network Queues

Heavy traffic analysis of network queues especially, multi class networks are gaining grounds recently. This is not unconnected with its numerous importance. Kimura [15] pointed out that multi-dimensional extension of server stations analysis is a natural diffusion model for a network of queues in computer systems. Similarly, Bertsekas and Gallagher [29] wrote that multi class network queues are used in analyzing problems of congestions and delays in computer systems, communication and complex productive systems. Their importance cannot be over emphasized. Unfortunately, several studies for instance have shown that not all multi class networks

[3]Another interesting queue to model under distinct structuring.

especially those with feed backs under heavy traffic can be approximated using the reflected Brownian motion see Williams [30]. However, the open multi class type under heavy traffic supports the diffusion approximation. Already, Reiser and Kobayashi [18] in the 70's have proved that network measurements via the diffusion approximation are quite adequate. Williams [30] studied a multi class open queuing network using the semi martingale Brownian motion process and provides sufficient conditions for which heavy traffic limit theorem holds for such queues, see Williams [30] for details. Similarly, Dai and Dai [31] studied an open queuing network with finite buffers consisting of d-finite server stations. Given that a server stops working when the downstream buffer is full and all customers served at a station are homogeneous in terms of service requirements and routing. They proved that the normalized d-dimensional queue length process converges in distribution to a semi martingale reflecting Brownian motion in a d-dimensional box under a heavy traffic condition. Pekoz and Joglekar [32] considered a ./G/k finite buffer queue with a stationary ergodic arrival process and a general service with delayed feedbacks and obtained that under certain mild conditions, the feedback flow of the class of customers re-entering the queue converges to the Poisson distribution when the delay waiting time distributions is scaled up. Similar studies on the network queues followed these developments especially via the martingale representations for limiting distributions of many server systems. Already, researches have shown that the reflected Ornstein-Uhlenbeck, the geometric Brownian motion, the reflected Levy and the reflected affine diffusion processes could be used to model successfully queuing systems with noisy processes such as reneging, balking and shunting process which are in effect measurable noises[4], see Ward and Glynn [33] and [34]. Guodong *et al.* [16] worked on the Palm model and a finite capacity $M/M/N/M_n + M$ model with reneging via the martingale diffusion approach and provided limit proofs for the heavy traffic approximations of the queue length distribution. The martingale approach applied on the queue length process of customers involves random time changes and random thinings of the stochastic queue length process. They established a key central limit theorem and a key functional weak law of large numbers for the Palm model and the finite capacity M/M/C model respectively. The result shows that the limiting queue length distribution in both models is a reflected Ornstein-Uhlenbeck diffusion process, an adapted stochastic process in which the stochastic variable changes more with time in addition to a finite variation component of the process. It is worthy to note that the shape of the reflected OU-graph to a greater extend depends on the initial customer size at time zero.

5. Modeling Heavy Data Traffic

Modeling in heavy data traffic queuing systems is challenged by two important factors; the nature of data traffic itself and model selection. The former creates a controversy that saddles on the later. Researches on internet and telecommunications traffic processes revealed that data traffic is characterized by properties such as regular variation, long-range dependencies, self-similarity and heavy tail distributions see Leland *et al.* [35], Park *et al.* [36] and Stralka *et al.* [37]. Long-range dependency and self-similarity in essence are associated with heavy tail distributions. The combination of the two defines how burst a traffic system is. A heavy data traffic process may be bursty or not depending on the time scales it is considered. Though, Medhi [1] wrote that a self-similar process arising from long-range dependent process like the internet traffic decays much slower than the exponential distribution in addition to a hyperbolic decay of autocorrelations, this does not render the Poisson traffic models that is non-bursty inadequate or better still out of the internet domain as some researches tried to portray.

The Poisson Traffic Arguments: The Bouncing Back

Since the pioneering works of Leland *et al.* [35], Willinger and Paxson [38] and many others on the new network traffic description that appeared alternatives to the Poisson models, it appears the Poisson distribution has lost its place as a suitable distribution for describing the internet traffic today. Amidst its advantage, history and effectiveness, a lot has been published about its inadequacy without attending to scaling issues in both time and space. For instance in 1994, Leland *et al.* [35] using long, high resolution traces of Ethernet packets indicates that arrival rates of the Internet Protocol (IP) packets on a Local Area Network (LAN) exhibit not Poisson but self-similar behavior. However, similar studies have shown that within a given time scale, the two traffic networks coexist together. For instance, Boxma and Cohen in [25] observing the plots of Ethernet traffic measurements on LAN of Willinger *et al.* [39], WAN of Paxson and Floyd [40] and VBR (Variable-Bit Rate) of Beran

[4]The two references above are good examples of works on modeling via the O-U and similar processes in queuing systems with noisy structures.

et al. [43], Boxma and Cohen [25] watched that bursty sub periods are alternated by less bursty sub periods in each of these traffic processes indicating the coexistence of the Poisson traffic and self-similar traffic processes. Similarly, Karagianis *et al.* [5] have shown that there exist a Poisson process and a long range dependence in heavy Internet backbone traffic. Furthermore, Karagianis *et al.* [5] indicated that the User Datagram Protocol/ Transmission Control Protocol (TCP/UDP) packets obey a Poisson process at sub-second time scales while they are long range dependent at large time scales. This suggests that relatively simple statistical theories of the Poisson process can still be applicable to the design and optimization of the internet. Similarly, Lee and Kim [4] proved that at small time scales, inter-arrival times of protocols such as the simple mail transfer protocol (smtp) is a Poisson process. Finally, Karagiannis *et al.* [5] have shown that the mighty internet traffic in the center of this controversy at sub-second time scales appears well described by Poisson packet arrivals and evident that, the ongoing pattern of Internet evolution may eventually renewed its Poisson tendencies even for the super second time scale. Time scaling factor appears to be a decisive factor in defining suitability in this modeling case. Consequently, the below lemma follows:

Lemma 5.1

The necessary and sufficient condition for the internet traffic or a similar process to fit in the Poisson process is that, the time scaling is sub-second.

Consequently, modeling in the heavy traffic sense may well be done with the Poisson model at sub-second time interval and at large time scales, modern day data traffic models such as the self-similar model that can capture important features such as traffic burstiness with long tail distributions are effective.

6. Applications

The role heavy traffic analysis plays in the development and advancement of service systems especially computing and telecommunications systems cannot be over emphasized. Initially, even the work of Kingman on the asymptotic waiting time distribution in a G/G/1 queue is a modeling of delay time distribution in a general service system just about to reach its service capacity. In heavy traffic analysis, any limit theorem derived is to provide understanding and approximation of distributions and tail behavior of measures for bettering performance of service systems. The general motive is advancement of corresponding service systems and application depends on the reality of model. For instance, heavy traffic analysis of networks queues of various priorities is for problem analysis of modern systems. As Bertsekas and Gallagher [29] indicated; such models are used in analyzing problems of congestions and delays in computer systems, communications and complex productive systems. Priority queuing analyses and those with service interruptions may be classical models of computer systems. In the context of queuing synthesis, as Kimura [15] indicated; diffusion models are for reliability and control problems in computer and telecommunication systems. Without such analysis, problems such as those mentioned could have down our systems. The processor sharing discipline queues are used in modeling time-sharing protocols in computer systems. High speed wireless networks carrying multimedia applications under long range dependence and heavy tailed properties are troubled by excess usage. Buche *et al.* [41] indicated that heavy traffic analysis of long range dependence in wireless internet traffic provides relief to troubles such as large file sizes downloads from the internet and multimedia applications for instance streaming a video. The processor sharing models such as those studied in Kleinrock [42] and Ritchie and Thompson [44] in the 70's and more recently, the limited processor model of Zhang and Zwart [45] has been widely used in the analysis of computer systems, network servers and data transmission over the internet. Kruk *et al.* [27] indicated that in the last decade, substantial attention has been paid to queuing systems in which customers have deadlines for service in heavy traffic (EDF queues). These types of queuing models feature in telecommunication systems carrying digitized voice or video traffic, tracking systems and real-time control systems. In the case of voice or video traffic, the packet information must be received, processed and displayed within stringent timing bounds so that the integrity of the transmission is maintained. Similarly, there are processing requirements for tracking systems that guarantee that a track can be successfully followed. Real-time control systems for instance, those associated with modern avionics systems, manufacturing plants or automobiles also gather data that must be processed within stringent timing requirements in order for the system to maintain stability or react to changes in the operating environment. Another class of heavy traffic queuing models of varied applications is the polling models. Levy and Sidi [46] indicated that these models were first introduced in 1970 precisely when the cyclic queues where used in modeling time allocations in computer systems. In these models, queues are visited by a single server in a cyclic discipline. Such models are applied in the performance analysis of communication systems such as token rings and packet switches, where a single server resource is shared among many traffic stream demands on the resource, see Coffman *et al.* [28]. In addition, Levy and Sidi [46] provided more areas of applications of polling queues in heavy traffic to include random access protocols of computer systems, robotics and manufacturing systems. In other fields such as transportation, where heavy traffic analysis has several applications, the work of David *et al.* [47] on the diffusion modeling of an airport queue is an excellent use of modeling via diffusion approximation in the transportation sector. Other relevant areas of application of the heavy traffic modeling in approximations queuing systems include the repairman problem etc. Already, the works of Iglehart [12] in the 60's has derived a diffusion limit approximations for several server case.

7. Open Problems

In all, one can identify the following areas as open problems for further research in heavy traffic queuing modeling and analysis:

1) On waiting time analysis in multi-server queues of computer and telecommunication systems.

2) Design of new queuing models that captures wide range of service systems with less complicated mode of analysis.

3) Analyzing queuing systems with similar structures as the one discussed in this review with essentially higher degree of control of system processes.

4) Designing new queuing schedules to describe service processes and routings of new or existing models and analyzing the behaviors and performance of models

8. Conclusion

In this article, mathematical modeling in heavy traffic queuing systems is generally surveyed. Initially, the onset of modeling in heavy traffic queues and asymptotic behaviors for different models were reviewed and distributions uncovered. We also looked at the diffusion approximation as a remedy to queuing analysis and approximations. Modeling both in network and heavy traffic data systems and matters arising from the internet and telecommunication traffic modeling via Poisson models and assumptions were studied and a justification on the suitability of the Poisson arrival process in addition to the new network traffic in capturing the internet traffic was supported. Finally, we provide real areas of application of heavy traffic models developed for the benefits of service systems.

Acknowledgements

The authors are grateful to Prof. Sivasamy, R., of the Statistics Department, University of Botswana for his suggestions, corrections and editing this article. Also, to the anonymous reviewer(s) of this article to this stage and to all authors of the literature reviewed in this work.

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American Journal of Operations Research, 2014, 4, 351-364
Published Online November 2014 in SciRes. http://www.scirp.org/journal/ajor
http://dx.doi.org/10.4236/ajor.2014.46034

Repealing the LIFO Inventory Accounting Choice? A Review of LIFO and Inventory Management

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Received 23 August 2014; revised 20 September 2014; accepted 6 October 2014

Academic Editor: Haiyan Li, Chongqing University of Technology, China

Abstract

Researchers in the disciplines of both Operations and Accounting have studied Inventory Management, though in relative isolation. In this paper, one of our goals is to help inform researchers in Operations Management about an extensively debated question in inventory accounting: whether to repeal the LIFO (Last-In-First-Out) inventory accounting choice? This question has received extensive scrutiny from various stakeholders including academics, businesses, and different levels of governmental agencies such as US Congress. Specifically, we provide a literature review on how LIFO affects and is affected by inventory management. This is done by first reviewing the potential determinants of LIFO inventory accounting choice and then reviewing potential interactions between LIFO and inventory management. It is our hope that this review will help stakeholders have a more comprehensive understanding of LIFO before making their decisions.

Keywords

Inventory Management, Inventory Accounting, FIFO, LIFO

1. Introduction

There have been two major inventory accounting systems in the world: FIFO and LIFO, standing for First-In-First-Out and Last-In-First-Out, respectively. In Operations Management, FIFO usually means the first item entering a system (e.g., a queue) will physically exit the system first, and LIFO means the opposite. In Accounting,

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How to cite this paper: Zhang, Y.B., Shi, C.M., Gao, P. and Wang, F. (2014) Repealing the LIFO Inventory Accounting Choice? A Review of LIFO and Inventory Management. *American Journal of Operations Research*, **4**, 351-364.
http://dx.doi.org/10.4236/ajor.2014.46034

however, FIFO and LIFO could mean something quite different: FIFO means items exiting a system will be evaluated at the purchase price of the oldest items in the system, and LIFO means items exiting a system will be evaluated at the purchase price of the newest items in the system. In other words, FIFO and LIFO in Operations Management imply something physical and in Accounting imply something financial.

Today, while FIFO (first-in-first-out) is widely adopted in almost all countries (e.g., see China Accounting Standards and International Accounting Standards), LIFO (last-in-first-out) is still used in US Generally Accepted Accounting Principles (US GAAP). Therefore, LIFO repealing debate now becomes the central topic of the Congress in the United States and academic research ([1]-[4]). Strong industry lobby groups argue for or against this inventory accounting system based on their industry interests. For the businesses involved, retailing is the most proactive industry against LIFO while oil industry strongly advocates for the retaining of LIFO. So the questions need to be answered are: should LIFO be abandoned after its use for 70 years and which potential determinants could decide the attitudes of the industry? First, some basic concepts of inventory accounting systems are to be clarified. Normally, LIFO and FIFO are the frequently used inventory valuation accounting systems. During inventory cost inflation, under LIFO, the recent higher cost is matched with the revenue, so the taxable income is reduced, thus reducing the tax and increasing the cash flow. Under FIFO, the lower cost is matched with the revenue, so the financial performance of profit increases. LIFO reserve is defined in the literature as an inventory account describing the inventory valuation difference between LIFO and FIFO. The first part of this literature review summarizes the potential determinants behind LIFO inventory accounting choice and analyzes the significance of these variables on LIFO choice in univariate and multivariate analysis. The second part mainly focuses on the interaction between LIFO choice and inventory management. Various empirical approaches and stochastic models are presented. The main purpose of this literature review is to provide a more comprehensive perspective regarding LIFO choice to the researchers from operational area. As no paper about the LIFO choice with a focus on LIFO determinants and interaction with inventory management has ever been published, this literature review is intended to fill the gap and make a contribution to this area.

The remainder of this paper is organized as follows. Section 2 introduces the determinants of LIFO choice. Section 3 includes the analysis of inventory liquidation and the inventory management strategy taken to mitigate the effect of inventory liquidation under LIFO choice. Section 4 describes the effect of LIFO reserve on just-in-time manufacturing system adoption. The key challenges and future development directions are given in Section 5. Concluding remarks are presented in Section 6.

2. The Determinants of the FIFO/LIFO Choice

Current research is devoted to analyze the determinants behind LIFO choice. As commonly discussed, reducing tax is the most important incentive for many firms, especially large ones, to adopt LIFO inventory accounting system. Meanwhile, it was found that even though the tax saving benefit is obvious, companies remain reluctant to adopt LIFO inventory accounting choice implying that the potential non-tax reasons may force managers to balance the tax benefit with the cost incurred from shifting to the LIFO from the FIFO accounting choice.

2.1. Tax Incentive

It is well understood that during an inflationary period, when companies shift from FIFO to LIFO accounting, the cash flow of the firm increases due to the fact that by matching the revenue with the recent higher cost the taxable income decreases and reduces the tax. Even though there are other non-tax factors constraining firms from adopting the LIFO method, tax incentive is always a very significant variable for a firm's accounting choice. During periods of inflation, product costs and non-decreasing inventory levels, LIFO may produce the highest cost of goods sold, the lowest net income, and therefore the lowest tax payments if used for tax purposes.

[5] states that compared with other determinants, tax related cash flow will always be one of the most important factors determining the inventory accounting choices. Normally cash flow difference between LIFO and FIFO is influenced by four factors: end inventory level, inventory into price changes, marginal corporate tax rate and discount rate. Managers can only use the inventory management strategy to influence the end inventory level, so the tax incentive greatly relies on the end inventory variability.

[6] presents two models to prove that the tax saving is much larger at the year of shifting to LIFO than at other years and meanwhile, the tax saving uncertainty is strongly correlated with the managerial intention to shift. The core idea of [6] is to analyze the possible effects caused by the time shifting option and the tax saving un-

certainty on the inventory accounting choice. Tax saving is calculated as the product of cost of goods difference between LIFO and FIFO with the marginal corporate tax rate. [6] takes six majors steps in the process to calculate the final tax savings. The first four steps divided the overall cost of the goods sold into labor cost, raw material cost and the overall overhead costs. The fifth step tries to analyze the price changing effects of labor cost and raw material cost. The final step finally combines all the mentioned cost factors into integrity of the cost of goods sold. A nonparametric Wilcoxon test was used to compare the tax savings of the switch year and non-switching year. Three out of the four data groups are highly significant implying that current tax saving is strongly related to the LIFO/FIFO cost-of-goods-sold difference.

[7] uses the collected data from the inventory levels under their current inventory accounting method and compared the data with the possible alternative as if data which is obtained if another inventory accounting choice is used. Applying univariate analysis, [7] proves that tax saving is considerably related to inventory accounting choice. Due to the limitation of assumptions and available data, the quantitative benefits of non-tax factors can hardly be determined and the tax rate is assumed constant through the period. A more consistent conclusion cannot be made.

[8] uses time serious statistical method and comparison between different industries to test the significant variables which may have an influence on the final inventory accounting choice. The key finding is that only size and tax savings are the significant variables of inventory accounting choice. [8] further analyzes the time shifting option and demonstrated that only when the tax saving reach a certain point will LIFO inventory accounting method been chosen.

Common ground regarding the determinant of tax saving has been reached by most researchers that tax saving is one of the most significant factors involved into the inventory accounting choice. But in most cases, the tax rate is assumed to be constant and the non-tax factors were not accurately quantified due to data limitations in analysis. A consistent and stronger conclusion will be made if researchers incorporate the stochastic tax rate into models and additional data could be collected to obtain more accurate proxies of non-tax factors.

2.2. Political Cost

Political cost arises when the companies try to use LIFO to defer tax which is prohibited by the government which uses various methods such as anti-trust policy, nationalization as well as other regulatory methods to deter the intention of tax deferral ([9] [10]). Normally the firm's size is used as the proxy to represent the political cost in the statistical significance analysis. It was found, that the larger the firm is, the more likely LIFO would be used. In order to reduce the negative influence as the main monopoly of certain industry or excessive large profits, companies used various methods to hide the monopolistic behavior ([8] [10]). The majority of researchers show that LIFO-firms are larger than the FIFO-firms (Eller, 1989; [7]-[10]). However, [11] in their LIFO survey rejects the so called size hypothesis and allege that the second data is not reliable to be used to demonstrated the significant relationship between company size and the inventory accounting choice.

2.3. Agency Cost

Executives don't always align their interests along with that of the shareholders. They act either to increase the total value of the company or try to increase their personal interests at the sacrifice of the interests of the shareholders ([12]). It is generally accepted by previous studies that the interest conflict between managers and shareholders is caused by two main reasons: 1) the managers are worried about the underperformance of the company which could lead the take-over risk by its competitors; and 2) the compensation plan of the managers is proportionate to the financial performance of the company ([13]). The two main streams, which have been moderately analyzed, are ownership structure and compensation plans. Both of them can affect the relationship between managers and shareholders and are part of the possible solutions to the problems. [8] states that the misaligned incentives among the shareholders would damage the relationship between shareholders and managers, and between shareholders and bondholders. The ownership structure, compensation plan, and capital structure of the firm are the three main proxies of the agency costs.

2.3.1. Ownership Structure

As stated by [12] and [13], an association between ownership structure and inventory accounting choice is based on two premises: 1) the mangers and shareholders possibly conflict with each other over the inventory accounting choice; and 2) the ownership of the firm can partly solve the problem of the conflict.

[13] divides the ownership structure into two main groups: managerial ownership and outside ownership; meanwhile, he considers a lot of other factors in his empirical model. He set up a regression model to test the significance of the independent variables of managerial ownership (MGR), outside ownership (OUTCON), leverage (LEV), inventory variability (INVVAR), and size (ASSETS), and indicator variables of wood products industry, food industry, electronics industry, and publishing industry.

$$\begin{aligned}\text{Prob}(\text{LIFO}) = a + b_1\text{MGR} + b_2\text{MGR}^2 + b_3\text{OUTCON} + b_4(\text{OUTCON}\cdot\text{MGR}) + b_5\text{ASSETS} + b_6\text{LEV} \\ + b_7\text{INVVAR} + b_8\text{WOOD} + b_9\text{ELECT} + b_{10}\text{FOOD} + b_{11}\text{PUB}.\end{aligned} \tag{1}$$

[13] finds that managerial ownership and outside ownership do interact with each other to influence the inventory accounting choice, so an interaction variable $(\text{OUTCON}\cdot\text{MGR})$ is also added. The univariate analysis rejects the outside ownership hypothesis that LIFO choice is positively related to the increase of outside ownership and shows that LIFO choice is more preferred when the managerial ownership is high or low than when it is in between. To further test the results, a multivariate analysis is used. The results of that analysis agree with the univariate analysis that LIFO is preferred when managerial ownership is low and high. Compared with other regression models, Niehaus's model takes a more comprehensive perspective by incorporating almost all the potential determinants with a focus on the factor of ownership structure.

[14] takes a similar viewpoint but divide the companies into private and public owned. It is widely stated that public companies are less inclined to use the LIFO method to improve their financial performance while private companies prefer LIFO to increase their cash flows because of the lower tax paid. [14] uses the matched-pair test with controls of size and industry type to test the significant of ownership structure. The private companies are used as the benchmark to evaluate the accounting choice behavior of public firms.

[15] presents the discriminative multivariate model to analyze the independent variables of size, ownership variable and capital structure. The empirical results show that size is the most significant variable of the LIFO choice. Ownership structure is also a significant factor at the level of 0.05 in the multivariate analysis compared with the significance level of 0.10 at the univariate analysis when size is controlled. It implies that size interacts with ownership structure to affect the LIFO choice.

However, the most significant drawback of current research is the assumption that the company is owner-controlled if the largest shareholder has 20% or more of the total shares and the company is manager-controlled if no shareholder owns 5% or more of the total shares. Neither the justification for creating such an artificial dichotomy from a sample of continuous data on the concentration of stock ownership, nor the statistical implication of this practice, is evident ([15]). [15] proposes that in order to avoid the arbitrary classification of firm ownership, a continuous variable measuring the degree to which a firm is closely held should be employed.

2.3.2. Compensation Plans

As the compensation plans for managers are dependent on the financial performance of the firm, the managers are less inclined to adopt LIFO which can lead to a lower financial performance ([16]). Current research has found a positive relationship between managers' compensation plans and firms' financial performance. [13] states that if the company does not adjust the inventory accounting method, it is likely that a positive correlation between managerial compensation and company income. Consequently, managers may have less incentive to choose the income decreasing inventory method of LIFO. A conflict may arise between the managers and shareholders because managers will prefer not to use LIFO so as to increase their compensation while shareholders will intend to adopt LIFO to reduce the tax paid so as to increase the cash flow for the company.

However, contradicting to the theoretical allegation, it was proven that, in average, changes to LIFO do not have significant negative effects on executives' compensation, no matter of whether kind of forms of compensation was provided ([17]). [17] presents a regression model:

$$\text{LnP}_i = a_0 + a_1\text{LnINC}_i + a_2\text{OM}_i + a_3\text{EC}_i + a_4\text{OM}_i\cdot\text{LnINC}_i + a_5\text{EC}\cdot\text{LnINC}_i + a_6\text{OM}_i\cdot\text{EC}_i\cdot\text{LnINC}_i + e_i \tag{2}$$

where the dependent variable is the dollar amount of the executive's compensation and the independent variables are the annual profit reported by the companies INC_i and two dummy variables OM_i and EC_i indicating whether the company is owner-controlled or manager-controlled and whether the company shifted to LIFO or not. The empirical results are shown in three aspects:

1) Before changing to LIFO choice, the companies which shift to LIFO compared with the companies still retain at FIFO enjoy a higher bonus plan.

2) At the changing year, companies successfully align the benefits of the managers with the shareholders to significantly increase the proportion of the bonus plan in the overall compensation plan.

3) After the changing to LIFO, the interaction term EC (Executive Compensation) × lnINC (income) is significant which demonstrate that the companies which shift to LIFO are quite more flexible in reconfiguration of the compensation compared with the companies which are still stick to the FIFO choice.

[18] takes a similar view that the compensation plan includes various incentive plans so the managers need to balance various compensation plans to see what the ultimate effect on their overall compensation amount. [18] uses three independent variables: income-based bonus plan, ownership structure, and leverage to test their significance on independent LIFO choice. The final empirical results show that the bonus plan is not correlated with the LIFO choice, the same with previous studies. One reason is that when the shares become a part of the compensation plan, managers are usually caught in a dilemma, such as stated by [13], that LIFO results in a higher stock price and therefore greater value for the managers' shares, meanwhile LIFO decreases the reported income and income-based bonus compensation. The mere existence of the compensation plan cannot determine the preferred accounting choice anymore, as the compensation plan structure now becomes quite comprehensive, incorporating different incentive plans for the managers. Meanwhile, the compensation plan becomes very flexible to adapt for the interest conflict between managers and shareholders.

2.3.3. Capital Structure

[19] finds that the capital structure affects the inventory accounting choice as well. Since then, [8] [15] [18] have stated that companies which heavily rely on external financing try to avoid the income decreasing inventory accounting method which will lead to a more restrictive covenants on contracts levied by financial institutions. Different papers try to analyze the relationship between the components of the capital structure and the inventory accounting choice to provide more practical insights for the capital and inventory accounting policy. [19] selects two groups of companies according to two different costing methods—full costing method and successful efforts to test the significance of long-term debt. [19] concludes that firms using a relatively large amount of long-term debt tend to oppose LIFO which reduces the financial performance level. [8] also uses the proxy of net tangible assets/long-term debt to test the significance of its positive relationship with LIFO adoption. Meanwhile, other proxies are also proposed for the debt covenant constraint. [8] and [18] test the hypothesis that LIFO adopters have a larger pool of retained earnings available for dividend payments relative to recent dividend payments, than non-adopters. [18] also states that the level of working capital and credit policy are required by most covenant contracts.

2.4. Investments and Production Character

There are two streams for the determinants of the accounting choice. One is focused on the deterministic factors such as the firm size, capital structure, and covenant constraints, etc. The other one is concentrated on a more realistic, dynamic and interactive accounting, investment and operational policy choice.

[8] states that difference in production and purchasing opportunity provides firms with different comparative advantages in adopting LIFO or FIFO. One of the key issues involved is how to use the production and purchasing policy to keep the end inventory level constant. Because each year, as the inventory purchased or produced, the new inventory will be added to the base inventory. It looks like a new layer of the inventory is added on the top. Therefore, the researchers treat the total inventory as integrity with many layers. During a new year, if the inventory consumption is larger than the inventory purchased or produced at that year, the base inventory from previous years will be used or called liquidated. In common, the inventory cost is increasing. The lower inventory cost of the liquidated inventory matches with the revenue so the taxable income increases resulting into a higher tax and lower cash flow which is against the original tax purpose of LIFO adoption. As the end-inventory level significantly influences the tax, operation policy, accounting choice and tax should be viewed in a dynamic and integrated manner. Specifically, companies should use production and purchasing policies to keep the end inventory at the optimal and stabilized level, meanwhile still trying to minimize the tax.

As introduced by [9], the Ricardian Hypothesis states that as firms are endowed with different production-investment opportunity sets, then the managers' inventory accounting choice can be predicted by the comparative advantage in tax cost minimization associated with the production-investment opportunity set of each firm. IFRS (International Financial Reporting Standards) allows only cost method to be used to calculate the inventory valuation while mark-down is prohibited. Such regulation implies that companies with large inventory price fluctuations may have advantage in adopting FIFO while companies with steady inventory price may prefer to use LIFO. Meanwhile, the achievement of tax minimization under LIFO requires a stable end inventory level. [9] supposes that comparative advantage in adopting LIFO is greatly dependent on the next three variables: stochastic inventory price, stochastic inventory variability, and bookkeeping and tax reporting cost. [9] uses eight proxy variables to reflect the characteristics of the production-investment opportunity set that are relevant to the choice of inventory accounting method. The eight variables are price variability, inventory variability, accounting income variability, absolute firm size, relative firm size, capital intensity, inventory intensity and industry classification. The univariate analysis shows that the size variable is a very significant factor for the LIFO choice, but the size is also significantly related with other determinants such as capital intensity which are also significantly related to LIFO choice. Hence, the conclusion about the significance of size would be spurious. However, the analysis shows that inventory variability is independently significant for LIFO choice implying the significance to keep end inventory level constant. According to multivariate analysis, the industry classification variable is significant what could be explained by the fact that different industry has different inventory price movements and companies in different industry possess different production and investment opportunity set. Basically, the multivariate analysis rejects the political cost and agency cost assumptions, but supports the Ricardian hypothesis because the significant variables left are inventory variability, industry classification, and inventory price variability.

[7] also indicates that the inventory accounting choice is based on the assumption that company makes its strategic inventory accounting choice according to the characteristics of its financing, operation and accounting systems. [7] presents three variables which are related to the LIFO choice. Total sales or holding gains are used to measure the size of the company. Fixed asset/total asset ratio is used to reflect the capital intensiveness. Covariance of inventory variation describes end inventory fluctuation. The results prove that companies which adopt LIFO tend to be larger, with more stable end inventory and stronger capital intensiveness.

To summarize the determinants of LIFO inventory accounting choice, the major potential variables which have been extensively studied are listed. Univariate and multivariate analysis is used to test their significance on the LIFO choice. Due to the limit of the current research, common ground has not been reached for all of the analyzed variables and some variables only occur in the analysis of one specific paper, so we don't put all the possible variables but those which are usually used and largely agreed upon in the current research. Authors and measurement are also included into the table for the convenience of reference and comparison with further research.

3. Inventory Liquidation

The extensively studied area of the interplay between inventory management and LIFO inventory accounting choice is inventory liquidation. As mentioned, the total inventory is treated as integrity with many inventory layers which are the inventory accumulated in the previous years. If the inventory consumption is higher than procurement or production of the new inventory in the new period, the older layer of inventory will be consumed. Such consumption of the previous base inventory is called inventory layer liquidation. Unlike the general understanding of inventory liquidation where inventory are sold at a cheaper price than original price in order to increase the sales volume, the inventory liquidation concept of LIFO analysis here is more focused on the inventory liquidation quantity which is the depletion of the base inventory quantity of the previous years. Papers analyzing inventory liquidation mainly focus on four aspects: the measurement and determinants of inventory liquidation, the consequences of inventory liquidation, the dynamic ordering models to mitigate inventory liquidation, and inventory pooling strategy to interact with inventory liquidation. Papers considering the last two aspects aim to reduce the negative effect of inventory liquidation on LIFO tax minimization purpose by using inventory management strategy consisting of dynamic order policy with purchasing and production strategy, and inventory pooling policy.

3.1. Inventory Liquidation Determinants

As the end inventory level can significantly influence the tax benefit under the LIFO inventory accounting method, it should be treated as a part of the function to maximize the cash flow. When inventory liquidation occurs, current sales revenue matches with the previous lower inventory cost during the inflation period, so taxable income increases respectively. This leads to the decreased current cash flow due to the increased tax. Companies try to avoid the inventory liquidation by means of intra-year purchasing or production in order to keep the end inventory level stable for the tax concern while some companies otherwise liquidate their end inventory for other reasons.

The literature regarding the determinants of the inventory liquidation could be classified into three main groups. Generally speaking, the managers need to consider and balance three relationships before making the inventory liquidation decision: holding cost vs. liquidation cost, cost of capital vs. liquidation cost, and economic changes vs. liquidation lost. Dynamically, the managers need to combine the accounting policy and operation policy simultaneously in terms of integrating inventory management, capital investment management and earnings management.

3.1.1. Holding Cost

The most significant consequence of holding an excessive end inventory to avoid inventory liquidation is the increase of inventory holding cost. [20] states that the trade-offs involved in the purchasing decision to replace the liquidated inventory is reduced federal income tax against higher carrying costs and order or setup costs. Generally, there are eight components of inventory carrying cost:

1) the cost of capital for purchased inventory,
2) additional property taxes,
3) warehousing costs,
4) insurance cost,
5) transportation cost,
6) handling cost,
7) deterioration cost,
8) obsolete cost.

Since some of the holding costs are tax deductible, such as cost of capital and property taxes, the decision rule is based on a comparison between the present value of these costs and the present value of the cost of the LIFO layer liquidation.

3.1.2. Capital Cost of Investment

Before managers make the investment choice, the risks of investment, interest rates and rate of return are the important factors to be considered. The main risk of inventory investment is inventory obsolescence. High interest rates will result in high inventory carrying costs. These high rates also present alternative economic opportunities for funds invested in inventories if there is a belief that the inflation rate will decrease in relation to the interest rate ([21]). [22] uses the questionnaire to test the significance of key factors determining inventory accounting choice. Among the executives, most of them identify cost of capital as one of the most critical important factors.

3.1.3. Economic Condition Changes

The overall economic condition significantly influences the inventory liquidation choice. The economy in recession leads to a dip of the demand causing the company to reduce its inventory in order to save cost ([21] [23]). [23] identifies the variables of sales' volumes and earnings to reflect the firm size and economic performance in the inventory liquidation choice. [21] states that the decreasing demands caused by the sluggish economy as a whole pushes the company to reduce the cost or to improve the reported income. Inventory liquidation can greatly help to alleviate the cash flow strain and relax the debt covenant constraint especially in the economic recession.

Besides the three main considerations, industry specificity also draws much attention in the literature because certain industries are more sensible to the business cycle phenomenon than others. [22] indicates that LIFO companies in cyclical industries should take a special care to manage their LIFO inventory levels better. Meanwhile, the characteristics of the industrial products also determine the inventory accounting choice. The retail industry is much more likely to take the FIFO method because the fast inventory turnover significantly constraints the possibility of tax manipulation and FIFO can better reflect the operational performance of the retailers. While the oil& gas industry prefer the LIFO industry ,due to the long time storage of its inventory and the fluctuations of the oil price in the international market, both of which provide the possibility of tax manipulation.

3.2. Inventory Liquidation Consequences

The extant literature provides contradictory perspectives regarding the incentives behind the inventory liquidation and its consequences. On the one side, [8] implies that companies are foregoing the tax benefits of switching to LIFO due to the potential cost caused by possible future LIFO layer liquidations. [24] finds that cash planning difficulty mostly follows the possible inventory liquidation after LIFO is adopted. [25] holds the same view that the sudden increase of tax liability can quickly diminish the tax deferral benefits from previous period and cause severe budget planning and allocation problems. On the other side, [26] argues that inventory liquidation is beneficial to the stabilized reported income and help to alleviate the restrictive debt contract covenants.

3.3. Dynamic Order Policy

The conventional dynamic order policy problem has already been extensively analyzed in the past. The cost factors normally involved are order cost, carrying cost, obsolete cost, stock-out cost along with other considerations such as customer satisfaction level. However, few studies have ever incorporated inflation and tax, the two most important economic factors, into the models. When inventory cost changes, the optimal order size will also be influenced due to tax concerns. Optimal order size, in turn, will be dependent on the cash flow changes. Firms need to consider the cash flow choice and the order size decision simultaneously. The existing literature treats the end inventory level as an exogenous variable where management has no control. However, since the LIFO firms' tax liability is partly a function of year-end inventory level, management can control the year-end inventory levels based on the LIFO or FIFO choice.

[5] proposes the anticipation hypothesis and incentive hypothesis, showing that the accounting choice and operation choice are mutually influenced by each other. The accounting choice should consider the future anticipated operational condition such as inventory levels which could be changed due to purchasing and production policy while the operation policy, such as the purchasing or manufacturing policy, could be altered and adjusted for the end inventory level to cater to the chosen accounting choice in order to maximize the tax benefit. [5] establishes the one-period optimal inventory policy model considering the LIFO and FIFO tax incentives. The cost, decisions and outcomes are treated and updated on an annual basis. The variables involved in the model are the order-up-to quantity and demand, which are notified only at the end of the year. The model is used to test the following testable implications:

- during inventory price inflation, LIFO end inventory level will be higher than that of FIFO;
- during inventory price deflation, LIFO end inventory level will be lower than that of FIFO;
- the existence of additional layer of LIFO inventory price changes will increase the optimal inventory level for sale under LIFO and FIFO.

For the previous models such as the one established by [5], the end inventory is considered as an exogenous variable and managers can't have an influence on it. However, under LIFO, tax liability is a function of the end inventory. Managers try to keep a constant end inventory to avoid liquidation, though the increased inventory level leads to a higher holding cost. Meanwhile, the purchasing and production cost should also be considered. Based on these considerations, the models set up by [27] and [28] attempt to treat the purchasing and production under control and do not consider the end inventory as an exogenous variable.

[27] sets up the model of expected after tax profit, under LIFO and FIFO in both single-period and multi-period:

$$\begin{aligned}\Pi(z) = {} & pE\left[\min(z,D)\right] - c_1(z-x) - hE\left[(z-D)^+\right] - sE\left[(D-z)^+\right] \\ & - M\left\{(p-c_0)E\left[\min(x,D)\right] + (p-c_1)E\left[\min\left(z-x,(D-x)^+\right)\right]\right\} + \alpha c_2 E\left[(z-D)^+\right].\end{aligned} \tag{3}$$

where $pE\left[\min(z,D)\right]$ is a stochastic revenue term, and D is the random demand. The cost of goods purchased is $c_1(z-x)$. Annual holding and shortage cost are defined by $hE\left[(z-D)^+\right]$ and $sE\left[(D-z)^+\right]$ respectively. The salvage value of goods remaining at the end of the year is $\alpha c_2 E\left[(z-D)^+\right]$, where $\alpha \in [0,1]$

is the discount factor. The results present the relationship between tax, inventory order policy and valuation convention. For the single period model, demand is treated as a non-fixed variable and will only be known at the end of the period. Annual order is aggregated into a single quantity which is the total volume of the expected demand of the next periods. Optimal inventory order policy under FIFO inventory accounting choice is a deviation of the typical Newsboy problem. While the optimal inventory order model under LIFO is diminished to be a first-order equation. The single period models of FIFO and LIFO also consider the relationship between relative optimal inventory level under LIFO/FIFO and various responses to the changes in tax rate, inventory costs and inventory level. The single period models of LIFO and FIFO are extended into multiple period problems. For FIFO problem, [27] add an additional assumption that inventory turnover fluctuates across the periods; the optimal solution is achieved as a closed form myopic policy. For LIFO, the expression is a function of the optimal inventory level of each inventory layer. Given the sequence of inventory levels and ordering policy, the after tax profit under LIFO could be found in the multi-period model. Finally, [27] set a two-period setting, where dynamic order policy is implemented under LIFO and FIFO, and then the optimal inventory level under LIFO and FIFO is compared. A search algorithm is used to get the best solution for LIFO, while a variation of the best solution from the FIFO multi-period model is achieved for FIFO model in this case.

[28] takes a very special perspective in looking at the end inventory problem. The purpose of his paper is to analyze the effects caused by the dynamic ordering policy which is taken to keep the end inventory constant from a social welfare standpoint. [28] introduces the concept of dollar value of LIFO inventory liquidation. The measuring of the inventory in dollars provides conveniences to put different kinds of products into the same inventory pool even though they are not economically or physically correlated. Under LIFO inventory liquidation, the main purpose of companies is to maintain the dollar value of the inventory at the end of the new period to be not less than the beginning value. Even though physical products get liquidated or new products are purchased during the new period, as long as the end inventory into dollars is equal or larger than the beginning inventory values, the inventory is not financially liquidated. [28] strives to establish a model, in order to see the LIFO end inventory controlling effect on resource allocation. A maximum taxable income model has been set up to calculate the taxable income even though it is slightly different from the economic income. An after-tax economic model is also built to measure the total social welfare. The final production effects and Pareto-Optimality shows that the measures used to keep the end inventory stable for LIFO companies introduce ineffectiveness into economic resources allocation within the society.

[29] extends the models established by [27] by considering both the intra-year and end-year purchasing and production decisions in a nonstationary price and demand environment. [29] maximizes the following function in a T-year period:

$$\sum_{t=1}^{T}\alpha^{t}\left\{\sum_{i=1}^{p}\left[\left(1-\tau_{t}\right)\left(s_{i}^{t}D_{i}^{t}-h_{i}^{t}I_{i}^{t}\right)-p_{i}^{t}y_{i}^{t}\right]+\tau_{t}\left(\sum_{j=1}^{N}p_{j}^{0}Z_{j}^{t}+\sum_{j=N+1}^{N+1}\sum_{i=1}^{P}p_{i}^{j-N}Z_{j,i}^{t}\right)\right\}. \tag{4}$$

Within each year there are i time periods, when the managers have to make a purchasing or production decision. The objective of the model is to maximize the after-tax profit of the company. The cost structure includes unit purchase (production) cost p_i^t, unit holding cost h_i^t, tax rate τ_t and sales prices s_i^t. The model involves some assumptions to facilitate the trade-off between tax and the cost incurred to carry excessive end inventory. Demand D_i^t is assumed to be known with certainty or, alternatively, demand could be forecasted with 100% accuracy. Stock-out cost is totally eliminated from the model implying that the demand variance is captured by the safety stock. Meanwhile, production capability, distribution channel, etc. are excluded from the model for simplicity. The mathematic model is solved by a non-linear formulation which reduced to be a linear formulation due to the assumption of cost inflation.

[30] holds the same view by using a mathematical model to prove the mutual relationship between the purchasing policy and inventory accounting choice where the purchasing policy will have a huge influence on the final inventory level meanwhile the end inventory level determines the further purchasing quantity. [30] establishes a three-period model with two demand scenario: low demand and high demand. In order to maximize the manager's compensation as well as the cash flow of the company, the managers have to decide the purchasing quantity based on the forecast of demand of the next period. The key issue here is to choose the optimal inventory level which not only maximize the utility of the manager's compensation but also minimize the divergence of interested between the managers and shareholders. Two contract settings are given: contract based on income

and contract based on income and inventory liquidation loss. Two inventory accounting scenarios are further presented: inventory accounting choice without tax effect and inventory accounting choice with tax effect. The results of the mathematical modeling show that given managers' incentive, LIFO tax incentive distorts efficient purchasing and production policy. As [30] generally agrees with most of the established arguments in the literature, the noteworthy thing is acquisition cost on the inventory accounting choice. As the model can't reflect this factor, further research has to focus on the effect of purchasing price changes on LIFO choice. Meanwhile, [30] suggests the mix inventory strategy, where some of the inventory adopts LIFO and rest for FIFO. An optimal inventory strategy for FIFO or LIFO alone should be further extended under inventory accounting incentives.

In order to better fit the stochastic demand settings, [31] introduces an alternative method which incorporated a second variable:

$$\alpha^{n-1}\left\{s_n z_n - c_n\left(Y_n - x_n\right) - \overline{c}_n\left(z_n - Y_n + I_n\right)^+ - h_n \max\left(\left(Y_n - z_n\right), I_n\right) - \tau\left[s_n z_n - h_n \max\left(\left(Y_n - z_n\right), I_n\right) - \mathrm{COGS}_n\right]\right\} \quad (5)$$

where the procurement cost at year n is $c_n\left(Y_n - x_n\right) - \overline{c}_n\left(z_n - Y_n + I_n\right)^+$, which is comprised of the cost of first order and second order, when $z_n > Y_n - I_n$; holding $\mathrm{costs}_n = h_n \max\left(\left(Y_n - z_n\right), I_n\right)$, and $\mathrm{Taxes}_n = \tau\left[s_n z_n - h_n \max\left(\left(Y_n - z_n\right), I_n\right) - \mathrm{COGS}_n\right]$. The second order guarantees that the inventory level not only satisfied the excessive inventory which is over the previous up-to-level inventory but also help to maintain the desired minimum end inventory level. In this case the fixed order cost is ignored for simplicity. However, in reality the accumulation of the fixed order cost incurred by multiple second orders across multiple years may outweigh the tax saving from the avoidance of inventory liquidation. In future research, a dynamic variable of the second order cost should be incorporated.

Considered the tax concern of LIFO, the purchasing and operation decisions are intended to keep the end inventory floated to avoid inventory layer liquidation. Such decisions are not always wise to minimize the total cost as far as the holding cost is high if more buffer inventory has to be carried. The intra-year and end-year production and purchasing decisions should be made into a more dynamic way by combing the tax liability, possible holding cost, production and purchasing cost as well as potential inventory cost changes. Under so many assumptions for the current models, further research regarding the end inventory level control could be extended to different areas, considering the tax liability of LIFO. The extended models could reflect a longer planning period, production capacity, stock-out cost and stochastic demand changes. The models should include the factor of fixed order cost which is a major factor for the function of after-tax profit maximization if companies are free to make several additional orders due to demand changes. An integer programming model could be expected if fix cost and decreasing inventory price are included. In order to better catch the interplay between intra-year and end-year operation decision, a more comprehensive stochastic model is going to be established.

3.4. Inventory Pooling Strategy

3.4.1. Inventory Pooling Measurement and Determinants

In Operation Management area, the concept of inventory pooling stands for the risk pooling strategy in order to diminish the risks and losses caused by the variation of customer demand. The main forms taken are centralized warehousing, common components, delayed differentiation, product substitution and e-tailing. While in LIFO inventory analysis, the main function of inventory pooling is intended to pool the inventory so as to reduce the loss of tax benefit caused by inventory liquidation. As mentioned before, inventory liquidation will match the recent revenue with the previous cost, so the profit will increase so as to increase the tax levied. The most important prerequisite of the inventory pooling strategy in LIFO is the introduction of inventory pooling into dollar values. Different from inventory pooling in units, dollar value provides the possibility to group the products which are counted based on different measurement, such as tons, kilos, etc. Normally, the inventory pooling of units is based on the classification of characteristics of the products, while the subdivision of inventory pooling of dollar values has resulted at least in part from the habits or customs of business management, government policy and, in some cases, mechanical convenience in inventory computation ([32]). The dollar value of inventory into multiple-item pools under LIFO allows for the reduction of inventory of some products to be offset by the increase of inventory of other products. The similarity of purpose between the inventory pooling strategy under general Operation Management and the inventory pooling strategy under LIFO inventory management is that both strategies are intended to diminish the risk of variation of customer demand by aggregating the inventory levels. The justification of the expansion of pooling should be either that the extended pooling can better allocate the cost under LIFO or the pooling better serves the objective of LIFO to match the cost with revenue.

3.4.2. Single Pool

The IFRS has set different pooling regulations for the manufacturing/processing and retailer/wholesale industries. Manufacturing/processing is allowed to have one pooling, while retailer/wholesale is required to adopt the pooling subdivisions to reflect the diversification of the product portfolio ([33]). This regulation sets the maximum limit for the degree of freedom of managerial discretion, while the managers still have a moderate freedom to pool the inventory for tax incentives. A widely accepted wisdom within the industry holds a view that if managers follow a strategy of deferring income taxes by the use of LIFO, the objective should be to minimize the number of LIFO pools subject to these regulatory constraints. The justification and rationale behind this is following: the more items a company combines in a single pool, the more likely it is that a decrease in some items will be offset by an increase in others. Moreover, the use of fewer pools tends to reduce the administrative burden of applying LIFO ([34]).

3.4.3. Multiple Pools

[35] directly attacks the general wisdom, that single pooling is a good solution for inventory liquidation. Even though single pooling is not always inaccurate, but it violates the inventory pooling decision process. As suggested by [35], there are four questions to answer in order to make a right inventory pooling decision:

1) How the purchasing decision is going to be influenced by the future inventory cost changes?
2) How the prices of different products move together?
3) How to stock different products?
4) How the inventory will be liquidated in the future?

The key determinant of the pooling decision is how the inventory prices' changes are interacted with inventory quantity changes. [35] compares the advantages of a single pool with multiple pools based on four observations where the inventory price change interacts with the inventory quantity changes. Single pooling is always preferred when the inventory has both highest price changes and highest quantity changes; multiple pooling is always preferred when the inventory has both lowest price changes and highest quantity changes. If inventory price and quantity level remain stable across time, there is no difference between single pooling and multiple pooling. After years of price instability the inventory price starts to stabilize while inventory quantity still keeps growing, the cost-of-goods difference between single pooling and multiple pooling will keep growing as well. The main purpose of these four observations is to provide a portfolio of scenarios to the managers and facilitate in making the right pooling decision under various situations of inventory price and quantity changes. Note, that no fixed pooling strategy is right all the time. The pooling strategy should be constantly updated considering the inventory changes.

However, [35] considers only four possible scenarios. Meanwhile there are many other possibilities of the interaction between inventory price and quantity changes. For example, the inventory price keeps decreasing before entering stability, then price suddenly drops. Under this condition, which inventory pooling strategy is preferred? At the same time, for those four observations, it is assumed that inventory quantity of each product keeps increasing. It means, inventory liquidation is not allowed. However, inventory liquidation is possible for many products in reality. Thus, further research could be extended by relaxing this assumption to see how inventory liquidation could be involved into the inventory pooling strategy.

4. Just-in-Time and Inventory Accounting Choice

Though there is no specified concept of a just-in-time manufacturing system, the most important characteristic of this new system aims to reduce unnecessary waste and improve operational productivity. One of the most significant achievements of the system is to significantly reduce the WIP (work-in-process) inventory. Inventory accounting choice and end inventory transaction policy are usually combined in order to manage earnings. The significant reduction of the inventory constrains the ability of using earnings management tool to manipulate the profit for the company. [36] finds that the LIFO reserve level is correlated with the intention of the company to a adopt just-in-time system. They suggest that companies which have limited LIFO reserve are highly involved in earnings management, have less intent to adopt the just-in-time system.

Normally, the inherent tax incentive behind the LIFO encourages the build-up of the inventory, which increases the cost of holding, maintenance, and obsolescence as well as other inventory related costs. [37] examines the end of the period purchasing as a function of the inventory accounting choice and find that the purchasing decision could be driven by the tax incentive. The tax incentive ultimately leads to inefficient inventory management. According to the just-in-time theory, purchasing will only be activated when necessary. This is the most significant cost saving method in inventory management. The inventory management inefficiency caused by the tax incentive totally counters the just-in-time principle. Further research could quantify the difference between the tax savings of the LIFO method without adopting the just-in-time savings and the inventory management inefficiency that could have been avoided.

5. Future Development

Current research is primarily dedicated to find and analyze the potential determinants of LIFO inventory accounting choice. Till now, the variables representing firm's physical characteristics, operational features, investment, tax policy, etc. have been proposed and analyzed in the univariate and multivariate analysis. Several factors are considered to have significant effect on LIFO choice even though further research is still anticipated for the rest of factors which have not been extensively studied. The univariate and multivariate analysis of the potential determinants is quite helpful because it provides a key list for the companies to consider before an inventory accounting choice is made. Even though multivariate analysis reached some conclusions regarding the collinearity among potential determinants, the interaction effect among factors still needs further attention from the researchers. The incentive behind is that the significance test of the individual determinant benefits the companies before they choose the inventory accounting system based on specified characteristics. The key task for the companies during the using of LIFO system is to make a trade-off between the tax benefit loss and the cost incurred to change their organizational features. Further analysis of the interaction between the potential determinants provides a clear picture of the necessary adjustments to compound the key characteristics at the lowest opportunity cost. Thus, further research on the potential determinants of LIFO choice should be directed into establishing the procedures managing the primary determinants to minimize the tax in the process of using the LIFO system.

Tax and inflation factors have been included into the dynamic order models to reflect the tax liability of LIFO inventory accounting system under inflation. To maximize the after-tax profit, the end inventory should be stable and not less than the beginning inventory level. Various stochastic models were established by including intra-year and end year purchasing and production decisions to consider a trade-off between inventory holding cost, tax cost, purchasing and operation cost, etc. and to minimize the total cost of the company under LIFO. Although some of the models treat inventory price and demand as stochastic variables, none of them intended to provide any method to mitigate the demand and price variance under LIFO. Further research is required in this area in order to lower the demand variance risks and increase customer satisfaction. The possible approach is to find the optimal safety stock level in the dynamic order models to catch the demand variance under LIFO. Meanwhile, models could be used to analyze the interaction between inventory pooling strategy and dynamic order strategy in order to diminish the associated risk.

Inventory pooling decision is one of the strategies to mitigate inventory liquidation risk. Current research discusses the choice of single or multiple inventory pools strategy. However no papers have ever been published regarding the optimal number of inventory pools in order to reduce inventory liquidation. Further research can focus on establishing the models of the interplay between inventory price changes and inventory quantity changes to get the optimal inventory pool number and minimize the total cost considering inventory liquidation minimization requirement.

6. Summary

This study is partially motivated by the current debate in the US Congress regarding the repeal of the LIFO inventory accounting choice. Hence, a detailed literature review of LIFO inventory accounting choice is necessary to provide a more comprehensive view to various stakeholders. Furthermore, this review aims to inform researchers in Operations Management, many of whom are unware of this debate. To this end, this paper classifies the previous research into two areas: the determinants of LIFO inventory accounting choice; the interplay between LIFO inventory accounting choice and inventory management. The effect of LIFO reserve on the just-in-time manufacturing system adoption is also presented. Further research is suggested to provide a more generalized picture to the operation managers regarding the possible determinants behind their inventory accounting choice in order to better fit the company in terms of its investment, operation, and other physical characteristics. The interplay between LIFO inventory accounting choice and inventory management research should be further extended by considering more potential inventory related factors.

Last but not least, although inventory management is a fundamental issue in both Operations Management and Accounting, research on the interface between these two disciplines has been limited. The recent establishment of the Department of POM-Accounting Interface at Production and Operations Management Journal indicates that there has been an increasing interest in this research direction. It is our hope that this research can encourage more research on the Interface of Operations and Accounting.

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American Journal of Operations Research, 2014, 4, 365-374
Published Online November 2014 in SciRes. http://www.scirp.org/journal/ajor
http://dx.doi.org/10.4236/ajor.2014.46035

Scientific Research

Using Multi-Attribute Decision Methods in Mathematical Modeling to Produce an Order of Merit List of High Valued Terrorists

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Received 3 September 2014; revised 28 September 2014; accepted 15 October 2014

Abstract

The authors present a methodology and an example of preparing an order of merit list to rank terrorist based upon decision maker weights. This research used an old terrorist data set as our base data to keep the information unclassified. This data is used to demonstrate this methodology. The authors perform numerical iterative criteria weight sensitivity analysis to show the effects on the model's outputs in changes in the weights. Through their analysis the most critical criterion is identified.

Keywords

Ranking Terrorists, TOPSIS, AHP, Criteria Weights, Pairwise Comparisons, Sensitivity Analysis

1. Introduction

The United States of America is fighting a war against terrorism. The National Strategy for Combating Terror (NSCT) [1] states that the United States Government's (USG) intent is to obtain victory in the long war against terror, and that the goal and objective in that intent is to defeat terrorists and their organizations. The United States (US) has termed this strategy as the Global War on Terror (GWOT) and the targeting of terrorist organizations' key personnel is an integral part of this effort.

According to Department of Defense (DoD) doctrine in Army FM 34-8-2 [2]:

> Targeting is the process of selecting targets and matching the appropriate response to them, including operational requirements and capabilities. The purpose of targeting is to disrupt, delay, or limit threat inter-

How to cite this paper: Fox, W.P. (2014) Using Multi-Attribute Decision Methods in Mathematical Modeling to Produce an Order of Merit List of High Valued Terrorists. *American Journal of Operations Research*, **4**, 365-374.
http://dx.doi.org/10.4236/ajor.2014.46035

ference with friendly Course of Action (COAs).

Human-targeting, the process of selecting a human target exists as a subset of this more general targeting doctrine. This human targeting research is being applied to terrorists.

A common misconception is that human-targeting denotes either a hard-power and soft-power strategy that involves either kinetic or non-kinetic power. Human-targeting is instead intent or objective neutral. It does not specify the type of action taken nor the counterterrorism (CT) objective desired. Human-targeting, rather, represents an analytical *process* that assigns a heuristic value to a target. This assignment of value allows for the prioritization of multiple targets and this prioritization permits CT organizations to direct efforts and allocate resources. Consequently, every government agency, unit, or official whose function serves to counter terrorism remains dependent on the human-targeting process [3].

To mitigate this risk of terrorist, we propose the development of a systematic method for the conduct of human targeting. We test the proposition using mathematical modeling and multi-attribute decision making tools. These methods are extensively tested and used for finding key network nodes, [4] [5], and ranking phase targeting of terrorist activities [6] [7] as well as commercial disciplines. These methods may be successfully applied to prioritize human targeting. This research is a preliminary example of this concept.

The current targeting process involves numerous complex and dynamic interactions filled with ambiguities. Minor variations in the process dramatically affect human-targeting decisions producing essentially unpredictable results. In other words, CT organizations may be targeting the wrong (or a less valuable) terrorist. This inefficiency is not only a misuse of intelligence, but wastes limited national resources, which inevitably places lives unnecessarily at risk. Left unaddressed, this critical USG decision-making process with systemic problems could result in a catastrophic intelligence failure [3].

In previous work by Twedell and Edmonds [8], they used a series of six linear regression models to ultimately model and obtain a series of terrorist rank orderings. We believe this proposed methodology is better suited to obtain a rank ordering.

2. Proposed Methodology: The Technique of Order Preference by Similarity to Ideal Solution (TOPSIS)

TOPSIS was the result of research and work done by Yoon and Hwang [9]. TOPSIS has been used in a wide spectrum of comparisons of alternatives including: item selection from among alternatives, ranking leaders or entities, remote sensing in regions, data mining, and supply chain operations. TOPSIS is chosen over other methods because it orders the feasible alternatives according to their closeness to an ideal solution [10].

Napier [11] provided some analysis of the use of TOPSIS for the department of defense in industrial base planning and item selection. For years the military used TOPSIS to rank order the systems' request from all the branches within the service for the annual budget review process as well as being taught again in as part of decision analysis. Current work is being done to show the ability of TOPSIS to rank order nodes of a dark or social network across all the metrics of social network analysis.

In manufacturing analysis, Wang [12] proposed two methods to improve TOPSIS for multi-response optimization using Taguchi's loss function. Ozturk and Batuk [13] used TOPSIS for spatial decisions and then linked to geographical information systems (GIS) operations for flood vulnerability. Olson and Wu [14] have shown how TOPSIS may be used for data mining and analysis in credit card score data. Olson and Wu [14] presented a comparison of weights (centroid weights, equal weights, and weights by linear regression) in TOPSIS models using baseball data where their conclusion is that accurate weights in TOPSIS are crucial to success.

In a business setting it has been applied to a large number of application cases in advanced manufacturing processes [15]-[17], purchasing and outsourcing [18] [19], and financial performance measurement [20].

2.1. TOPSIS Methodology

We describe the TOPSIS process is carried out through the following steps.

Step 1

Create an evaluation matrix consisting of m alternatives and n criteria, with the intersection of each alternative and criteria given as x_{ij}, giving us a matrix $\left(\boldsymbol{X}_{ij}\right)_{m\times n}$.

$$
\boldsymbol{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_M \end{array}
\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_n \end{array} \\
\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix} \end{array}
$$

Step 2

The matrix shown as $\boldsymbol{D}$ above then normalized to form the matrix $\boldsymbol{R} = \left(\boldsymbol{R}_{ij}\right)_{m\times n}$ using the normalization method

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum x_{ij}^2}} \quad \text{for } i = 1, 2\cdots, m; \quad j = 1, 2, \cdots, n$$

Step 3

Calculate the weighted normalized decision matrix. First we need the weights. Weights can come from either the decision maker or by computation.

Step 3(a)

Use either the decision maker's weights for the attributes $x_1, x_2, \cdots, x_n$ or compute the weights through the use Saaty's (1980) AHP's decision maker weights method to obtain the weights as the eigenvector to the attributes versus attribute pair-wise comparison matrix.

$$\sum_{j=1}^{n} w_j = 1$$

The sum of the weights over all attributes must equal 1 regardless of the method used.

Step 3(b)

Multiply the weights to each of the column entries in the matrix from *Step* 2 to obtain the matrix, $\boldsymbol{T}$.

$$\boldsymbol{T} = \left(t_{ij}\right)_{m\times n} = \left(w_j r_{ij}\right)_{m\times n}, \quad i = 1, 2, \cdots, m$$

Step 4

Determine the worst alternative $\left(A_w\right)$ and the best alternative $\left(A_b\right)$: Examine each attribute's column and select the largest and smallest values appropriately. If the values imply larger is better (profit) then the best alternatives are the largest values and if the values imply smaller is better (such as cost) then the best alternative is the smallest value.

$$A_w = \left\{\left\langle \max\left(t_{ij} \middle| i = 1, 2, \cdots, m\right) j \in J_- \right\rangle, \left\langle \min\left(t_{ij} \middle| i = 1, 2, \cdots, m\right) j \in J_+ \right\rangle\right\} \equiv \left\{t_{wj} \middle| j = 1, 2, \cdots, n\right\}$$

$$A_{wb} = \left\{\left\langle \min\left(t_{ij} \middle| i = 1, 2, \cdots, m\right) j \in J_- \right\rangle, \left\langle \max\left(t_{ij} \middle| i = 1, 2, \cdots, m\right) j \in J_+ \right\rangle\right\} \equiv \left\{t_{bj} \middle| j = 1, 2, \cdots, n\right\}$$

where,

$J_+ = \left\{ j = 1, 2, \cdots, n \middle| j \right\}$ associated with the criteria having a positive impact, and

$J_- = \left\{ j = 1, 2, \cdots, n \middle| j \right\}$ associated with the criteria having a negative impact.

We suggest that if possible make all entry values in terms of positive impacts.

Step 5

Calculate the L2-distance between the target alternative i and the worst condition A_w

$$d_{iw} = \sqrt{\sum_{j=1}^{n} \left(t_{ij} - t_{wj}\right)^2}, \quad i = 1, 2, \cdots, m$$

and the distance between the alternative i and the best condition A_b

$$d_{ib} = \sqrt{\sum_{j=1}^{n} \left(t_{ij} - t_{bj}\right)^2}, \quad i = 1, 2, \cdots, m$$

where d_{iw} and d_{ib} are L2-norm distances from the target alternative i to the worst and best conditions, respectively.

Step 6

Calculate the similarity to the worst condition:

$$s_{iw} = \frac{d_{iw}}{\left(d_{iw} + d_{ib}\right)}, \quad 0 \le s_{iw} \le 1, \quad i = 1, 2, \cdots, m$$

$S_{iw} = 1$ if and only if the alternative solution has the worst condition; and

$S_{iw} = 0$ if and only if the alternative solution has the best condition.

Step 7

Rank the alternatives according to their value from S_{iw} $\left(i = 1, 2, \cdots, m\right)$.

2.2. Sensitivity Analysis

Since AHP, at least in the pairwise comparisons, is based upon subjective inputs using the 9 point scale then sensitivity analysis is extremely important. Leonelli [21] in his master's thesis, outlines procedures for sensitivity analysis to enhance decision support tools including numerical incremental analysis of a weight, probabilistic simulations, and mathematical models. How often do we change our minds about the relative importance of an object, place, or thing? Often enough that we should alter the pairwise comparison values to determine how robust our rankings are in the AHP process. We suggest doing enough sensitivity analysis to find the "break-point" values, if they exist, of the decision maker weights that change the rankings of our alternatives. Since the pairwise comparisons are subjective matrices compiled using the Saaty's method, we suggest as a minimum a "trial and error" sensitivity analysis using the numerical incremental analysis of the weights.

Chen [22] grouped sensitivity analysis into three main groups: numerical incremental analysis, probabilistic simulations, and mathematical models The numerical incremental analysis, also known as One-at-a-time (OAT) or "trial and error" works by incrementally changing one parameter at a time, finding the new solution and showing graphically how the ranks change. There exist several variations of this method [23] [24]. Probabilistic simulation employs Monte Carlo simulation [25] that allows random changes in the weights and simultaneously explores the effect on the ranks. Modeling may be used when it is possible to express the relationship between the input data and the solution results.

The decision weights are subject to sensitivity analysis to determine how the affect the final ranking. Sensitivity analysis is essential to good analysis. Additionally, Alinezhad [26] suggests sensitivity analysis for TOPSIS for changing an attribute weight. Equation (1) was developed for adjusting weights based upon a single weight change that we used is:

$$w'_j = \frac{\left(1 - w'_p\right)}{\left(1 - w_p\right)} w_j \tag{1}$$

where w'_j is the future weight of criteria j, w_p the current selected weight to be changed, w'_p the new value of the selected weight, w_j is the current weight of criteria j.

3. Application to Ranking Terrorist

A CT analyst produced both target lists (blue and green) between 2004-2005 [8]. After refinement, the blue target list consisted of 21 terrorists rank-ordered in importance. Additionally, the blue target list assigns the 21 individuals into "Tier" 1 through 5. After refining the second list, the green target list, it contained 31 rank ordered terrorists.

3.1. Criteria Variables: Terrorist Attributes

Based on a review of relevant literature as well as our combined experience of personnel in defense analysis department, we identify 96 critical attributes of terrorists to initially use in the modeling process. We organize these 96 critical attributes to test as predictive variables. Many of these variables were categorical (binary) variables, so we tried to consolidate and refine the number of variables to consider. We felt that initially concentrating on the decision criteria might provide useful information. To maintain organization, we subdivided the criteria into four main categories: Cell Membership/Experience Variables; Other Individual Variables; Worldliness Variables; and SNA/Graph Measures Variables that we refer to as Level 1 criteria. We then broke each of these into sub-criteria with their own respective data that we refer to as Level 2 criteria. The Level 2 criteria were used in the OML process. This is highlighted in Table 1.

We further propose a hierarchy for our analysis.

Objective: Find the Most Dangerous Terrorist

Alternatives: List of terrorists active in 2008

Criteria: Level 1: Level 2 breakdown

Step 1. Obtaining the decision maker weights by level.

Level 1: Priorities: Social Network Analysis, Individual Variables, Cell membership/experience, Worldliness. A begin the pairwise comparisons using our Excel template.

	Element		
A	Compared with B	More Important	Intensity (1 - 9)
Social Network	Individual Variables	A	3
	Cell membership	A	4
	Worldliness	A	5
Individual Variables	Cell membership	A	2
	Worldliness	A	4
Cell Membership	Worldliness	A	4

The decision matrix is

	Social Networks	Individual Variables	Cell Membership	Worldliness
Social Networks	1	3	4	5
Individual Variables	1/3	1	2	4
Cell Membership	1/4	1/2	1	4
Worldliness	1/5	1/4	1/4	1

The consistency ratio, $CR = 0.0372$, which is less than 0.1 implies the decision matrix is consistent. The decision weights for Level 1 are:

Eigenvector Criterion Weights	
Social Network	0.55728387
Individual Variables	0.21319939
Cell/Organizational Variables	0.14475047
Worldliness	0.08476628

Next, we proceed to do similar analyses for Level 2. We will take each set of Level 2 variables and obtain their respective weights. In show how we did this in more detail for only one of the Level 1 criteria, Social Networks.

Table 1. Criteria breakdown.

Criteria				
Level 1	Cell Membership/Experience	Individual Variables	Worldliness Variables	Social Network Analysis
Level 2	State Sponsorship	Versatility	languages	Degree Centrality
	Safe Havens	References	Countries	Eigenvector Centrality
	Unity	Age	Speaks English	Closeness
	Funds	Months as a Terrorist		Propagation Fit
	Criminal Activity	Number of Aliases		Bunker Score
	Organ. Structure			

For example, we start with the breakdown of Level 1 social network into specific Level 2 criteria shown to be valid variables and follow the same methods to obtain our decision weights.

The decision maker matrix for these sub-criteria based upon pairwise comparisons is

	Degree	Closeness	Eigenvector	Bunker	Propagation
Degree	1	2	3	4	5
Closeness	1/2	1	2	3	4
Eigenvector	1/3	1/2	1	2	3
Bunker	1/4	1/3	1/2	1	2
Propagation	1/5	1/4	1/3	1/2	1

The resulting weights were found and above matrix is consistent ($CR = 0.00318$).

Degree	0.43799368
Closeness	0.24613871
Eigenvector	0.14836386
Bunker	0.09752303
Propagation	0.06998072

We multiply these by the Level 1 weight of 0.55728387 to obtain the weights to be used in our TOPSIS model of

Degree	0.24405008
Closeness	0.13714849
Eigenvector	0.08266834
Bunker	0.05433983
Propagation	0.03899326

We followed this technique this for all Level 2 variables. We present the results only by criteria main level.

Individual Variables ($CR = 0.011$)

Versatility	0.032969674
Number of Alias	0.022282055
Months as a Terrorist	0.076963373
Number References	0.008325744
Age	0.014376266

Cell Membership/Experience ($CR = 0.02753$)

Type	0.006426886
Structure	0.0489001
State Sponsor	0.013992024
Safe Havens	0.009257188
Funds	0.007394782
Criminal Activity	0.025117386
Unity	0.033661635
Leader	0.047679325
Logistics	0.010593563

Worldliness ($CR = 0.003$)

Languages	0.04621939
Countries	0.02429715
English	0.01424946

We apply the TOPSIS seven steps as described in Section 2 with the data collected for our terrorists. We present our top 25 terrorist ranking in **Table 2**.

3.2. Sensitivity Analysis

We apply sensitivity analysis. The sensitivity analysis should be applied to the decision maker weights because they result from subjective pairwise comparison using Saaty's 9 point process.

We used the suggested sensitivity approach suggested by Alinezhad [26]. In their article they present mathematical formulas for many sensitivity results. We only use the incremental adjusted weights and with the speed of our computer template we quickly changed the weights and obtained new ranking. The formula used is repeated here:

$$w'_j = \frac{1-w'_p}{1-w_p} w_j$$

where w'_j is the new weight and w_p is the original weight of the criterion to be adjusted and w'_p is the value after the criterion was adjusted. We plotted the top 10 alternatives using several major adjustments in criteria weighting each time insuring a different criterion was the most heavily weighted. It is seen from the graph, **Figure 1**, that the top 2 did never changed positions.

A complete sensitivity analysis would concern each decision weight being incrementally changed and finding the range over which changes in ranking did or did not occur.

We present a side by side comparison showing the top 25 are still about the same with order adjustments. The top5 are identical and the top 10 are still the top 10 with only terrorist #42, #55, #25 having slight ranking changes as shown in **Table 3**.

Table 2. Ranking of the top 25 terrorists.

Alternative	TOPSIS Value	Terrorist # Code	Subjetive Tier Rank	Model Rank
22	0.675218	54	1	1
1	0.675216	12	1	2
26	0.54184	24	7	3
3	0.47225	53	4	4
24	0.47225	52	3	5
53	0.465736	40	7	6
65	0.388934	5	4	7
23	0.348206	3	3	8
42	0.331119	33	46	9
45	0.326806	90	65	10
2	0.318377	91	3	11
25	0.305574	50	6	12
55	0.288408	97	47	13
49	0.255626	23	40	14
63	0.1955147	16	62	15
40	0.192414	25	23	16
60	0.185771	30	52	17
34	0.180796	19	15	18
30	0.154171	6	11	19
18	0.137166	58	26	20
41	0.132053	27	25	21
10	0.10009	7	17	22
59	0.097761	15	51	23
21	0.088592	56	31	24
33	0.087089	103	14	25

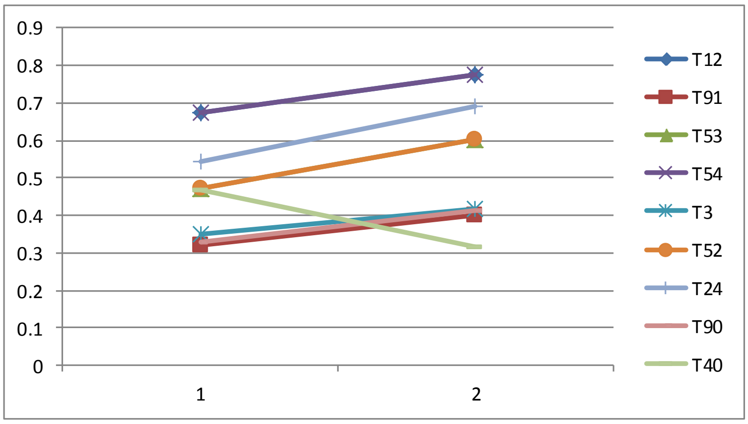

Figure 1. Sensitivity analysis for OML of one decision weight.

Table 3. Updated ranking of terrorists.

Terrorist		Sensitivity Analysis	
# Code	Rank	TOPSIS	Rank
54	1	54	1
12	2	12	2
24	3	24	3
53	4	53	4
52	5	52	5
40	6	3	6
5	7	90	7
3	8	91	8
33	9	40	9
90	10	5	10
91	11	33	11
50	12	50	12
97	13	97	13
23	14	23	14
16	15	25	15
25	16	16	16
30	17	30	17
19	18	19	18
6	19	15	19
58	20	58	20
27	21	6	21
7	22	99	22
15	23	98	23
56	24	27	24
103	25	77	25

This does indicate the model results are sensitive to the decision maker's pairwise comparisons that are used to find the decision maker weights.

4. Discussion

Based on our analysis, we see substantial benefits of applying our methodology to ordering the targeting of terrorist. However, since our MADM research was primarily focused on explaining and demonstrating this methodology, we first recommend that additional research be conducted in the form of applying this methodology to an active target set that can serve as a further proof of concept. Once our methodology can be verified and validated, we recommend integration into the targeting process of both counter-terrorist focused units and the larger force. We provide a conceptual framework for developing decision support tools for all types of decision problems beyond just the target prioritization problem. We envision an eventual suite of decision support tools and larger decision support systems to assist decision makers with a wide range of problems.

This process provides leadership at all levels with a methodology to produce a key target list among terrorist and terrorist organizations based upon quantitative analysis. We feel that having a quantitative process is better than either a totally subjective approach or a linear regression modeling approach offered by Twedell and Edmond's research.

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