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Wavelet Packet Frames on a Half Line Using the Walsh-Fourier Transform

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ABSTRACT

In this paper, we study the construction of dyadic wavelet packet frames on a positive half line $\Re^+$ using the Walsh-Fourier transform.

Keywords: Wavelet Packet; Frame; Walsh Function; Walsh-Fourier Transform; Dilation; Translation; Modulation

1. Introduction and Preliminaries

Frames in Hilbert space were introduced by Duffin and Schaffer [1] in 1952, in the context of non-harmonic Fourier series. Couple of years later, frames were brought to life by Daubechies, Grossmann and Meyer [2]. Frames are generalizations of orthonormal basis. The linear independence property for a basis, which allows every vector to be uniquely represented as a linear combination.

The theory of frames are widely used in signal processing, data analysis, image compression and enhancement, coding theory, filtering of signals and many more.

In recent years, wavelets have been generalized in many different setting see for instance Dahlke [3], Holschneider [4], Papadakis [5], Lang [6-8].

Various authors studied the wavelet frames and dyadic wavelet frames such as Daubechies [9], Chui and Shi [10], Casazza and Christensen [11], Christensen [12,13], Protosov and Farkov [14], Farkov [15], Shah and Debnath [16], Ahmad and Iqbal [17,18]. Motivated by these authors, in this paper, we extended our results to dyadic wavelet packet frames on the positive half line $\Re^+$.

Let $\Re^+ = [0,+\infty)$ be the positive half line and $\square^+ = \{0,1,2,\cdots\}$. Let us denote the integer and fractional parts of a number $x \in \Re^+$ by $[x]$ and $\{x\}$, respectively. Then, for each $x \in \Re^+$ and any positive integer *j*, we set

$$x_j = \left[2^j x\right](\bmod 2),\ x_{-j} = \left[2^{1-j} x\right](\bmod 2). \quad (1)$$

For each $x \in \Re^+$, these numbers are the digits of a binary expansion

$$x = [x] + \{x\} = \sum_{j<0} x_j 2^{-j-1} + \sum_{j>0} x_j 2^{-j}.$$

It is clear that

$$[x] = \sum_{j=1}^{\infty} x_{-j} 2^{-j-1} \text{ and } \{x\} = \sum_{j=1}^{\infty} x_j 2^{-j}$$

and there exists $k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0$ for all $j > k$.

The binary dyadic addition on $\Re^+$ is defined by

$$x \oplus y = \sum_{j<0} \left|x_j - y_j\right| 2^{-j-1} + \sum_{j>0} \left|x_j - y_j\right| 2^{-j},$$

where x_j, y_j are defined in (1). Moreover, we note that $x \oplus y = x \ominus y = 0$, where $\ominus$ denotes the substitution modulo 2 on $\Re^+$.

For $x \in [0,1)$, let $\omega_1(x)$ be given by

$$\omega_1(x) = \begin{cases} 1, & \text{if } x \in [0,1/2), \\ -1, & \text{if } x \in [1/2,1). \end{cases}$$

The extension of the function ω_1 to $\Re^+$ is denoted by the equality $\omega_1(x+1) = \omega_1(x)$ for all $x \in \Re^+$. Then, the generalized Walsh functions $\{\omega_n(x) : n \in \square^+\}$ are defined by

$$\omega_0 \equiv 1,\ \omega_n(x) = \Pi_{j=0}^{k} \left(\omega_1\left(2^j x\right)\right)^{\mu_j},\ n \in \mathbb{N}, x \in \Re^+, \quad (2)$$

where

$$n = \sum_{j=0}^{k} \mu_j 2^j,\ \mu_j \in \{0,1\},\ \mu_k = 1,\ k = k(n).$$

Note that the Walsh functions almost behave like characters with respect to dyadic addition, namely

$$\omega_n(x \oplus y) = \omega_n(x)\omega_n(y),\ n \in \square,\ x, y \in [0,1). \quad (3)$$

Thus, for each fixed *y*, equality (3) is valid for all $x \in \Re^+$ except countably many of them.

For $x, y \in \Re^+$, let

$$\chi(x,y)=(-1)^{\sigma(x,y)}, \text{ where}$$
$$\sigma(x,y)=\sum_{j=1}^{\infty}\left(x_j y_{-j}+x_{-j}y_j\right), \tag{4}$$

and x_j, y_j are given by (1). Note that

$$\chi\left(x,2^{-n}m\right)=\chi\left(2^{-n}x,m\right)=\omega_m\left(2^{-n}x\right),$$

for all $x\in\left[0,2^{-n}\right)$ and $m,n\in\mathbb{Z}^{+}$. It is shown by Golubov *et al.* [19] that both the system $\left\{\chi(\alpha,\cdot)\right\}_{\alpha=0}^{\infty}$ and $\left\{\chi(\cdot,\alpha)\right\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1)$.

The Walsh Fourier transform of a function $f\in L^1\left(\Re^+\right)$ is defined by

$$\hat{f}(\omega)=\int_{\Re^+}f(x)\overline{\chi(x,\omega)}\,\mathrm{d}x,$$

where $\chi(x,\omega)$ is given by (4). The properties of the Walsh Fourier transform are quite similiar to those of the classical Fourier transform [19-21]. In particular, $f\in L^2\left(\Re^+\right)$, then $\hat{f}\in L^2\left(\Re^+\right)$ and

$$\left\|\hat{f}\right\|_{L^2\left(\Re^+\right)}=\left\|f\right\|_{L^2\left(\Re^+\right)}. \tag{5}$$

By a dyadic interval of range n in $[0,1)$, we mean intervals of the form

$$I_{(n)}^k=\left[k2^{-n},(k+1)2^{-n}\right),\ k\in\mathbb{Z}^{+}.$$

It is easy to verify that

$$I_{(n)}^k\cap I_{(n)}^l=\varnothing,\ \ k\neq l \text{ and } \bigcup_{k=0}^{2^n-1}I_{(n)}^k=[0,1). \tag{6}$$

Moreover, the dyadic topology is generated by the collection of dyadic intervals and each dyadic interval is both open and closed under the dyadic topology. Therefore, it follows that for each $0\le j<2^j$, the Walsh function $\omega_j(x)$ is piecewise constant and hence continuous. Thus $\omega_j(x)=1$ for $x\in I_{(n)}^0$. For each $x\in[0,1)$ and $n\in\mathbb{N}$, we denote the dyadic interval of length 2^{-n} which contains x by $I_{(n)}(x)$. Thus,

$$I_{(n)}(x)=I_{(n)}^k(x),$$

where $0\le k<2^n$ is uniquely determined by the relationship $x\in I_{(n)}(x)$.

By a Walsh polynomial, we mean a finite linear combination of Walsh functions. Thus, an arbitrary Walsh polynomial of order n can be written as

$$\omega(x)=\sum_{j=0}^{n}b_j\omega_j(x), \tag{7}$$

where the b_j are complex coefficients. Since $\omega_j(x)$ is constant on $I_{(n)}(x)$, for each $0\le j<2^n$, therefore, each Walsh polynomial is a dyadic step function and vice versa [19,21].

Let $\varepsilon_n\left(\Re^+\right)$ be the space of dyadic entire functions of order n, that is, the set of all functions which are constant on all intervals of range n. Thus, for every $f\in\varepsilon_n\left(\Re^+\right)$, we have

$$f(x)=\sum_{k=0}^{\infty}f\left(2^{-j}k\right)\chi_{I_n}(x),\ \ x\in\Re^+. \tag{8}$$

Clearly, each Walsh polynomial of order 2^{n-1} belongs to $\varepsilon_n\left(\Re^+\right)$. The set $\varepsilon\left(\Re^+\right)$ of dyadic entire functions on $\Re^+$ is the union of all the spaces $\varepsilon_n\left(\Re^+\right)$. It is clear that $\varepsilon\left(\Re^+\right)$ is dense in $L^p\left(\Re^+\right)$, $1\le p<\infty$, and each function in $\varepsilon\left(\Re^+\right)$ is of compact support and so is its Walsh Fourier transform. Thus , we will consider the following set of functions:

$$\varepsilon^0\left(\Re^+\right)=\left\{f\in\varepsilon\left(\Re^+\right):\operatorname{supp}\hat{f}\subset\Re^+\setminus\{0\}\right\}. \tag{9}$$

A system of elements $\left\{f_n\right\}_{n\in\Lambda}$ in a Hilbert space H is called a frame for H if there exists two +ve numbers A and B such that for any $f\in H$,

$$A\|f\|^2\le\sum_{n\in\Lambda}\left|\left\langle f,f_n\right\rangle\right|^2\le B\|f\|^2.$$

The numbers A and B are called frame bounds. If $A=B$, the frame is said to be tight. The frame is called exact if it ceases to be a frame whenever any single element is deleted from the frame.

The continuous wavelet transformation of a L^2-function f with respect to the wavelet ψ, which satisfies admissibility condition, is defined as:

$$\left(T^{wav}f\right)(a,b)=|a|^{-1/2}\int_{-\infty}^{\infty}f(t)\overline{\psi\left(\frac{t-b}{a}\right)}\mathrm{d}t,$$
$$a,b\in\Re^+,\ a\neq 0.$$

The term wavelet denotes a family of functions of the form $\psi_{a,b}=|a|^{-1/2}\psi\left((t-b)/a\right)$, obtained from a single function ψ by the operation of dilation and translation.

For a function $\psi\in L^2\left(\Re^+\right)$, we define the following operators as follows:

Translation: $T_a\psi(x)=\psi(x-a), x\in\Re^+, a>0$.
Modulation: $E_a\psi(x)=\mathrm{e}^{2\pi iax}\psi(x), x\in\Re^+, a>0$.
Dilation: $D_a\psi(x)=|a|^{-1/2}\psi(x/a)$, for all $x\in\Re^+, a>0$.

2. Wavelet Packets on $\Re^+$

We have the following sequence of functions due to Wickerhauser [22]. For $l\in\mathbb{Z}^+$

$$\psi_{2l}(x)=\sqrt{2}\sum_{k\in\mathbb{Z}}a_k\psi_l(2x-k) \text{ and}$$
$$\psi_{2l+1}(x)=\sqrt{2}\sum_{k\in\mathbb{Z}}b_k\psi_l(2x-k), \tag{i}$$

where $a=\left\{a_k\right\}$ is the filter such that $\sum_{n\in\mathbb{Z}}a_{n-2k}a_{n-2l}=\delta_{kl},\ \sum_{n\in\mathbb{Z}}a_n=\sqrt{2}$ and $b_k=(-1)^k a_{1-k}$.

For $l=0$ in (i), we get

$$\psi_0(x)=\psi_0(2x)+\psi_0(2x-1),$$

$$\psi_1(x)=\psi_0(2x)-\psi_0(2x-1),$$

where ψ_0 is a scaling function and may be taken as a characteristic function. If we increase *l*, we get the following

$$\psi_2(x)=\psi_1(2x)+\psi_1(2x-1),$$

$$\psi_3(x)=\psi_1(2x)-\psi_1(2x-1)$$

$$\psi_4(x)=\psi_1(4x)+\psi_1(4x-1)+\psi_1(4x-2)+\psi_1(4x-3)$$

and so on.

Here ψ_l 's have a fixed scale but different frequencies. They are Walsh functions in $[0,1)$. The functions $\psi_l(t-k)$, for integers *k*, *l* with $l\geq 0$, form an orthonormal basis of $L^2(\Re^+)$.

Theorem 2.1. For every partition *P* of the non-negative integers into the sets of the form $I_{lj}=\{2^j l,\cdots,2^j(l+1)-1\}$, the collection of functions $\psi_{l;j,k}=2^{j/2}\psi_l(2^j x-k)$, $l\in\mathbb{Z}^+$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^+$, is an orthonormal basis of $L^2(\Re^+)$.

Dyadic Wavelet Packet Frames on $\Re^+$

For any function $\psi_l\in L^2(\Re^+)$, we consider the system of function $\{\psi_{l;j,k}\}$, $j,k\in\mathbb{Z}\times\mathbb{Z}^+$ in $L^2(\Re^2)$ and $l\in\mathbb{Z}^+$ as,

$$\left\{\psi_{l;j,k}(x)=2^{j/2}\psi_l\left(2^j x\ominus k\right):\ j,k\in\mathbb{Z}\times\mathbb{Z}^+,l\in\mathbb{Z}^+,x\in\Re^+\right\}. \tag{10}$$

By taking Walsh Fourier transform to (10), we obtain,

$$\hat{\psi}_{l;j,k}(\zeta)=2^{-j/2}\hat{\psi}_l\left(2^{-j}\zeta\right)\omega_k\left(2^{-j}\zeta\right).$$

Then, we call system (10) wavelet packet frame for $L^2(\Re^+)$ if there exists constants *C* and *D*, $0<C\leq D<\infty$ such that

$$C\|f\|^2\leq\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2\leq D\|f\|^2, \tag{11}$$

for all $f\in L^2(\Re^+)$. The constants *C* and *D* are called frame bounds. If *C* = *D*, the frame is said to be tight. The frame is called exact if it ceases to be a frame whenever any single element is deleted from the frame.

Since the set $\varepsilon(\Re^+)$ is dense in $L^p(\Re^+)$, $1\leq p<\infty$ and is closed under Walsh-Fourier transform, the set $\varepsilon^0(\Re^+)$ defined by (9) is also dense in $L^2(\Re^+)$.

Therefore, the system given in (10) is frame for $L^2(\Re^+)$ if the inequalities in (11) holds for all $f\in\varepsilon^0(\Re^+)$.

For $j\in\mathbb{Z}, m\in\mathbb{Z}^+$, we have

$$\int_{2^j m}^{2^j(m+1)}\omega_k\left(2^{-j}\zeta\right)\mathrm{d}\zeta=\int_0^{2^j}\omega_k\left(2^{-j}\zeta+2^j m\right)\mathrm{d}\zeta$$
$$=\int_0^{2^j}\omega_k\left(2^{-j}\zeta\right)\mathrm{d}\zeta.$$

Let $f\in\varepsilon(\Re^+)$ and ψ_l be in $L^2(\Re^+)$, then

$$\left\langle f,\psi_{l;j,k}\right\rangle=2^{-j/2}\int_0^{2^j}\left[\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\mathbf{F}_{l;j,m}(\zeta)\right]\omega_k\left(2^{-j}\zeta\right)\mathrm{d}\zeta$$

where $\mathbf{F}_{l;j,m}(\zeta)=\hat{f}\left(\zeta\oplus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)}$. Now for each $j\in\mathbb{Z}$, let $\mathbf{F}_j$ be the function defined by

$$\mathbf{F}_j(\zeta)=\sum_{l\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}F_{l;j,m}(\zeta).$$

Then, clearly $\mathbf{F}_j\left(\zeta\oplus 2^j\right)=\mathbf{F}_j(\zeta)$, for all $\zeta\in\Re^+$ and in view of (8), we have ,

$$\mathbf{F}_j(\zeta)=\sum_{k\in\square^+}d_k\left(\mathbf{F}_j\right)\omega_k\left(2^{-j}\zeta\right),\ \zeta\in\left[0,2^j\right)$$

where

$$d_k\left(\mathbf{F}_j\right)=2^{-j}\int_0^{2^j}\mathbf{F}_j(\zeta)\omega_k\left(2^{-j}\zeta\right)\mathrm{d}\zeta.$$

Applying Parseval's formula and the fact that $\{\omega_n:n\geq 0\}$ forms n orthonormal basis for $L^2[0,1]$, we obtain,

$$\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2=\int_{\Re^+}\overline{\hat{f}(\zeta)\hat{\psi}_l\left(2^{-j}\zeta\right)}\left\{\sum_{l\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}\hat{f}\left(\zeta\oplus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)}\right\}\mathrm{d}\zeta. \tag{12}$$

Lemma 2.1. Let $f\in\varepsilon^0(\Re^+)$ and ψ_l be in $L^2(\Re^+)$. If $\operatorname{ess\,sup}_{\zeta\in\Re^+}\sum_{l\in\square^+}\sum_{j\in\square}\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2<+\infty$, then

$$\sum_{l\in\square^+}\sum_{j\in\square}\sum_{k\in\square^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2=\int_{\Re^+}\left|\hat{f}(\zeta)\right|^2\sum_{l\in\square^+}\sum_{j\in\square}\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2\mathrm{d}\zeta+R(f). \tag{13}$$

where

$$R(f)=\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\int_{\Re^+}\overline{\hat{f}(\zeta)\hat{\psi}_l\left(2^{-j}\zeta\right)}\hat{f}\left(\zeta\oplus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)}\mathrm{d}\zeta. \tag{14}$$

Furthermore, the iterated series in (14) is absolutely convergent.

Proof. From (12) we have

$$\sum_{l\in\square^+}\sum_{j\in\square}\sum_{k\in\square^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2=\sum_{l\in\square^+}\sum_{j\in\square}\int_{\Re^+}\left\{\left|\hat{f}(\zeta)\right|^2\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2+\overline{\hat{f}(\zeta)\hat{\psi}_l\left(2^{-j}\zeta\right)}\times\sum_{m\in\mathbb{N}}\hat{f}\left(\zeta\oplus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)}\mathrm{d}\zeta\right\}$$
$$=\sum_{l\in\square^+}\sum_{j\in\square}\int_{\Re^+}\left|\hat{f}(\zeta)\right|^2\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2\mathrm{d}\zeta+R(f).$$

Since $\operatorname{ess\,sup}_{\zeta\in\Re^+}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2<+\infty$, and therefore, by the Levi's Lemma, we obtain

$$\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2=\int_{\Re^+}\left|\hat{f}(\zeta)\right|^2\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2\mathrm{d}\zeta+R(f).$$

Now we claim that the itrated series in (14) is absolutely convergent. To do this, let

$$\begin{aligned}I&=\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\int_{\Re^+}\left|\hat{f}(\zeta)\hat{\psi}_l\left(2^{-j}\zeta\right)\hat{f}\left(\zeta\oplus 2^j m\right)\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)\right|\mathrm{d}\zeta\\&=\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{N}}2^j\int_{\Re^+}\left|\hat{f}\left(2^j\zeta\right)\hat{\psi}_l(\zeta)\hat{f}\left(2^j(\zeta\oplus m)\right)\hat{\psi}_l(\zeta\oplus m)\right|\mathrm{d}\zeta.\end{aligned}$$

Note that

$$\left|\hat{\psi}_l(\zeta)\hat{\psi}_l(\zeta\oplus m)\right|\le 12\left(\left|\hat{\psi}_l(\zeta)\right|^2+\left|\hat{\psi}_l(\zeta\oplus m)\right|^2\right).$$

Therefore, it suffices to prove that

$$\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{N}}2^j\int_{\Re^+}\left|\hat{f}\left(2^j\zeta\right)\hat{f}\left(2^j\zeta\oplus 2^j m\right)\right|\left|\hat{\psi}_l(\zeta)\right|^2\mathrm{d}\zeta<\infty. \tag{15}$$

Since $m\neq 0\,(m\in\mathbb{N})$ and $f\in\varepsilon^0\left(\Re^+\right)$, therefore $J>0$ such that for all $|j|>J$,

$$\hat{f}\left(2^j\zeta\right)\hat{f}\left(2^j\zeta\oplus 2^j m\right)=0.$$

On the other hand for each fixed $|j|\le J$ and $\zeta\in\Re^+$, there exists a constant M such that for all $m>M$

$$\hat{f}\left(2^j\zeta\oplus 2^j m\right)=0.$$

Thus, it follows that only a finite number of terms of the iterated series in (15) are non zero. Consequently, there exists constant C such that

$$I\le C\left\|\hat{f}\right\|_\infty^2\left\|\psi_l\right\|_2^2.$$

This fact shows that iterated series in (14) is absolutely convergent.

3. Main Results

Theorem 3.1. If $\left\{\psi_{l;j,k}(x):(j,k)\in\mathbb{Z}\times\mathbb{Z}^+,l\in\mathbb{Z}^+\right\}$ is a wavelet packet frame in $L^2\left(\Re^+\right)$ with frame bound C and D, then

$$C\le\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2\le D,\ a.e.\ \zeta\in\Re^+. \tag{16}$$

Proof. For $f\in\varepsilon\left(\Re^+\right)$ and $\psi_l\in L^2\left(\Re^+\right)$, now by Equation (12),

$$\begin{aligned}&\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2\\&=\int_{\Re^+}\overline{\hat{f}(\zeta)}\hat{\psi}_l\left(2^{-j}\zeta\right)\\&\quad\cdot\left\{\sum_{l\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}\hat{f}\left(\zeta\oplus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)}\right\}\mathrm{d}\zeta.\end{aligned}$$

Let S_j be the set of all regular points of $\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2$, which means that for each $x\in S_j$,

$$2^n\int_{\zeta-x\in I_n(x)}\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2\mathrm{d}\zeta\to\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2,\ \text{as } n\to\infty.$$

Then means $\left(S_j^c\right)=0$. Thus, means $\left(\bigcup_{j\in\mathbb{Z}}S_j^c\right)=0$ [23], now let $\zeta_0\to\Re^+\setminus\bigcup_{j\in\mathbb{Z}}S_j^c$. Then, for each fixed positive integer T, we consider

$$\hat{f}(\zeta)=2^{t/2}\omega_t\left(\zeta-\zeta_0\right),\ t\ge T,$$

where $\omega_t\left(\zeta-\zeta_0\right)$ are the Walsh function of $\zeta_0+I_t(x)$ where $I_t's$ are mutually disjoint , therefore, for $m\in\mathbb{N}$ and $j\ge -T$, we have

$$\hat{f}\left(2^j\zeta\right)\hat{f}\left(\zeta\oplus 2^j m\right)=0$$

and hence $\|f\|_2^2=1$.

Furthermore, we have,

$$\begin{aligned}&\sum_{j\ge -T}\sum_{l\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2\\&=\sum_{j\ge -T}\int_{\zeta+I_t(x)}2^t\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2\mathrm{d}\zeta\le D.\end{aligned}$$

By letting $t\to\infty$ and $T\to\infty$ consecutively, we obtain

$$\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^{-j}\zeta_0\right)\right|^2\le D,$$

which is the right inequality of (16).

In order to prove the left inequality of (16), let

$$\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2=I_1+I_2,$$

where

$$I_1=\sum_{l\in\mathbb{Z}^+}\sum_{j>-T}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2\ \text{and}$$

$$I_2=\sum_{l\in\mathbb{Z}^+}\sum_{j\le -T}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{l;j,k}\right\rangle\right|^2.$$

Since $\left\{\psi_{l;j,k}(x)\right\}$ is a frame for $L^2\left(\Re^+\right)$, so, $I_1\ge C-I_2$. As we have already shown it that $I_1=\sum_{l\in\mathbb{Z}^+}\sum_{j>-T}\left|\hat{\psi}_l\left(2^{-j}\zeta_0\right)\right|^2$, therefore, it is enough to prove that $I_2\to 0$ as $T\to\infty$. By (12) and the Cauchy-Schwartz inequality, we obtain

$$0 \le I_2 \le \sum_{l\in\mathbb{Z}^+}\sum_{j\le -T}\sum_{m\in\mathbb{Z}^+}\left\{\int_{\Re^+}\left|\overline{\hat{f}(\zeta)}\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2 \mathrm{d}\zeta\right\}^{1/2}\left\{\int_{\Re^+}\left|\hat{f}\left(\zeta\oplus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\oplus m\right)}\right|^2 \mathrm{d}\zeta\right\}^{1/2}.$$

If $\zeta + 2^j m \in \zeta_0 \oplus I_t$, then for each fixed $j \le -T$, we have $\left|2^j m\right| \le 2^{-t}$ and hence $|m| \le 2^{-t-j}$. Consequently, by the Walsh Fourier transform of f, the number of summation index m is bounded by 2^{-t-j}. Thus

$$I_2 \le \sum_{l\in\mathbb{Z}^+}\sum_{j\le -T}\sum_{m\in\mathbb{Z}^+}2^{-t-j}\int_{\Re^+}\left|\overline{\hat{f}(\zeta)}\hat{\psi}_l\left(2^{-j}\zeta\right)\right|^2 \mathrm{d}\zeta \le \sum_{l\in\mathbb{Z}^+}\sum_{j\le -T}\int_{2^j\zeta_0+I_{-j+t}}\left|\hat{\psi}_l(\zeta)\right|^2 \mathrm{d}\zeta. \tag{17}$$

For given $\varepsilon > 0$ and $\zeta_0 \ne 0$, we choose T such that

$$2^{-T} < \left|\zeta_0\right| = 2^s \text{ and } \int_{I_{T-s}}\left|\hat{\psi}_l(\zeta)\right|^2 \mathrm{d}\zeta < \epsilon.$$

Then, we have

$$2^j\zeta_0 + I_{-j+t}(x) \subset I_{T-s}(x),\ \forall j \le T \tag{18}$$

as

$$\left|2^j\zeta_0\right| = 2^j 2^s \le 2^{-T}2^s \text{ and } I_{-j+t}(x) \subset I_{T-s}(x).$$

Since $I_t'{s}$ are mutual disjoint, therefore, it can be easily verified that

$$\left\{2^{j_1}\zeta_0 + I_{-j_1+t}(x)\right\} \cap \left\{2^{j_2}\zeta_0 + I_{-j_2+t}(x)\right\} = \varnothing, \tag{19}$$

for each $j_1 < j_2 \le -T$.

Applying (18) and (19) in (17), we obtain,

$$I_2 \le \int_{I_{T-s}}\left|\hat{\psi}_l(\zeta)\right|^2 \mathrm{d}\zeta < \epsilon.$$

Now by Chui and Shi [10] and Shah and Debnath [16], we get the desired results.

Theorem 3.2. Let $\psi_l \in L^2(\Re^+)$, $l \in \mathbb{Z}^+$ be such that

$$C_{\psi_l} = \inf_{|\zeta|\in[1,2]}\left[\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 - \sum_{m\ne 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^j\zeta\right)\hat{\psi}_l\left(2^j\zeta\oplus m\right)\right|\right] > 0$$

$$D_{\psi_l} = \sup_{|\zeta|\in[1,2]}\left[\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 + \sum_{m\ne 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^j\zeta\right)\hat{\psi}_l\left(2^j\zeta\oplus m\right)\right|\right] < \infty.$$

Then $\left\{D_{2^j}T_k\psi_l(x)_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^+,l\in\mathbb{Z}^+}\right\}$ is a wavelet packet frame for $L^2(\Re^+)$ with bounds C_{ψ_l}, D_{ψ_l}.

Proof. For a function $f \in L^2(\Re^+)$, $m \in \mathbb{Z}^+$, we have

$$\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f, D_{2^j}T_k\psi_l\right\rangle\right|^2 = \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle \hat{f}, D_{2^{-j}}E_{-k}\hat{\psi}_l\right\rangle\right|^2 = \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle \hat{f}, E_{-k2^j}D_{2^{-j}}\hat{\psi}_l\right\rangle\right|^2$$

$$= \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\int_{\Re^+}\hat{f}(\zeta)\overline{E_{-k2^j}D_{2^{-j}}\hat{\psi}_l(\zeta)}\mathrm{d}\zeta\right|^2 = \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}2^{-j}\sum_{k\in\mathbb{Z}^+}\left|\int_{\Re^+}\hat{f}(\zeta)\overline{\hat{\psi}_l\left(2^{-j}\zeta\right)}\mathrm{e}^{2\pi \mathrm{i}k2^{-j}\zeta}\mathrm{d}\zeta\right|^2$$

$$= \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}2^{-j}2^j\int_0^{2^j}\left|\sum_m \hat{f}\left(\zeta\ominus 2^j m\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)}\right|^2 \mathrm{d}\zeta$$

$$= \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\int_0^{2^j}\sum_h \hat{f}\left(\zeta\ominus 2^j h\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\ominus h\right)}\sum_m \overline{\hat{f}\left(\zeta\ominus 2^j m\right)}\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\mathrm{d}\zeta$$

$$= \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_h\int_0^{2^j}\hat{f}\left(\zeta\ominus 2^j h\right)\overline{\hat{\psi}_l\left(2^{-j}\zeta\ominus h\right)}\sum_m \overline{\hat{f}\left(\zeta\ominus 2^j m\right)}\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\mathrm{d}\zeta$$

$$= \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\int_{\Re^+}\hat{f}(\zeta)\overline{\hat{\psi}_l(2^{-j}\zeta)}\sum_m \overline{\hat{f}\left(\zeta\ominus 2^j m\right)}\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\mathrm{d}\zeta$$

$$= \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_m\int_{\Re^+}\hat{f}(\zeta)\overline{\hat{f}\left(\zeta\ominus 2^j m\right)}\overline{\hat{\psi}_l\left(2^{-j}\zeta\right)}\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\mathrm{d}\zeta$$

$$= \int_{\Re^+}\left|\hat{f}(\zeta)\right|^2\sum_l\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2\mathrm{d}\zeta + \sum_{m\ne 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\int_{\Re^+}\hat{f}(\zeta)\overline{\hat{f}\left(\zeta\ominus 2^j m\right)}\,\overline{\hat{\psi}_l\left(2^{-j}\zeta\right)}\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\mathrm{d}\zeta = (*).$$

Applying the Cauchy-Schwarz inequality twice, we have

$$
\begin{aligned}
(*) \le & \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 \mathrm{d}\zeta + \sum_{m\neq 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \left(\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|\left|\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\right|\right)^{1/2} \\
& \cdot\left|\hat{f}\left(\zeta\ominus 2^j m\right)\right|\left(\left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|\left|\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\right|\right)^{1/2}\mathrm{d}\zeta \\
\le & \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 \mathrm{d}\zeta + \sum_{m\neq 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left(\int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|\left|\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\right|\mathrm{d}\zeta\right)^{1/2} \\
& \cdot\left(\int_{\mathfrak{R}^+} \left|\hat{f}\left(\zeta\ominus 2^j m\right)\right|^2 \left|\hat{\psi}_l\left(2^{-j}\zeta\right)\right|\left|\hat{\psi}_l\left(2^{-j}\zeta\ominus m\right)\right|\mathrm{d}\zeta\right)^{1/2} \\
\le & \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 \mathrm{d}\zeta + (a')(a'').
\end{aligned}
$$

The terms (a') and (a'') are actually identical (use the change of variable $\zeta \to \zeta + 2^j m$ in (a'')), so by changing the summation index $j \to -j$, $m \to -m$, we have

$$
\begin{aligned}
(*) \le & \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 \mathrm{d}\zeta + \sum_{m\neq 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \left|\hat{\psi}_l\left(2^j\zeta\right)\right|\left|\hat{\psi}_l\left(2^j\zeta\oplus m\right)\right|\mathrm{d}\zeta \\
\le & \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \left(\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 + \sum_{m\neq 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|\left|\hat{\psi}_l\left(2^j\zeta\oplus m\right)\right|\right)\mathrm{d}\zeta.
\end{aligned}
$$

Thus,

$$
\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+} \left|\left\langle f, D_{2^j} T_k \psi_l\right\rangle\right|^2 \le D_{\psi_l}\|f\|^2 .
$$

A similar conclusion shows

$$
(*) \ge \int_{\mathfrak{R}^+} \left|\hat{f}(\zeta)\right|^2 \left(\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|^2 - \sum_{m\neq 0}\sum_{l\in\mathbb{Z}^+}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l\left(2^j\zeta\right)\right|\left|\hat{\psi}_l\left(2^j\zeta\oplus m\right)\right|\right)\mathrm{d}\zeta.
$$

Thus result follows.

RETRACTED

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