An Asymptotic Formula on an F.Smarandache Function

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Abstract: For any prime p, let $e_p(n)$ denotes the largest exponent of power p which divides n. the arithmetical function $e_{pq}(n)$ defined as follows: for any two primes p and q with $(p,q)=1$, let $e_{pq}(n)$ denotes the largest exponent of power pq which divides n. In this paper, we use the elementary methods to study the mean value properties of $e_{pq}(n)$, and give an interesting asymptotic formula for it.

Keywords: the perfect k-th power number; asymptotic formula; Mean value.

1 Introduction

For any prime p, let $e_p(n)$ denotes the largest exponent of power p which divides n. In problem 68 of reference [1], Professor F. Smarandache asked us to study the properties of this arithmetical function. About this problem, many scholars showed great interest in it, and obtained some interesting results, see references [2] and [3].Similarly, we will define arithmetical function $e_{pq}(n)$ as follows: for any two primes p and q with (p, q) = 1, let $e_{pq}(n)$ denotes the largest exponent of power pq which divides n. That is, $e_{pq}(n)=\max\{a:(pq)^a \mid n, a\in N^+\}$

According to [1], a number n is called a perfect k-th power number if it satisfied $k\mid a$ for all $p^a \| a$, where $p^a \| a$ denotes $p^a \mid a$ but $p^{a+1} \dagger$ n. Let A denotes the set of all the perfect k-th power numbers. It seems that no one knows the relations between these two arithmetical functions before. The main purpose of this paper is using the elementary methods to study the mean value properties of $e_{pq}(n)$ acting on the set A, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let p and q are two primes with (p, q) = 1, then for any real number $x\ge 1$, we have the asymptotic formula

$$l/l_0$$

$$\sum_{\substack{n\le x\\ n\in A}} e_{pq}(n)=C_{P,q}kx^{\frac{1}{k}}+\left(x^{\frac{1}{2k}+e}\right)$$

,Where $C_{P,q}=\frac{(p-1)(q-1)}{pq}\sum_{n=1}^{\infty}\frac{n}{(pq)^n}$ is a computable positive constant, and e denotes any fixed positive number.

2 Proof of the Theorem

In this section, we shall complete the proof of the theorem. First we define arithmetical function $a(n)$ as follows:

$$a(n)=\begin{cases}1, if\ n\ is\ a\ perfect\ k-th\ power\ number\\ 0, \quad otherwise\end{cases}$$

In order to complete the proof of Theorem, we need the following:

Lemma. For any real number $x\ge 1$, we have the asymptotic formula

$$\sum_{\substack{n\le x\\ (n,pq)=1}} a(n)=x^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}+O\left(x^{\frac{1}{2k}+e}\right)1$$

Proof. Let $f(s)=\sum_{\substack{n=1\\ (n,pq)=1}}\frac{a(n)}{n^s}$ where Re(s) > 1. From the Euler product formula [4] and the multiplicative properties of $a(n)$, we have

$$f(s)=\prod_{\substack{P\\ (P,pq)=1}}\left(1+\frac{a(P^k)}{P^{ks}}+\frac{a(P^{2k})}{P^{2ks}}+\mathbf{L}\right)$$

$$=\prod_{P}\left(1+\frac{1}{P^{ks}}+\frac{1}{P^{2ks}}+\mathbf{L}\right)\times\left(1-\frac{1}{p^{ks}}\right)\times\left(1-\frac{1}{q^{ks}}\right)$$

$$=z(ks)\left(1-\frac{1}{p^{ks}}\right)\times\left(1-\frac{1}{q^{ks}}\right)$$

where $z(s)$ is the Riemann zeta-function, and $\prod_P$ denotes the product over all primes.

Now by Perron formula [5] with

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$s_0=0$, $b=\frac{1}{k}+\frac{1}{\log x}$, $T=x^{\frac{1}{2k}}$, $H(x)=x$,and $B(s)=\frac{1}{s-\frac{1}{k}}$, we have

$$\sum_{\substack{n\le x\\(n,pq)=1}} a(n)=$$

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(ks)\frac{(p^{ks}-1)(q^{ks}-1)}{(pq)^{ks}}\frac{x^s}{s}ds+O\left(x^{\frac{1}{2k}+\varepsilon}\right)$$

To estimate the main term, we move the integral line from

$b=\frac{1}{k}+\frac{1}{\log x}$ to $a=\frac{1}{2k}+\frac{1}{\log x}$ Therefore,

$$\frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT}+\int_{b+iT}^{a+iT}+\int_{a+iT}^{a-iT}+\int_{a-iT}^{b-iT}\right)f(s)\frac{x^s}{s}ds$$

$$=\operatorname{Res}\left[f(s)\frac{x^s}{s},\frac{1}{k}\right]$$

Note that $\lim_{s\to 1}\zeta(s)(s-1)=1$, we may immediately get

$$\operatorname{Res}\left[f(s)\frac{x^s}{s},\frac{1}{k}\right]=x^{\frac{1}{k}}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)$$

Now from the estimate

$\left|\frac{1}{2\pi i}\left(\int_{b+iT}^{a+iT}+\int_{a+iT}^{a-iT}+\int_{a-iT}^{b-iT}\right)f(s)\frac{x^s}{s}ds\right| \ll x^{\frac{1}{2k}+\varepsilon}$ we can easily get

$$\sum_{\substack{n\le x\\(n,pq)=1}} a(n)=x^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}+O\left(x^{\frac{1}{2k}+\varepsilon}\right)$$

This proves the lemma.

Now we prove the theorem. From the properties of geometrical series and the definition of $e_{pq}(n)$, combining the lemma we have

$$\sum_{\substack{n\le x\\n\in A}} e_{pq}(n)=\sum_{\substack{a\le\log_{pq}x\\k|a}} a\sum_{\substack{n\le\frac{x}{(pq)^a}\\(n,pq)=1}} a(n)$$

$$=\sum_{a\le\frac{\log_{pq}x}{k}} ka\left(\left(\frac{x}{(pq)^{ka}}\right)^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}+\left(\left(\frac{x}{(pq)^{ka}}\right)^{\frac{1}{2k}+\varepsilon}\right)\right)$$

$$=kx^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}\times$$

$$\left(\sum_{n=1}^{\infty}\frac{n}{(pq)^n}-\sum_{a>\frac{\log_{pq}x}{k}}\frac{a}{(pq)^a}\right)+O\left(x^{\frac{1}{2k}+\varepsilon}\right)$$

$$=kx^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}\left(\sum_{n=1}^{\infty}\frac{n}{(pq)^n}-\frac{1}{(pq)^{\left[\frac{\log_{pq}x}{k}\right]}}\sum_{a=1}^{\infty}\frac{a+\left[\frac{[\log_{pq}x]}{k}\right]}{(pq)^a}\right)+O\left(x^{\frac{1}{2k}+\varepsilon}\right)$$

$$=kx^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}\left(a_{p,q}+O\left(x^{-\frac{1}{k}}\log x\right)\right)+O\left(x^{\frac{1}{2k}+\varepsilon}\right)$$

$$=\frac{(p-1)(q-1)}{pq}a_{p,q}kx^{\frac{1}{k}}+O\left(x^{\frac{1}{2k}+\varepsilon}\right)$$

where $a_{p,q}=\sum_{n=1}^{\infty}\frac{n}{(pq)^n}$ is a computable positive constant.

This completes the proof of Theorem.

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