A New Approach to Inversion of a Cauchy Matrix

Jianguo QIN, Guizhen ZHI

Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, P.R. China, 450002

Email: *qinjianguo@zzuli.edu.cn, zhiguizhen@zzuli.edu.cn*

Abstract: By means of Liouvill theorem of Function of a Complex Variable, the inversion formula for Cauchy matrix is given. As a by-product, the inversion formula of the so-called Hilbert matrix is also obtained.

Keywords: Cauchy matrix; Liouvill theorem; inversion formula

1 Introduction

As an important structured matrix, Cauchy matrix and its variety have found many applications, in the areas of signal processing[1], interpolation of Nevanlinna-type[2], rational approximation, among others.

Let $\omega_1, \omega_2, \dots, \omega_m$, $z_1, z_2, \dots, z_n$ be m+n pairwise distinct complex numbers throughout the paper. By definition, the Cauchy matrix is of the form

$$K = [\frac{1}{\omega_j - z_i}]_{i,j=1}^{n,m} . \tag{1}$$

It is well known that K is invertible if and only if n=m.

In the present note, a proof to the inversion formula of K is given, essentially based on Liouvill Theorem of Function of Complex Variable. This method can be used in a more general case, i.e., the Loewner matrices case. For more information about Cauchy matrices, see, e.g., [3] and [4].

First of all, we take up some notation, which will be used throughout the paper.

$$A_\pi = \text{diag}[\omega_1, \omega_2, \dots, \omega_n],$$

$$A_\xi = \text{diag}[z_1, z_2, \dots, z_n],$$

$$C_\pi = row(i/i)_{i=1}^n = B_\xi^T,$$

$$V_\omega = \left(\omega_i^j\right)_{j=0,i=1}^{n-1,n},$$

$$V_z = \left(z_i^j\right)_{i=1,j=0}^{n,n-1}. \tag{2}$$

2 The Main Results

Lemma 1. Let m=n. Then K is the only solution of the Sylvester equation

$$XA_\pi - A_\xi X = B_\xi C_\pi \tag{3}$$

Proof This is immediate from the result of Theorem 3 of [5], since $\sigma(A_\pi) \cap \sigma(A_\xi) = \Phi$.

Lemma 2 Let m=n. Then

(a) K is invertible if and only if $y\text{K} = C_\pi$ has a solution $\text{S} \in M_{1,n}(C)$;

(b) K is invertible if and only if $\text{K}x = B_\xi$ has a solution $\text{T} \in C^n$.

Proof To prove (a), we assume first that S solves $y\text{K}=C_\pi$: $\text{SK}=C_\pi$.

Put

$$U = A_\xi + B_\xi \text{S}. \tag{4}$$

By Lemma 1，K fulfils

$$\text{K}A_\pi = A_\xi \text{K} + B_\xi \text{SK} = \text{UK},$$

and further,

$$KA_\pi^j = U^j K, \qquad \forall j \geq 1.$$

A simple calculation shows that $col(C_\pi A_\pi^j)_{j=o}^{n-1} = V_\omega$, where V_ω is as in (2). Thus,

$$\begin{aligned} V_\omega &= col(C_\pi A_\pi^j)_{j=0}^{n-1} \\ &= col(SKA_\pi^j)_{j=0}^{n-1} \\ &= col(SU^j K)_{j=0}^{n-1} \\ &= col(SU^j)_{j=0}^{n-1} K . \end{aligned} \tag{5}$$

Observe that V_ω is invertible on account of $\omega_1, \omega_2, \dots, \omega_n$ pairwise distinct, so are both K and $col(SU^j)_{j=0}^{n-1}$.

The necessity of the condition is plain.

The proof of (b) is similar to that just given.

Lemma 3 Let $m = n$. Then

(a) $Kx = B_\xi$ has a solution $\text{P}=col(a_i)_{i=1}^n$, where

$$a_\alpha = \frac{\prod_{k=1}^n (\omega_\alpha - z_k)}{\prod_{k \neq \alpha} (\omega_\alpha - \omega_k)},$$

$$\alpha = 1, 2, \cdots, n. \tag{6}$$

(b) $yK = C_\pi$ has a solution $\text{Q}=row(b_i)_{i=1}^n$, where

$$b_\beta = \frac{\prod_{k=1}^{n}(\omega_k - z_\beta)}{\prod_{k\neq\beta}(z_k - z_\beta)},$$

$$\beta = 1,2,\cdots,n. \quad (7)$$

Proof (a) we need only to verify that $a_1, a_2, ..., a_n$ defined by (6) are subject to

$$m\times 1\sum_{\alpha=1}^{n}\frac{a_\alpha}{\omega_\alpha - z_\beta} = 1,$$

$$\beta = 1,2,...,n,$$

that is

$$\sum_{\alpha=1}^{n}\frac{\Pi_{k\neq\beta}(\omega_\alpha - z_k)}{\Pi_{k\neq\alpha}(\omega_\alpha - \omega_k)} = 1,$$

$$\beta = 1,2,...,n \quad (8)$$

Consider ω_n as a complex variable λ in (8), and the other parameters $\omega_1, \omega_2, ..., \omega_{n-1}$, $z_1, z_2, \ldots z_n$ are fixed. The left side of Eq.(8) can be transformed into

$$F(\lambda) = \sum_{\alpha=1}^{n-1}\frac{\prod_{k\neq\beta}(\omega_\alpha - z_k)}{\prod_{k\neq\alpha,1\le k\le n-1}(\omega_\alpha - \omega_k)} \times\frac{1}{\omega_\alpha - \lambda} + \frac{\Pi_{k\neq\beta}(\lambda - z_k)}{\Pi_{k=1}^{n-1}(\lambda - \omega_k)}.$$

Observe that $F(\lambda)$ is a rational function in λ with $F(\infty) = 1$ and with possible poles of order at most 1 occurring at $\lambda = \omega_1, \omega_2, ..., \omega_{n-1}$. However, the residue of $F(\lambda)$ at the pole $\lambda = \omega_\beta$ is equal to $\lim_{\lambda -> \omega_\beta}(\lambda - \omega_\beta)F(\lambda) = 0$, $\beta = 1,2,...,n-1$, so that $F(\lambda)$ with $F(\infty) = 1$ has no pole in the complex plane. Hence $F(\lambda)$ is analytic and bounded in the complex plane. By Liouvill theorem of Function of Complex Variable[6], $F(\lambda) \equiv 1$. Taking up $\lambda = \omega_n$, we get (8), this is, P=col(a_i)$_{i=1}^{n}$ is a solution of $Kx = B_\varsigma$.

(b) The fact that $row(b_i)_{i=1}^{n} K = C_\pi$, where the $b_j^{'}s$ are as in (7), amount to

$$\sum_{\beta=1}^{n}\frac{b_\beta}{\omega_\alpha - z_\beta} = 1, \quad \alpha = 1,2,...,n,$$

or equivalently,

$$-\sum_{\alpha=1}^{n}\frac{b_\alpha}{z_\alpha - \omega_\beta} = 1, \quad \beta = 1,2,...n.$$

However, this is a simple consequence of the assertion (a) with ω_α , z_β and a_α wherein replaced by z_α , ω_β and - b_α , respectively.

Theorem 4 Let $m = n$. Then K is always invertible, and K^{-1} has an expression

$$K^{-1} = \left[t_{\alpha\beta}\right]_{\alpha,\beta=1}^{n},$$

$$t_{\alpha,\beta} = \frac{\prod_{j\neq\alpha}(\omega_j - z_\beta)}{\prod_{j\neq\beta}(z_\beta - z_j)} \times\frac{\Pi_{k=1}^{n}(\omega_\alpha - z_k)}{\Pi_{k\neq\alpha}^{n}(\omega_k - \omega_\alpha)}. \quad (9)$$

Proof By Lemma 3, $P = col(a_i)_{i=1}^{n}$ and Q=row(b_i)$_{i=1}^{n}$ with components $a_1, a_2, ..., a_n$ and $b_1, b_2, \cdots, b_n$ defined by (6)and (7) are solutions of $Kx = B_\varsigma$ and $yK = C_\pi$ respectively. Then K is invertible by Lemma 2.

From $KA_\pi - A_\xi K = B_\xi C_\pi$, we deduce in turn

$$A_\pi K^{-1} - K^{-1}A_\xi = K^{-1}B_\xi C_\pi K^{-1}$$

$$= \text{PQ= col(} a_i)_{i=1}^{n} \text{ row(b}_i)_{i=1}^{n}.$$

By comparison of the (α, β) components on both sides of the last equation, this yields

$$\omega_\alpha t_{\alpha\beta} - t_{\alpha\beta}z_\beta = a_\alpha b_\beta, \quad 1 \le \alpha, \beta \le n,$$

where $t_{\alpha\beta}$ stands for the (α, β) components of K^{-1}. Therefore,

$$K^{-1} = \left[t_{\alpha\beta}\right]_{\alpha,\beta=1}^{n},$$

$$t_{\alpha,\beta} = \frac{\prod_{j\neq\alpha}(\omega_j - z_\beta)}{\prod_{j\neq\beta}(z_\beta - z_j)} \times \frac{\Pi_{k=1}^{n}(\omega_\alpha - z_k)}{\Pi_{k\neq\alpha}^{n}(\omega_k - \omega_\alpha)}$$

as required.

3 An Application

Consider the so-called Hilbert-Schur matrix of order n

$$\Gamma_\gamma = [\frac{1}{j+i+\gamma-1}]_{i,j=1}^{n}, \quad (10)$$

in which γ is either 0 or a fixed non-integral real parameter.

By using j and $1-i-\gamma$ in place of ω_j and z_i in (1), we see that no number j coincides with any number as $1-i-\gamma$, so that Γ_γ is an invertible Cauchy matrix of order n. By Theorem 4, Γ_γ^{-1} can be expressed as

$$\Gamma_\gamma^{-1} = (v_{\alpha\beta}{}^{(\gamma)})_{\alpha,\beta=1}^{n},$$

where

978-1-935068-10-5 © 2010 SciRes.

$$v_{\alpha\beta}^{(\gamma)} = \frac{\prod_{j\neq\alpha}(j+\beta+\gamma-1)}{\prod_{j\neq\beta}(j-\beta)}$$

$$\times\frac{\Pi_{k=1}^{n}(\alpha+k+\gamma-1)}{\Pi_{k\neq\alpha}(k-\alpha)}.$$

In particular, the Hilbert matrix Γ_0 of order n is invertible with

$$\Gamma_0^{-1} = (v_{\alpha\beta}^{(0)})_{\alpha,\beta=1}^{n},$$

where

$$v_{\alpha\beta}^{(0)} = \frac{\Pi_{j\neq\alpha}(j+\beta-1)}{j_{j\neq\beta}(j-\beta)}$$

$$\times\frac{\Pi_{k=1}^{n}(\alpha+k-1)}{\Pi_{k\neq\alpha}(k-\alpha)}.$$

Acknowledgement

The authors are grateful to professor CHEN Gong-ning for the help he has given.

References

[1] Heinig G. Inversion of generalized Cauchy matrices and other classes of structured matrices, in Linear Algebra for Signal processing (Minneapolis, MN, 1992), The IMA Volumes in Mathematics and its Applications, vol.69, Springer, New York, 1995, pp. 63–81.

[2] Qin J G, Tan R M. An application of Cauchy matrices in Nevanlinna-Pick interpolation problem, Proceedings of the third international workshop on matrix analysis and applications (Volume 2) [C]. Edited by Chang-Qing Xu, Guang-Hui Xu, Ju-Li Zhang. 2009, pp. 211–214.

[3] Polya G and Szego G. Problems and theorems in Analysis [M], Vol II, Springer-Verlag, New York, 1976.

[4] Jolia G. Principles G'eometrique d'anlyse, Gauthiers-Villers, Paris, Vol. II (1932).

[5] Chen G N. Matrix Theory and Its Applications [M]. Beijing: Science Press, 2007, 146
陈公宁, 矩阵理论及其应用 [M]. 北京：科学出版社，2007, 146.

[6] Li Q Z，Jiao B C, Wang A, Wang Y S. Function of a Complex Variable [M]. Beijing: Science Press, 2000, 173.
李庆忠主编，焦保聪，王安，王燕生编. 复变函数[M]. 北京：科学出版社，2000, 173.