Negative Implicative BCK-Algebras

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Abstract: The notion of BCK-algebras was formulated first in 1966 by K. Iséki, Japanese, Mathemation. The notion of positive implicative BCK-algebras was introduced by K. Iséki in 1975. In previous studies, scholars gave the definition of the positive implicative BCK-algebras, its characterizations and the relationship between other BCK-algebra, but it does not give the definition of the negative implicative BCK-algebras, thus in this article, we will give a definition of negative implicative BCK-algebra as well as some characterizations.

Keywords: BCK-algebra, negative implicative BCK-algebra, positive implicative BCK-algebra, binary operation

1 Introduction

The notion of BCK-algebras was formulated in 1966 by K. Iséki, Japanese, Mathemation. This notion is originated from two different ways. One of the motivations is based on set theory. In set theory, there are three most elementary and fundarmental operations among various operations including the general analytical operation introduced by L.Kantorovic and E. Livenson to make a new set from old sets. These fundamental operations are to make the union, the intersection and the set difference, then, as a generalization of those three operations and properties, we have the notion of Booltean algebra. If we take both of the union and the intersection, then, as a generalization of this notion, for example, there is the notion of semirings. Moreover, if we consider the notion of the union or the intersection, we have the notion of an upper semilattice or a lower semilattice. But the notion of set difference was not considered systematically before K. Iséki.

Another motivation is from classical and non-classical propositional calculi. There are some systems which contain the only implication functor among the logical functors. These examples are the system of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C. A. Meredith.

There are many classes of BCK-algebras, for example, subalgebras, bounded BCK-algebras, positive implicative BCK-algebra, implicative BCK-algebra, commutative BCK-algebra, BCK-algebras with condition (S), Griss (and semi-Brouwerian) algebras, quasicommutative BCK-algebras, direct product of BCK-algebras, and so on. They gave a theorem of estimating the number of subalgebras in a finite BCK-algebras, and gave a way extending BCK-algebras, and also provided characterizations of commutative, positive, implicative BCK-algebras.

They gave ideals in BCK-algebras. The ideal theory plays an important role for the general development of BCK-algebras, they discussed ideals, implicative ideals, commutative ideals, positive impolicative ideals, maximal ideals, finitely generated ideals, principal ideals, prime and irreducible ideals, Varlet ideals, additive ideals,and minimal prime ideals; they also gave basicproperties and some characterizations of suchideals; they considered quotient algebras, Noetherian BCK-algebras, lower BCK-semilattices, decomposition properties of ideals and ideal lattices.

Here we will give a new class of BCK-algebra, which is called negative implicative BCK-algebra.

2 Basic Knowledge

Definition 1 Let X be a subset with a binary operation $*$ and a constant 0. Then $(X;*,0)$ is called a BCK-algebra if it satisfies the following conditions:

BCI-1 $((x*y)*(x*z))*(z*y)=0$,

BCI-2 $(x*(x*y))*y=0$,

BCI-3 $x*x=0$,

BCI-4 $x*y=0$ and $y*x=0$

Imply $x=y$,

BCK-5 $0*x=0$.

In X we can define a binary operation $\le$ by $x\le y$ if and only if $x*y=0$.Then

$(X;*,0)$ is called a BCK-algebra if it satisfies the following conditions:

BCI-1′ $(x*y)*(x*z)\le z*y$

BCI-2′ $x*(x*y)\le y$

BCI-3′ $x\le x$

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BCI-$4'$ $x \le y$ and $y \le x$ imply $x = y$

BCK-$5'$ $0 \le x$

BCI-$6'$ $x \le y$ if and only if $x * y = 0$

For any BCK-algebra $(X;*,0)$,* and $\le$ are called a BCK-operation and BCK-ordering on X respectively.

Example 1 Let $X = \{0,1,2\}$ in which * is defined by the following table

*	0	1	2
0	0	0	0
1	2	2	0
2	2	2	0

Figure 1 The definition Table of $X = \{0,1,2\}$

Then $(X;*,0)$ satisfies BCI-1, BCI-2, BCI-4 and BCK-5, but it does not satisfy BCI-3, so $(X;*,0)$ is not a BCK-algebra.

Lemma 1 In a BCK-algebra $(X;*,0)$ we have $(x*y)*z = (x*z)*y$ for all x, y and z in X.

We can find the proof in [1].

Lemma 2 In a BCK-algebra $(X;*,0)$, we have the following properties:

1) $x \le y$ implies $z*y \le z*x$

2) $x \le y$ and $y \le z$ imply $x \le z$

Lemma 3 In a BCK-algebra $(X;*,0)$, then for any x, y, z in X, the following hold:

1) $x*y \le z$ implies $x*z \le y$

2) $(x*z)*(y*z) \le x*y$

3) $x \le y$ implies $x*z \le y*z$

4) $x*y \le x$

5) $x*0 = x$

Lemma 4 In any BCK-algebra, we have

$$x*(y \wedge x) = x*y$$

here $x \wedge y = y*(y*x)$. Obviously, $x \wedge y$ is a lower bound of x and y, and $x \wedge x = x$, $x \wedge 0 = 0 \wedge x = 0$. But in general, $x \wedge y \ne y \wedge x$

In the following, we give equivalent definit-ions for BCK-algebra.

Lemma 5 An algebra $(X;*,0)$ of type (2, 0) is a BCK-algebra if and only if it satisfies the following conditions:

BCI-1 $((x*y)*(x*z))*(z*y) = 0$,

(a) $x*(0*y) = x$

BCI-4 $x*y = 0$ and $y*x = 0$ imply $x = y$,

In combinatory logic, there are various combinators. The names combinators are attached to formulas of the algebra in the following way: If a combinator U has functionality V thecorresponding formula is obtained by replacing FZW in V by $W*Z$ and equating the result with 0.

The functionalty of the combinators K, B, C, I and W are given by

$$Fx(Fyx) \quad K$$

$$F(Fxy)(F(Fzx)(Fzy)) \quad B$$

$$F(Fx(Fyz))(Fy(Fxz)) \quad C$$

$$Fxx \quad I$$

$$F(Fx(Fxy))(Fxy) \quad W$$

Thus according to the above procedure the corresponding formulas are:

$$K \quad (x*y)*x = 0$$

$$B \quad ((y*z)*(x*z))*(y*x) = 0$$

$$C \quad ((z*x)*y)*((z*y)*x) = 0$$

$$I \quad x*x = 0$$

$$W \quad (y*x)*((y*x)*x) = 0$$

So we have the following theorem.

Theorem 1 An algebra $(X;*,0)$ of type $(2,0)$ is a BCK-algebra if and only if it satisfies the axioms K, B, C and the rule BCI-4.

Lemma 6 In any BCK-algebra X, we have

$$(x*(x*y))*(y*x) \le x*(x*(y*(y*x)))$$

for all x, y, z in X.

Definition 2 Let $(X;*,0)$ be a BCK-Algebra and let X_0 be a nonempty subset of X. Then X_0 is called to be a subalgebra of X, if for any x, y in X_0, X_0 is closed under the binary operation * of X.

Lemma 7 Suppose that $(X;*,0)$ is a BCK -algebra and let X_0 be a subalgebra of X. Then

1) $0 \in X_0$

2) $(X_0;*,0)$ is also a BCK-algebra

3) X_0 is a subalgebra of X

4) $\{0\}$ is a subalgebra of X

Definition 3 A BCK-algebra $(X;*,0)$ is called to be positive implicative if it satisfies for for all x, y, z in X.

Lemma 8 Let $(X;*,0)$ be a BCK-algebra

Then the following conditions are equivalent each other:

1) X is positive implicative;
2) $x*y=(x*y)*y$
3) $(x*(x*y))*(y*x)=x*(x*(y*(y*x)))$
4) $x*y=(x*y)*(x*(x*y))$
5) $x*(x*y)=(x*(x*y))*(x*y)$
6) $(x*(x*y))*(y*x)=(y*(y*x))*(x*y)$

Lemma 9 Let $(X;*,0)$ be a BCK-algebra

Then the following conditions are equivalent:

1) X is positive implicative;
2) $(x*y)*z=0$ implies $(x*z)*(y*z)=0$
3) $(x*y)*y=0$ implies $x*y=0$

Example 2 Let $X=\{0,1,2,3\}$ and * on X be given by the table

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Figure 2 The Result Table of $X=\{0,1,2,3\}$ by * operation

Then $(X;*,0)$ is a positive implicative BCK-algebra.

3 Conclusions

Definition 1 A BCK-algebra $(X;*,0)$ is called to be negative implicative if it satisfies $(z*x)*(z*y)=z*(x*y)$ for all x,y,z in X.

In order to give some equivalent conditions of negative implicative BCK-algebra, we nee the following theorem.

Theorem 1 In any BCK-algebra $(X;*,0)$ we have $y*(x*z)=z*(x*y)$ for such x,y,z with the condition $(x*y)*(x*z)=z*y$ in X.

Proof. By BCI-$2'$ we have

$x*(x*z)\le z, x*(x*y)\le y$ .By theorem 1.4 of [1] we get

$$(x*(x*z))*(x*y)=y*(x*z)\le z*(x*y)$$

$$(x*(x*y))*(x*z)=z*(x*y)\le y*(x*z)$$

Obviously $z*(x*y)=y*(x*z)$.

The proof is completed.

Theorem 2 Let $(X;*,0)$ be a BCK-algebra, then the following conditions are equivalent each other:

a) X is negative implicative,
b) $x*y=x*(y*x)$,
c) $x*y=x*((y*x)*x)$,
d) $x*(x*y)=x$,
e) $(x*(x*y))*(y*x)=x*(y*x)$

Proof (a)$\Longrightarrow$(b) By definition 2, we have $x*y=(x*y)*(x*x)=x*(y*x)$, Which is (b).

(b)$\Longrightarrow$(c)

$$(x*((y*x)*x))*(x*(y*x))=(y*x)*((y*x)*x)=0$$

$$(x*(y*x))*(x*((y*x)*x))=((y*x)*x)*(y*x)=0$$

then $x*((y*x)*x)=x*(y*x)$ by (b) we obtain $x*y=x*((y*x)*x)$. (c) holds.

(c)$\Longrightarrow$(d) Substituting $x*y$ or y in (c) we get

$$x*(x*y)=x*(((x*y)*x)*x)=x*(0*x)=x*0=x$$

which is (d).(d)$\Longrightarrow$(e) Right $*$-multiplying both side of (d) by $y*x$, we have

$$(x*(x*y))*(y*x)=x*(y*x),$$

thus (e) holds. (e)$\Longrightarrow$(b) Concerning (e) we get

$$(x*(x*y))*(y*x)=x*(y*x)\le x*y$$

If we substitute $x*(x*y)$ for x, and substitute $y*x$ for y, then we obtain

$$x*y=(x*(x*y))*((y*x)*(x*(x*y)))$$
$$\le x*((y*x)*(x*(x*y)))\le x*(y*x)$$

thus $x*y=x*(y*x)$, which is (b). (b) $\Longrightarrow$ (a) Suppose (b) holds. Then

$$((z*x)*(z*y))*(z*(x*y))$$
$$=((z*(x*z))*(z*y))*(z*(x*y))$$
$$=(y*(x*z))*(z*(x*y)),$$

by theorem 2 $y*(x*z)=z*(x*y)$,

thus $(y*(x*z))*(z*(x*y))=0$,

then $((z*x)*(z*y))*(z*(x*y))=0$. Similarly

$$(z*(x*y))*((z*x)*(z*y))$$
$$=(z*(x*y))*((z*(x*z))*(z*y))$$

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$$= (z*(x*y))*(y*(x*z)) = 0,$$

thus we obtain $(z*x)*(z*y) = z*(x*y)$, (a) holds.

This finished the proof.

Corollary1 A positive implicative BCK-algebra is not a negative implicative BCK-algebra.

Proof. Let $X = (0,a,b,1)$ and $*$ on X be given by the table:

*	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	1	1	0

Figure 3 The Result Table of $X = (0,a,b,1)$ by * operation

Then $(X;*,0)$ is a positive implicative BCK-algebra, but is not a negative implicative BCK-algebra. As

$$(a*0)*(a*b) = a*a = 0$$

$$a*(0*b) = a*0 = a$$

$$(a*0)*(a*b) \neq a*(0*b)$$

The proof is finished.

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