The Comparison Criterion of Stability of Impulsive Stochastic Differential Systems

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Abstract: The stability of nonlinear impulsive stochastic differential systems is discussed in this paper. The comparison criterion of stability in probability and almost sure stability of trivial solution of the systems with impulses at fixed times is established by using the impulsive stochastic comparison theorem based on vector Lyapunov function.

Keywords: stochastic differential systems; stability; comparison principle; impulsive

1. Introduction

The impulsive stochastic differential system (ISDS) has been widely used in sorts of technology and social science aspects such as information technology, communications, genetics, epidemiology, interest rate control and financial crisis precaution. Many works have been made on the stability of nonlinear ISDS in recent years, mainly reflects of the aspect of Moment Stability [1-3]. Besides, the stability problem of nonlinear differential systems with the random impulse is discussed in Wu [4] and Xiong [5], The condition of exponential stability and the almost-sure stability of the zero solution of system are obtained from the characteristics of Lyapunov function of deterministic differential system corresponding.

Comparison approach is one of the main methods to prove the stability of differential system. It was widely used in the study of the stability of deterministic nonlinear impulsive differential systems [6]. But it is rarely to apply comparison approach in the research of the stability of ISDS. A kind of impulsive Itô's stochastic differential system is considered in Liu [7]; the exponential stability criterion of trivial solution of system is given by the comparison theorem.

The stability problem of ISDS is studied using comparison theorems in this paper. By using the basic comparison theorem of impulsive stochastic differential system, the comparison criterion of stability in probability and almost sure stability of ISDS is established.

2. Preliminaries

We consider the nonlinear stochastic differential system with impulse given by

$$\begin{cases} x'(t,\omega) = f(t,x,\omega) & t \neq t_k, \\ \Delta x \triangleq x(t_k^+,\omega) - x(t_k,\omega) = I_k(x,\omega) & t = t_k, \\ x(t_0,\omega) = x_0(\omega) \end{cases} \quad (1)$$

where $0 < t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ as $k \to \infty$. Measurable random function $f: R_+ \times S(\rho) \to R^n$ is a *a.s.* continuous on $(t_k, t_{k+1}] \times R^n$ for each $x \in S(\rho)$, $\lim_{(t,y)\to(t_k^+,x)} f(t,y,\omega) = f(t_k^+, x,\omega)$ *a.s.* exists, $f(t,x,\omega)$ satisfies the conditions of Caratheodory existence theorem, $f(t,0,\omega) \equiv 0$. $S(\rho) \in R^n$ is the ρ-neighborhood of the origin.

$I_k: R^n \to R^n$ is *a.s.* continuous, $I_k(0,\omega) \equiv 0$, $k = 1,2,\cdots$.

In the following text, expression of relation with random variable, random function, random inequalities, and so on were established on the assumption with the probability 1.

The functions and notations referred in this paper are as follow:

Class-**K**、Dini derivative defined as usual.

$PC = \{\psi: R_+ \times D \to R^m, D \in R^n, \psi(t,x,\omega)$ denote the continuous decomposable stochastic function, and for each $x \in D$,

$$\lim_{(t,y)\to(t_k^+,x)} \psi(t,y,\omega) = \psi(t_k^+,x,\omega)$$

exists, k=1 ,2, }.

$$V_0 = \{V(t,x,\omega) \in V_0,$$

if V is satisfies the local Lipschitz condition about x, and $V(t,x,\omega) \in PC[R_+ \times S(\rho), R_+^l]\}$.

$$D_f^+ V \triangleq \limsup_{h\to 0^+} h^{-1}[t+h, V(x+hf(x,t,\omega),\omega) - V(t,x,\omega)], \quad V(t,x,\omega) \in V_0$$

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Definition 1 We say that the trivial solution of ISDS (1) is stable in probability, if for any given $\varepsilon>0$, $\eta>0$ and $t_0 \in R_+$, there exists $\delta=\delta(t_0,\varepsilon,\eta)>0$ which is continuous for t_0 such that

$$P\{|X_0(\omega)|>\delta\}<\eta \text{ implies } P\{|X(t,\omega)|\geq\varepsilon\}<\eta,\ t\geq t_0.$$

If the above δ is independent of t_0, then the trivial solution of system (1) is uniformly stable in probability.

Definition 2 We say that the trivial solution of ISDS (1) is almost sure stable. For any given $\varepsilon>0$ and $t_0 \in R_+$, there exists $\delta=\delta(t_0,\varepsilon)>0$ such that

$$|X_0(\omega)|<\delta \text{ implies } |X(t,t_0,\omega)|<\omega,\ \ t>t_0.$$

if the above δ is independent of t_0, then the trivial solution of system (1) is uniformly almost sure stable.

Consider the auxiliary compare system:

$$\begin{cases} y'(t,\omega)=g(t,y,\omega) & t\neq t_k, \\ y(t_k^+,\omega)=\psi_k(t_k,y(t_k),\omega) & t=t_k, \\ y(t_0,\omega)=y_0(\omega)(>0) \end{cases} \quad (2)$$

where, $g\in PC(R_+\times R^l,R^l)$, $g(t,y,\omega)$ is the random measurable function, and is quasi-monotone non-decreasing for each fixed $(t,\ y)$. ψ_k is continuous and non-decreasing with y, $g(t,0,\omega)\equiv 0$.

The stability definition of the trivial solution of auxiliary system (2) is defined by replacing the $|x_0(\omega)|$ and $|x(t,\omega)|$ in definition 1 and 2 by $\sum_i y_{0i}(\omega)$ and $\sum_i y_i(t,\omega)$, where $y_{0i}(\omega)$ and $y_i(t,\omega)$ are the components of $y_0(\omega)$ and $y(t,\omega)$, $i=1,\ldots,l$.

Lemma 1 (Ladde [8]),

Assume that the definition of function $g(t,y,\omega)$ is the same as in system (2), $m(t,\omega)$ is a continuous dividable random process in $[t_0,t_0+a)$, and

$$D^+m(t,\omega)\leq g(t,m(t),\omega),\ t\in[t_0,t_0+a)$$

$r(t,\omega)$ is the maxima solution of the system

$y'(t,\omega)=g(t,y,\omega), y(t_0,\omega)=y_0(\omega)$, $t\in[t_0,t_0+a)$.

If $m(t_0,\omega)\leq y_0(\omega)$, then

$$m(t,\omega)\leq r(t,\omega),\ \ t\in[t_0,t_0+a).$$

Lemma 2 (comparison theorem for impulse stochastic system)

For system (1) and auxiliary system (2), if there exists a function $V(t,x,\omega)\in V_0$, such that

H₁: $D_f^+V(t,x,\omega)\leq g(V(t,x,\omega),t,\omega)$, $t\neq t_k$

H₂: $V(t_k^+,x+I_k(x),\omega)\leq\psi_k(t_k,V(x,t_k,\omega),\omega)$, $t=t_k$

H₃: $x(t,t_0,x_0,\omega)$ is a solution of (1), such that $V(t_0,x_0,\omega)\leq y_0(\omega)$。

Let $r(t,\omega)$ is the maximal solution of auxiliary system (2) when $t\geq t_0$, then

$$V(t,x,\omega)\leq r(t,\omega)$$

3. Comparison Criterion of Stability

The main conclusions of this paper are as follow.

Theorem 1 (stability in probability comparison criterion)

For system (1) and auxiliary system (2), if exists a function $V(t,x,\omega)\in V_0$, such that

1) $D^+V(t,x,\omega)\leq g(t,V(t,x,\omega),\omega)$, $t\neq t_k$

2) $V(t_k^+,x+I_k(x),\omega)\leq\psi_k(t_k,V(t_k,x,\omega),\omega)$, $t=t_k$

3) $b(|x|)\leq\sum V_i(t,x,\omega)\leq a(t,|x|,\omega)$

$b(\cdot),a(t,\cdot,\omega)\in\mathbf{K}$

The stability in probability of the trivial solution of system (2) implies the stability in probability of the trivial solution of system (1).

Proof: Suppose the trivial solution of system (2) is stable in probability, for any given $0<\varepsilon<\rho,\eta>0$, and $t_0\in R^+$, there exists $\delta_1=\delta_1(t_0,\varepsilon,\eta)>0$, such that

$$\mathrm{P}(\sum_i y_i(t,t_0,\omega)\geq b(\varepsilon))<\eta,\ \ t\geq t_0$$

when $\mathrm{P}(\sum_i y_{0i}(\omega)>\delta_1)<\eta$, where $y(t,t_0,\omega)$ is any solution of system (2).

From condition 3), y_0 can be fond to satisfy $V(t_0,x_0,\omega)\leq y_0(\omega)$, and $\sum_i y_{0i}(\omega)=a(|x_0|,t_0,\omega)$.

$\because a\in\mathbf{K}$, there exists $\delta(t_0,\omega,\eta)$ such that

$$\begin{aligned} P(|x_0(\omega)|>\delta)&=P(a(|x_0|,t_0,\omega)>\delta_1) \\ &=P(\sum_i y_{0i}(\omega)>\delta_1). \end{aligned}$$

That is $P(|x_0(\omega)|>\delta)<\eta$, implies $\mathrm{P}(\sum_i y_{0i}(\omega)>\delta_1)<\eta$.

We say that the trivial solution of system (1) is stable in probability too.

Otherwise, we can find a solution $x(t,t_0,x_0,\omega)$ of system (1) which satisfies $P(|x_0(\omega)|>\delta)<\eta$ and $t'\in(t_k,t_{k+1}]$, such that $P(|x(t',\omega)|\geq\varepsilon)=\eta$, and $P(|x(t,\omega)|\geq\varepsilon)<\eta$, $t_0\leq t<t'$.

In addition, from Lemma 1, $V(t,x,\omega)\leq r(t,u_0,t_0,\omega)$, so

$$b(|x(t,\omega)|)\leq\sum_i V_i(t,x,\omega)\leq\sum_i r_i(t,u_0,t_0,\omega)$$

Therefore

$$\begin{aligned} \eta&=P(|x(t^{'},w)|\geq\varepsilon)=P(b(|x(t^{'},\omega)|)\geq b(\varepsilon)) \\ &\leq P(\sum_i V_i(t,x,\omega)\geq b(\varepsilon))\leq P(\sum_i r_i\geq b(\varepsilon))<\eta. \end{aligned}$$

is contradiction. Hence the theorem is proved.

Theorem 2 (almost sure stability comparison criterion)

If the condition in theorem 1 remains the same, the almost sure stability of the trivial solution of system (2) implies the almost sure stability of the trivial solution of system (1).

Proof: For any given $0 < \varepsilon < \rho, t_0 \in R_+$, from the almost sure stability of system (2), there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$, such that

$$\sum_i y_i(t, t_0, \omega) < b(\varepsilon), \quad t \geq t_0$$

when $\sum_i y_{0i}(\omega) < \delta_1$, where $y(t, t_0, \omega)$ is any solution of system (2)

From condition 3), we can find y_0 to satisfy $V(t_0, x_0, \omega) \leq y_0(\omega)$, and $\sum_i y_{0i}(\omega) = a(|x_0|, t_0, \omega)$.

$\because a \in \mathbf{K}$, there exists $\delta(t_0, \omega)$ to satisfy $|x_0(\omega)| \leq \delta$ and $a(|x_0|, t_0, \omega) \leq \delta_1$ simultaneously. So $|x_0(\omega)| \leq \delta$ implies $\sum_i y_{0i}(\omega) < \delta_1$.

We say that the trivial solution of system (1) is almost sure stability too.

Otherwise, we can find a solution $x(t, t_0, x_0, \omega)$ of system (1) which satisfies $|x_0(\omega)| \leq \delta$ and $t' \in (t_k, t_{k+1}]$, such that $|x(t', \omega)| = \varepsilon$ and $|x(t, \omega)| < \varepsilon,\ t_0 \leq t < t'$. So $x(t, \omega) \in S(\rho),\quad t_0 < t \leq t'$. From Lemma 1,

$$V(t, x, \omega) \leq r(t, u_0, t_0, \omega) \quad t_0 \leq t < t',$$

so

$$b(|x(t, \omega)|) \leq \sum_i V_i(t, x, \omega) \leq \sum_i r_i(t, u_0, t_0, \omega) \quad t_0 \leq t < t'$$

For the function is continuous on *t'*, the inequality is satisfied when *t*= *t'*, i.e.

$$b(|x(t', \omega)|) \leq \sum_i V_i(t', x, \omega) \leq \sum_i r_i(t', u_0, t_0, \omega)$$

Therefore

$$b(\varepsilon) = b(|x(t', \omega)|) \leq \sum_i V_i(t', x, \omega)$$

$$\leq \sum_i r_i(t', t_0, u_0, \omega) < b(\varepsilon)$$

is contradiction. Thus the theorem is proved.

Corollary 1 If the function a is independent of t in condition 3) of Theorem 1, i.e. $a(|x|, t, \omega) = a(|x|, \omega)$, Then the uniformly stability properties of the trivial solution of system (2) implies the uniformly stability properties of the trivial solution of system (1).

Proof: In the proof of Theorem 1, from the trivial solution uniformly stability in probability of system (2), δ_1 is independent of t_0, and from a is independent of *t*, we have δ is independent of t_0.

The proof of uniformly almost sure stability is similar.

4. Conclusions

The stability problem of a stochastic differential system with a fixed moment impulse is studied via comparison approaches in this paper. Comparison approach is virtually to convert the stability problem of Higher-order differential system into the stability problem of lower-order differential system. Firstly, the comparison theorem of impulsive stochastic system was generalized from comparison theorem based on Lyapunov function vector, by using which the comparison criterion of stability in probability and almost-sure stability of impulsive stochastic differential systems is established.

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