A New Approach of Local Utility Maximization to Hedging in Incomplete Markets*

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Abstract: In this paper a new approach of local utility maximization to hedging in incomplete markets is provided. Different from current local approach, the discounted financial wealth process is considered immediately by pointwise. The optimal portfolios are obtained by pointwise solution of equations involving the semimartingale characteristics of the securities price process in a general continuous-time market.

Keywords: Incomplete markets; Portfolio optimization; Hedging; Local utility; Special semimartingale

1. Introduction

It is well known that in an incomplete market there are several equivalent martingale measures and that perfect hedging is not always possible. In this situation, if you want to be as safe as you should invest in a superhedging strategy (cf. e.g. El Karoui and Quenez (1995)[6], Karatzas and Shreve (1998)[15]). However, even for simple European call options only trivial superhedging strategies exist in a number of reasonable market models (cf. Eberlein and Jacod (1997)[5], Frey and Sin (1999)[11]). This is unsatisfactory.

Alternatively, you may minimize some form of quadratic risk (cf. Föllmer llmer and Sondermann (1986)[16], Duffie and Richardson (1991)[4],Schweizer (1994, 2001)[19],[20] for an overview). This can be interpreted as a special case of the following third approach if we allow for quadratic utility functions.

In this paper we follow a third popular suggestion, namely, to maximize some expected utility among all portfolios that differ only in the underlying asset and have a fixed position in the contingent claim. Variations of this approach have been investigated by Duffie (1992)[3], Karatzas and Shreve (1998)[15],Cvitanic,Pham and Touzi (1999)[1], Kallsen (1999)[13], Föllmer and Leukert (2000)[7], Delbaen et al (2002). In particular, Kallsen (1999, 2002)[13], [14] proposes a solution to the hedging problem, in stead of maximization of the expected utility of terminal wealth as is usually done (Duffie (1992)[3], Karatzas and Shreve (1998)[15]), which is based on maximization of the expected utility of the gains over infinitesimal time intervals. The approach has several advantages. Firstly, it is much easier to determine optimal strategies than in the classical utility maximization frame work. Secondly, optimal strategies will usually be more robust against long term model misspecification since they depend only on the local behaviour of the securities price process. Thirdly, there is no dependence on a terminal date T.

Inspired by the local idea of above-mentioned, we provide an alternative approach of local utility maximization to hedging in incomplete markets. Different from the local utility maximization in Kallsen (1999, 2002) [13],[14], our approach is based on maximization of the expected utility of the discounted financial wealth process by pointwise. We think that the approach introduced in this paper is natural and reasonable, as a investor, you would like to maximize the expected utility of the wealth process step by step to determine optimal portfolios.

The paper is organized as follows. Section 2 introduces a new approach of the local utility maximization in a simple discrete-time setting. Section 3 generalizes the approach to continuous-time. In Section 4 we illustrate hedging by the approach given in this paper in a concrete model.

2. Discrete-time markets

We start with a general model of a frictionless security market where investors are allowed to trade continuous up to some fixed finite planning horizon T. Uncertainty in the financial market is modelled by a probability space （Ω，F，P ）and a filtration $F = (\mathrm{F}_t)_{0\le t\le T}$ satisfying

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the usual conditions of right-continuity and completeness. We assume that the σ-field $\mathcal{F}_0$ is trivial, and that $\mathcal{F}_T = \mathcal{F}$.

There are d+1 primary traded assets, whose price processes are given by stochastic processes $S^0,\cdots,S^d$. We assume that $S=(S^0,\cdots,S^d)$ follows an adapted, right-continuous with left-limits (RCLL) and strictly positive semimartingale on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$. Security 0 plays a special role. As a numera'ire by which all other assets are discounted it can be interpreted as the benchmark for risklessness. Form now on we consider only the discounted price process $\tilde{S}=\frac{1}{S^0}(S^0,\cdots,S^d)=(1,\frac{1}{S^0}S^1,\cdots,\frac{1}{S^0}S^d)$ Self-financing trading strategies are modelled by $\mathbb{R}^{d+1}$-valued, predictable, stochastic process $\varphi=(\varphi^0,\cdots,\varphi^d)$, where φ_t^i denotes the number of shares of security i in portfolio at time t. If the (vector) stochastic integral exists, we can define the real-valued discounted gain process $G(\varphi)$ by $G_t(\varphi):=\int_0^t \varphi_s\cdot d\tilde{S}_s$ and the discounted value process (wealth process) $V(\varphi)$ by $V_t(\varphi)=V_0(\varphi)+G_t(\varphi)$, where $V_0(\varphi)$ is called the initial investment or endowment of the investors. The trading constraints are given in form a set $M:=\{\psi\in\mathbb{R}^d : f^i(\psi)\le 0$, for $i=1,\cdots,r$ and $f^i(\psi)=0$ for $i=r+1,\cdots m\}$, where $f^1,\cdots,f^r:\mathbb{R}^d\to\mathbb{R}$ are differentiable, convex mappings and $f^{r+1},\cdots,f^m:\mathbb{R}^d\to\mathbb{R}$ are affine mappings. Typical choices are $M=\mathbb{R}^d$ (no constraints), $M=\mathbb{R}_+^d$ (no short sales), $M=\mathbb{R}^{d-1}\times\{\xi^d\}$ (fixed position in security d). By $\mathcal{M}$ we denote the set of all trading strategies φ such that $(\varphi_t^1,\cdots,\varphi_t^d)(\omega,t)\in M$ for any $(\omega,t)\in\Omega\times[0,T]$. Throughout this paper, we assume that that the subset $\{\psi\in\mathbb{R}^d : f^i(\psi)<0$, for $i=1,\cdots,r$ and $f^i(\psi)=0$ for $i=r+1,\cdots,m\}$ of M is non-empty and write $f^i(\psi):=f^i(\psi^1,\cdots,\psi^d)$ for $\psi=(\psi^0,\psi^1,\cdots,\psi^d)\in\mathbb{R}^{d+1}$

Throughout this section, we restrict ourselves to a discrete-time market, i.e., we assume that $\tilde{S}$ is piecewise constant on the open intervals between integer times. In this case we can define the real-valued discounted gain process $G(\varphi)$ by $G_t(\varphi)=\sum_{s=1}^t \varphi_s\cdot\Delta\tilde{S}_s$ (where $\Delta\tilde{S}_s=\tilde{S}_s-\tilde{S}_{s-1}$) and the discounted value process (wealth process) $V_t(\varphi)=V_0(\varphi)+G_t(\varphi)$.

The specification of the investor's attitude towards risk is done in terms of a utility function as follows:

2.1 Definition

A function $u:\mathbb{R}\to\mathbb{R}$ will be called the utility function if it satisfies

a) u is two times continuously differentiable.

b) The derivatives u',u'' are bounded and $\lim_{x\to\infty}u'(x)=0$.

c) $u'(x)>0, u''(x)<0$, for any $x\in\mathbb{R}$.

With $\mathcal{F}^1(\tilde{S})$ we denote the set of all trading strategies φ with $E(\sum_{t=1}^T|\varphi_t\cdot\Delta\tilde{S}_t|)<\infty$. In the following, we will provide an alternative approach to choose trading strategy.

2.2 Definition

A strategy $\varphi\in\mathcal{M}\cap\mathcal{F}^1(\tilde{S})$ will be called u-optimal for $\mathcal{M}$ if, for $t=1,\cdots,T$,

$$E(u(V_t(\varphi)))\ge E(u(V_{t-1}(\varphi)+\tilde{\varphi}_t\cdot\Delta\tilde{S}_t)), \quad (2.1)$$

where $\tilde{\varphi}$ is any strategy of $\mathcal{M}\cap\mathcal{F}^1(\tilde{S})$.

Remark It is well-known that the most popular optimality criterion is maximization of the terminal wealth $V_T(\tilde{\varphi})$, i.e., a strategy $\varphi\in\Phi_a$ will be called u-optimal if $E(u(V_T(\varphi)))\ge E(u(V_T(\tilde{\varphi})))$ for any strategy $\tilde{\varphi}\in\Phi_a$, where Φ_a is a suitable set of the set self-financing trading strategies. In addition, the local utility notion is given by Kallsen (1999, 2002), i.e., a strategy $\varphi\in\mathcal{M}\cap\mathcal{F}^1(\tilde{S})$ will be called u-optimal if $E(u(\Delta G_t(\varphi)))\ge E(u(\Delta G_t(\tilde{\varphi})))$, $t=1,\cdots,T$, for any strategy $\tilde{\varphi}\in\mathcal{M}\cap\mathcal{F}^1(\tilde{S})$, where $\Delta G_t(\varphi):=\varphi_t\cdot\Delta\tilde{S}_t$. Obviously, our concept defined in Definition 2.2 is related to maximization of expected local utility but, different from the above local utility, the discounted financial wealth process $V(\varphi)$ is also considered.

For the sake of the discussion in continuous-time markets, we will give the equivalent form of Definition 2.2 as follows:

2.2.1 Definition

We call a strategy $\varphi\in\mathcal{M}\cap\mathcal{F}^1(\tilde{S})$ u-optimal for $\mathcal{M}$ if there exist $a_t>0, t=1,\cdots,T$ such that

$$E\left(\sum_{t=1}^T a_t u(V_t(\varphi))\right)\ge E\left(\sum_{t=1}^T a_t u(V_{t-1}(\varphi)+\tilde{\varphi}_t\cdot\Delta\tilde{S}_t)\right), \quad (2.2)$$

where $\tilde{\varphi}$ is any strategy of $\mathcal{M}\cap\mathcal{F}^1(\tilde{S})$.

Remark Obviously, (2.2) follows from (2.1),where $a_t=1$, $t=1,2,\cdots,T$. Conversely, assume that (2.2) holds.

For any $1 \le t \le T$ fixed, let $\tilde{\varphi}_k^* = \varphi_k$ as $k \ne t$, $t = 1, 2, \cdots, T$, and $\tilde{\varphi}_t^* = \tilde{\varphi}_t$. Then, (2.2) yields that $E\left(\sum_{t=1}^{T} a_t u(V_t(\varphi))\right) \ge E\left(\sum_{t=1}^{T} a_t u(V_{t-1}(\varphi) + \tilde{\varphi}_t^* \cdot \Delta\tilde{S}_t)\right)$ This shows that $E(a_t u(V_t(\varphi))) \ge E(a_t u(V_{t-1}(\varphi) + \tilde{\varphi}_t \cdot \Delta\tilde{S}_t))$, i.e., (2.1) is true.

2.3 Theorem

Suppose that $M = \{\psi \in \mathbb{R}^d : \psi^i = \xi^i$ for $i = k+1, \cdots, d\}$ for some $k \in \{0, \cdots, d\}$, $\xi^{k+1}, \cdots \xi^d \in \mathbb{R}$. Then a trading strategy $\varphi \in \mathrm{M} \cap \mathrm{F}^1(\tilde{S})$ is u-optimal for M if and only if

$$E(u'(V_t(\varphi))\Delta\tilde{S}_t^i \mid \mathrm{F}_{t-1}) = 0,$$

where $i = 1, \cdots, k$ and $t = 1, \cdots, T$.

Proof. A short calculation using II.2.14 and II.1.26 of Jacod and Shiryaev (1987)[12] shows that this is special case of the more general Corollary 3.1 proved below.

3. Continuous-Time Markets

Now we turn to continuous-time case. The general mathematical framework is as in Section 2. Furthermore, we assume that $\tilde{S}$ is a $\mathbb{R}^{d+1}$-valued special semi-martingale with characteristics (B, C, ν) (cf. Definition II.2.6 in Jacod and Shiryaev (1987)[12]). By II.2.9 and 2.29 of Jacod and Shiryaev (1987)[12], one can write (B, C, ν) in the form

$$B_t + (x - h(x)) * \nu_t = \int_0^t b_s dA_s, C_t = \int_0^t c_s dA_s, \nu = A \otimes F,$$

where $A \in \Lambda_{loc}^+$ is a predictable process, b is a predictable $\mathbb{R}^{d+1}$-valued process, c is a predictable $\mathbb{R}^{(d+1)\times(d+1)}$-valued process whose values are symmetric, non-negative definite matrices, and F is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathrm{F})$ into $(\mathbb{R}^{(d+1)}, \mathrm{B}^{(d+1)})$. Typical choices for A are $A_t := t$ (e.g. for Lévy processes, diffusions, Itô processes etc.) and $A_t := \sum_{s \le t} 1_{N\setminus\{0\}}(s)$ (discrete-time processes). By $\mathrm{F}^1(\tilde{S})$ we denote the set of all trading strategies φ satisfying

$$\int_0^T (\left|\varphi_{t \cdot b_t}\right| + \varphi_t^{\mathrm{T}} c_t \varphi_t + \int ((\varphi_t \cdot x)^2 \wedge |\varphi_t \cdot x|) F_t(dx)) \; dA_t \in L^1(P)$$

In order to extend Definition 2.2' from discrete-time to continuous-time, we firstly give the following limit theorem.

3.1 Theorem

Let $\{\alpha_n\}_{n\ge1}$ be a positive sequence satisfying to $\sum_{n=1}^{\infty} \alpha_n = 1$, and $(\sum_n)_{n\in N}$ be a sequence of discrete sets

$$\sum_n = \{t_0^n, \cdots, t_{m_n}^n\}$$

with $0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = T$.

Suppose that

$\left\|\sum_n\right\| := \sup \{t_i^n - t_{i-1}^n : i \in \{1, \cdots, m_n\}\} \to 0$ as $n \to +\infty$.

Let φ be any trading strategy in $\mathrm{F}^1(\tilde{S})$. Then we have

$$E\left(\sum_{i=1}^{m_n} \alpha_{m_n - i+1} u(V_{t_i^n}(\varphi))\right) \to u(x_0) + E\left(\int_0^T \gamma(V_{t-}(\varphi), \varphi_t) dA_t\right) \quad (3.1)$$

as n → ∞, where $x_0 = V_0(\varphi)$ and γ(. , .) is defined as follows:

3.2 Definition

For any $\varphi \in \mathbb{R}^{d+1}, t \in \mathbb{R}_+$, we call the random variable

$$\gamma(V_{t-}(\varphi), \varphi_t) = u'(V_{t-}(\varphi))\varphi_t \cdot b_t + u''(V_{t-}(\varphi))\varphi_t^{\mathrm{T}} c_t \varphi_t / 2 + \int (u(V_{t-}(\varphi) + \varphi_t \cdot x) - u(V_{t-}(\varphi)) - u'(V_{t-}(\varphi))\varphi_t \cdot x) F_t(dx)$$

the local utility of φ in time t.

Remark It is surprising that the right hand of (3.1) only depends on $\sum_{n=1}^{\infty} \alpha_n = 1$ but does not depend on the concrete values of $\{\alpha_n\}_{n\ge1}$.

The previous theorem inspires the following definition.

3.3 Definition

We call a trading strategy $\varphi \in \mathrm{M} \cap \mathrm{F}^1(\tilde{S})$ u-optimal for M if

$$E\left(\int_0^T \gamma(V_{t-}(\varphi), \varphi_t) dA_t\right) \ge E\left(\int_0^T \gamma(V_{t-}(\varphi), \tilde{\varphi}_t) dA_t\right)$$

for any $\tilde{\varphi} \in \mathrm{M} \cap \mathrm{F}^1(\tilde{S})$.

Remark $\varphi \in \mathrm{M} \cap \mathrm{F}^1(\tilde{S})$ is u-optimal for M if and only if, for any $\tilde{\varphi} \in \mathrm{M} \cap \mathrm{F}^1(\tilde{S})$, $\gamma(V_{t-}(\varphi), \varphi_t) \ge \gamma(V_{t-}(\varphi), \tilde{\varphi}_t)(P \otimes A)$ -almost all $(\omega, t) \in \Omega \times [0, T]$.

3.4 Theorem

A strategy $\tilde{\varphi} \in \mathrm{M} \cap \mathrm{F}^1(\tilde{S})$ is u-optimal for M if and only if there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ with $\lambda_i \ge 0$ and

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$\lambda_j f^j(\varphi_t)=0$ for $j=1,\cdots,r$ such that

$$u'(V_{t-}(\varphi))b_t^i+u''(V_{t-}(\varphi))c_t^{i\cdot}\cdot\varphi_t$$
$$+\int x^i(u'(V_{t-}(\varphi)+\varphi_t\cdot x)-u'(V_{t-}(\varphi)))F_t(dx)$$
$$-\sum_{j=1}^m\lambda_j D_j f^j(\varphi_t)=0 \qquad (3.2)$$

for $(P\otimes A)$ -almost all $(\omega,t)\in\Omega\times[0,T]$ where $i=1,\cdots,d$, $D_i f$ denotes the i-th partial derivative of the function f.

3.5 Corollary

Suppose that

$$M=\{\psi\in\mathbb{R}^d:\psi^i=\xi^i \text{ for } i=k+1,\cdots,d\}.$$

Then a trading strategy $\varphi\in\mathrm{M}\cap\mathrm{F}^1(\tilde S)$ is u-optimal for M if and only if

$$u'(V_{t-}(\varphi))b_t^i+u''(V_{t-}(\varphi))c_t^{i\cdot}\cdot\varphi_t$$
$$+\int x^i(u'(V_{t-}(\varphi)+\varphi_t\cdot x)-u'(V_{t-}(\varphi)))F_t(dx)=0$$

for $(P\otimes A)$ -almost all $(\omega,t)\in\Omega\times[0,T]$,where $i=1,\cdots,k$.

Proofs

Throughout we use the notations of Jacod and Shiryaev (1987)[12].

3.6 Proof of Theorem

Since $\varphi\in\mathrm{F}^1(\tilde S)$, in the same way of the first and second steps of the proof of Theorem 3.1 in Kallsen (1999)[13], we easily see that $\varphi\in L(\tilde S)$. Moreover, let $\alpha:=\sup_{x\in R}(2u'(x))+\sup_{x\in R}|u''(x)|$, it is easy to prove that

$$|\gamma(V_{t-}(\varphi),\varphi_t)|\le$$
$$\alpha|\varphi_t\cdot b_t|+\frac{1}{2}\alpha\varphi_t^T c_t\varphi_t+\alpha\int((\varphi_t\cdot x)^2\wedge|\varphi_t\cdot x|)F_t(dx)$$

By

$\varphi\in\mathrm{F}^1(\tilde S)$,

we know that $\gamma(V_{t-}(\varphi),\varphi_t)\in L^1(\Omega\times\mathbb{R}_+,\mathcal{P},P\otimes A)$.

Since $\tilde S$ is a special semimartingale and $\varphi\in\mathrm{F}^1(\tilde S)$ is predictable, for fixed $n\in N$ and $i\in(1,\cdots,m_n)$,then $V_{t_i^n}(\varphi)=x_0+\int_0^{t_i^n}\varphi\cdot d\tilde S$ is a special semimartingale with predictable part of bounded variation $\int_0^{t_i^n}\varphi_t\cdot d\tilde B_t$ and local martingale part $\int_0^{t_i^n}\varphi\cdot d(M^c+M^d)_t$. One easily verifies that the measure of jumps $\mu^{u(V_{t_i^n}(\varphi))}$ of $u(V_{t_i^n}(\varphi))$ is given by

$$\mu^{u(V_{t_i^n}(\varphi))}([0,t]\times G)=\int 1_{[0,t]}1_G(u(V_{t_i^n-}(\varphi)+x)-u(V_{t_i^n-}(\varphi)))\mu^{V_{t_i^n}(\varphi)}(ds,dx)$$

for any $t\in\mathbb{R}_+,G\in\mathrm{B}$ with $0\notin G$ and $\nu^{u(V_{t_i^n}(\varphi))}$ is given by the same integral representation, but integrated with respect to $\nu^{V_{t_i^n}(\varphi)}$ instead of $\mu^{V_{t_i^n}(\varphi)}$. Since

$$(x^2\wedge|x|)*\nu^{u(V_{t_i^n}(\varphi))}\le(\alpha^2+\alpha)(x^2\wedge|x|)*\nu^{V_{t_i^n}(\varphi)},$$

it follows that $u(V_{t_i^n}(\varphi))$ is a special semimartingale. By Itô's formula, we can obtain that

$$u(V_{t_i^n}(\varphi))=\int_0^{t_i^n}u'(V_{t-}(\varphi))dV_t(\varphi)^c+\frac{1}{2}\int_0^{t_i^n}u''(V_{t-}(\varphi))d\left\langle V(\varphi)^c,V(\varphi)^c\right\rangle_t$$
$$+\int_{[0,t_i^n]\times\mathbb{R}^{d+1}}(u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi))-u'(V_{t-}(\varphi))\varphi_t\cdot x)$$
$$\times\nu^{\tilde S}(dt,dx)+\int_0^{t_i^n}u'(V_{t-}(\varphi))\varphi\cdot d\tilde B_t+u(x_0)$$

Again, let

$$\beta_t^{(n)}:=\sum_{i=1}^{m_n}\left(1_{(t_{i-1}^n,t_i^n]}(t)\sum_{j=1}^{m_n-i+1}\alpha_j\right)$$

we have

$$\sum_{i=1}^{m_n}\alpha_{m_n-i+1}u(V_{t_i^n}(\varphi))=\sum_{i=1}^{m_n}\alpha_{m_n-i+1}u(x_0)+$$
$$\int_0^T\gamma V_{t-}(\varphi),\varphi_t)dA_t+\int_0^T\beta_t^{(n)}u'(V_{t-}(\varphi))\varphi_t\cdot d\tilde S_t^c$$
$$+\int_{[0,T]\times\mathbb{R}^{d+1}}\beta_t^{(n)}(u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi)))(\mu^{\tilde S}-\nu^{\tilde S})(dt,dx)$$
$$+\int_0^T(\beta_t^{(n)}-1)u'(V_{t-}(\varphi))\varphi_t\cdot b_t dA_t+\frac{1}{2}\int_0^T(\beta_t^{(n)}-1)u''(V_{t-}(\varphi))\varphi^{\vdash}c_t\varphi dA_t$$
$$+\int_0^T\int(\beta_t^{(n)}-1)(u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi))-u'(V_{t-}(\varphi))\varphi_t\cdot x)\times F_t(dx)dA_t$$

Note that $\sum_{j=1}^{\infty}\alpha_j=1$, it follows that $0<\beta_t^{(n)}<1$ and $\beta_t^{(n)}\to1$, as n→ ∞ for any $t\in(0,T)$.

Define

$$X_\cdot^{(n)}:=\int_0^\cdot\beta_t^{(n)}u'(V_{t-}(\varphi))\varphi_t\cdot d\tilde S_t^c .$$

By $\varphi\in\mathrm{F}^1(\tilde S)$, we may get

$$\left[X^{(n)},X^{(n)}\right]_T=\int_0^T(\beta_t^{(n)}u'(V_{t-}(\varphi))^2\varphi_t^{\vdash}c_t\varphi_t dA_t\le\alpha^2\int_0^T\varphi_t^{\vdash}c_t\varphi_t dA_t\in L^1(P)$$

By I.4.50c in Jacod and Shiryaev (1987)[12], $X^{(n)}$ is a square-integrable martingale and $E(X_T^{(n)})=0$.

For

$$Y_t^{(n)}:=\beta_t^{(n)}(u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi)))$$
$$Y_t^{(n)}:=\beta_t^{(n)}(u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi))).$$

Then $\left|Y_t^{(n)}\right|\le\alpha\left|\varphi_t\cdot x\right|$ for any $x\in\mathbb{R}^{d+1}$. By $\varphi\in\mathrm{F}^1(\tilde S)$ and Proposition 3.9 of Kallsen (1999)[13], we know that $Y*(\mu-\upsilon)$ is a uniformly integrable martingale. Hence, we may get that

$$E\left(\int_{[0,T]\times\mathbb{R}^{d+1}}\beta_t^{(n)}(u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi)))(\mu^{\tilde S}-\upsilon^{\tilde S})(dt,dx)\right)=0$$

Setting

$Z_t:=u(V_{t-}(\varphi)+\varphi_t\cdot x)-u(V_{t-}(\varphi))-u'(V_{t-}(\varphi))\varphi_t\cdot x$,

It is easy to verity that $\left|Z_t\right|\le\alpha\left|\varphi_t\cdot x\right|$ and $\left|Z_t\right|\le\alpha\left|\varphi_t\cdot x\right|^2$. In addition, $(\beta^{(n)}-1)Z_t$ tends 0 for ($P\otimes A\otimes F$)-almost all $(\omega,t,x)\in\Omega\times\mathbb{R}_+\times\mathbb{R}^{d+1}$ as n

$\to \infty$. Since $\varphi \in F^1(S)$, by dominated convergence theorem, $\int_0^T \int (\beta_t^{(n)} - 1) Z_t F_t(dx) dA_t$ converges0 in $L^1(P)$ as n$\to \infty$.

Similar to the above proof, we can obtain that and

$$\int_0^T \int (\beta_t^{(n)} - 1) u''(V_{t-}(\varphi)) \varphi^\tau c_t \varphi dA_t$$

And

$$\int_0^T \int (\beta_t^{(n)} - 1) u''(V_{t-}(\varphi)) \varphi^\tau c_t \varphi dA_t$$

tend 0 in $L^1(P)$ for n $\to \infty$. This completes the proof of Theorem 3.1.

Similar to the proof of Proposition 3.10 in Kallsen (1999)[13], we can obtain the following proposition.

3.7 Proposition

$\varphi \in F^1(S)$ is u-optimal for M if and only if $\gamma(V_{t-}(\varphi), \varphi_t) \geq \gamma(V_{t-}(\varphi), \tilde{\varphi}_t)$ $P \otimes A$-almost all (ω, t) for any $\tilde{\varphi} \in \mathbb{R} \times M$.

3.8 Proof of Theorem

Notice that a strategy $\varphi \in M \cap F^1(S)$ is u-optimal for M if and only if, for any $\tilde{\varphi} \in M \cap F^1(S)$, $\gamma(V_{t-}(\varphi), \varphi_t) \geq \gamma(V_{t-}(\varphi), \tilde{\varphi}_t)(P \otimes A)$ -almost everywhere on $\Omega \times [0, T]$. Define the mapping $h: \mathbb{R}^{d+1} \to \mathbb{R}$ by $h(\psi_t) := -\gamma(V_{t-}(\varphi), \varphi_t)$. It is obvious that h is a convex mapping. Meanwhile, we may get that

$$D_i h(\psi) = u'(V_{t-}(\varphi)) b_t^i + u''(V_{t-}(\varphi)) c_t^{i\cdot} \cdot \psi_t$$
$$+ \int x^i (u'(V_{t-}(\varphi) + \psi_t \cdot x) - u'(V_{t-}(\varphi))) F_t(dx)$$

Let (P) denote the ordinary convex program in the sense of Rockafellar (1970)[18] corresponding to h, the constraints $f^j(\psi) \leq 0$, for $j = 1, \cdots, r$ and $f^j(\psi) = 0$ for $j = r+1, \cdots, m$.

Since $\varphi \in M \cap F^1(S)$ is u-optimal, by Proposition 3.1, we have that φ_t is an optimal solution to the corresponding convex program (P). Then, by Theorem 28.2, 28.3 and 25.1 in Rockafellar (1970)[18], there exists a Kuhn-Tucker vector $\lambda_1, \cdots \lambda_m$ with $\lambda_j \geq 0$ and $\lambda_j f^j(\varphi_t) = 0$ for $j = 1, \cdots, r$ such that (3.2) holds for ($P \otimes A$)-almost all $(\omega, t) \in \Omega \times [0, T]$, where $i = 1, \cdots, d$.

Conversely, suppose that there exist $\lambda_1, \cdots \lambda_m \in \mathbb{R}$ with $\lambda_j \geq 0$ and $\lambda_j f^j(\varphi_t) = 0$ for $j = 1, \cdots, r$ such that (3.2) holds for ($P \otimes A$)-almost all $(\omega, t) \in \Omega \times [0, T]$, where $i = 1, \cdots, d$. By Theorem 28.3 of Rockafellar (1970)[18], we know that φ_t is an optimal solution to the corresponding convex program (P). Hence $\gamma(V_{t-}(\varphi), \varphi_t) \geq \gamma(V_{t-}(\varphi), \tilde{\varphi}_t)$ for any $\tilde{\varphi} \in \mathbb{R} \times M$. By Proposition 3.1, which implies that $\varphi \in M \cap F^1(S)$ is u-optimal for M. This completes the proof of Theorem 3.2.

3.9 Proof of Corollary

This follows immediately from Theorem 3.2.

4. Example

The example given by Kallsen (1999)[13] is employed to illustrate our approach in practice. The classical Black-Scholes model will be considered. More specifically, we consider a bank account $S_t^0 = e^{rt}$ with constant interest rate $r \in \mathbb{R}$, a stock

$$S_t^1 := S_0^1 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

with $\mu \in \mathbb{R}, \sigma \in (0, \infty)$ and a standard Wiener process W. A European call option S^2 on the stock with expiration $T \in \mathbb{R}$ and strike price $K \in \mathbb{R}_+$. Suppose that the filtration is generated by S^1 and that S^2 is given by the Black-Scholes formula, i.e., $S_t^2 = f(t, \tilde{S}_t^1) e^{rt}$ for any $t \in [0, T]$,where

$$f(t,x) := x\Phi\left(\frac{\log(xr^{rT}/K}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}\right)$$
$$-Ke^{-rT}\Phi\left(\frac{\log(xr^{rT}/K}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t}\right)$$

and Φ denotes the cumulative distribution function of the standard normal distribution. $\tilde{S} = (\tilde{S}^0, \tilde{S}^1, \tilde{S}^2)$ is a special semimartingale with $A_t = t, b_t^0 = 0, b_t^1 = (\mu - r)\tilde{S}^1$,

$$b_t^2 = (\mu - r)\tilde{S}_t^1 D_2 f(t, \tilde{S}_t^1),$$
$$c_t^{00} = c_t^{01} = c_t^{02} = 0, c_t^{11} = (\sigma \tilde{S}_t^1)^2,$$
$$c_t^{12} = (\sigma \tilde{S}_t^1)^2 D_2 f(t, \tilde{S}_t^1),\ c_t^{22} = (\sigma \tilde{S}_t^1 D_2 f(t, \tilde{S}_t^1))^2$$

for any $t \in [0, T]$.

Assume that you have sold one European call option and you want to hedge the risk by trading only in stock and bond. In such case your trading constraints are given by $M := \mathbb{R} \times \{-1\}$. By Corollary 3.1, we know that the stock component of your optimal portfolio is

$$\varphi_t^1 = \frac{c_t^{12}}{c_t^{11}} - \frac{u'(V_{t-}(\varphi))}{u''(V_{t-}(\varphi))} \frac{b_t^1}{c_t^{11}} = \Phi\left(\frac{\log(\tilde{S}_t^1 r^{rT}/K}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}\right)$$

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$$+\left(-\frac{V_{t-}(\varphi)u''(V_{t-}(\varphi))}{u'(V_{t-}(\varphi))}\frac{b_t^1}{c_t^{11}}\right)^{-1}\frac{(\mu-r)V_{t-}(\varphi)}{S_t^1\sigma^2}$$

for $t\in[0,T]$. It is easy to verify $\varphi\in F^1(S)$.

It is well-known that the most commonly used examples of utility functions are $u(x):=x^{1-\kappa}/(1-\kappa)$ for $0<\kappa<1$ and $u(x):=\log x$ for $\kappa=1$. There, $\kappa=-(xu''(x))/u'(x)$ is the measure of constant proportional risk aversion. In this case,the coefficient of the second term of φ_t^1 is κ^{-1}. The greater κ is, the smaller the second term of φ_t^1 is. The first term of φ_t^1 equals the perfect Black-Scholes hedging strategy(cf. Lamberton and Lapeyre (1996)[17], Remark 4.3.6).

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