

Optimal Multiperiodic Control for Inventory Coupled Systems: A Multifrequency Second-Order Test

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Abstract

A complex autonomous inventory coupled system is considered. It can take, for example, the form of a network of chemical or biochemical reactors, where the inventory interactions perform the recycling of by-products between the subsystems. Because of the flexible subsystems interactions, each of them can be operated with their own periods utilizing advantageously their dynamic properties. A multifrequency second-order test generalizing the π -test for single systems is described. It can be used to decide which kind of the operation (the static one, the periodic one or the multiperiodic one) will intensify the productivity of a complex system. An illustrative example of the multiperiodic optimization of a complex chemical production system is presented.

Keywords

Optimal Multiperiodic Control, Complex Systems, Inventory Interactions, Nested Optimization, Multifrequency Second-Order Test

1. Introduction

We consider complex autonomous inventory coupled (IC) systems. Such systems can take, for example, the form of a network of chemical or biochemical networks, where the inventory interactions perform the recycling of by-products or by-streams from some subsystems to other subsystems as their input components or energy carriers [1]. Because of the flexible interactions of the subsystems, each of them can be operated with their own period utilizing advantageously its dynamic properties. In this context, we formulate the multiperiodic optimal control problem, which generalizes the periodic control approach finding much attention for the optimization of chemical and bio-

technological processes [2]-[6]. We analyze three kinds of operation for IC systems: the steady state one, the periodic one, and the multiperiodic one with possibly incommensurate operation frequencies of the subsystems. We develop a multifrequency second-order test, which can be used to ensure the best intensification of the productivity of IC systems preserving at the same time their advantageous ecological features: many by-products are recycled within a complex system. The justification of the test proposed is obtained by the approach avoiding the regularity conditions, which generalizes such an approach for single systems. We illustrate the theoretical considerations by the example of multiperiodic optimization of a complex chemical production system.

Notation: R_+ is the set of positive reals; $R^n(C^n)$ is the space of n -dimensional real (complex) vectors x with the norm $|x| \doteq \max_{0 \leq k \leq n} |x_k|$; $|x|_1 \doteq \sum_{k=1}^n |x_k|$; $\mathcal{L}_\infty^{\tau,n}$ is the space of τ -periodic n -dimensional essentially bounded functions x equipped with the norm $\|x\|_\infty \doteq \text{esssup}_{0 \leq t < \tau} |x(t)|$; $\mathcal{W}_{1,\infty}^{\tau,n}$ is the space of τ -periodic n -dimensional functions with the essentially bounded derivative and the norm $\|x\|_{1,\infty} \doteq \|x\|_\infty + \|\dot{x}\|_\infty$; $O_{n \times m}(I_n)$ is the zero (the identity) matrix of the dimension $n \times m$ ($n \times n$); $\text{Int}(X)$ is the interior of the set X ; n_x is the dimension of a variable x ; $n_x \doteq \sum_{1 \leq i \leq n} n_{x_i}$ for $x = (x_i)_{1 \leq i \leq n}$; $T_\kappa^{\tau,n}$ is the set of τ -periodic n -dimensional trigonometric polynomials of degree κ ; and $E_\tau[f(t)]$ is the average value of a τ -periodic function $f(t)$.

2. Optimal Multiperiodic Control Problem

Consider the following optimal multiperiodic control problem for IC systems (the problem M) composed of the set $\mathbb{N} \doteq \{1, 2, \dots, N\}$ of N subsystems: find for each subsystem its operation period $\tau_i \in R_+$, and its τ_i -periodic control process

$s_i^{\tau_i} \doteq (x_i^{\tau_i}, w_i^{\tau_i}) \in S_i^{\tau_i} \doteq \mathcal{W}_{1,\infty}^{\tau_i, n_{x_i}} \times \mathcal{L}_\infty^{\tau_i, n_{w_i}}$ encompassing its τ_i -periodic state trajectory $x_i^{\tau_i}$, its τ_i -periodic extended control $w_i^{\tau_i}$, which minimize the performance index

$$\mathcal{J}(\tau, s_\tau) \doteq \sum_{i \in \mathbb{N}} a_i(y_i) \tag{1}$$

being a *scalar* function of the τ_i -averaged outputs of the subsystems

$$y_i = \frac{1}{\tau_i} \int_0^{\tau_i} h_i(x_i^{\tau_i}(t), w_i^{\tau_i}(t)) dt \quad (i \in \mathbb{N}), \tag{2}$$

and subject to the resource-technological constraints of the subsystems

$$b_i(y_i) = 0, \quad c_i(y_i) \leq 0 \quad (i \in \mathbb{N}), \tag{3}$$

the state equations of the subsystems

$$\dot{x}_i^{\tau_i}(t) = f_i(x_i^{\tau_i}(t), w_i^{\tau_i}(t)), \quad t \in T_{\tau_i} \doteq [0, \tau_i] \quad (i \in \mathbb{N}), \tag{4}$$

the inventory constraints

$$\frac{1}{\tau_i} \int_0^{\tau_i} K_i w_i^{\tau_i}(t) dt \leq \sum_{j \in \mathbb{N}} K_{ij} y_j \quad (i \in \mathbb{N}), \tag{5}$$

and the box constraints for the process variables

$$\tau_i \in \mathcal{T}_i, \quad x_i^{\tau_i}(t) \in X_i, \quad w_i^{\tau_i}(t) \in W_i, \quad t \in T_{\tau_i} \quad (i \in \mathbb{N}), \tag{6}$$

where the extended control $w_i^{\tau_i}(t) \doteq (u_i^{\tau_i}(t)^T, v_i^{\tau_i}(t)^T)^T, (t \in T_{\tau_i})$ of the i th subsystem encompasses its local control $u_i^{\tau_i} \in \mathcal{L}_{\infty}^{\tau_i, n_{u_i}}$, and its inventory control $v_i^{\tau_i} \in \mathcal{L}_{\infty}^{\tau_i, n_{v_i}}$, and $K_i \doteq (O_{n_{v_i} \times n_{u_i}}, I_{n_{v_i}})$, and $\mathcal{T}_i \subset \mathbb{R}_+, X_i \subset \mathbb{R}^{n_{x_i}}, W_i \subset \mathbb{R}^{n_{w_i}}$ are the box sets, and

$$a_i : \mathbb{R}^{n_{y_i}} \rightarrow \mathbb{R}, b_i : \mathbb{R}^{n_{y_i}} \rightarrow \mathbb{R}^{n_{p_i}}, c_i : \mathbb{R}^{n_{y_i}} \rightarrow \mathbb{R}^{n_{r_i}},$$

$$h_i : \mathbb{R}^{n_{x_i}} \times \mathbb{R}^{n_{w_i}} \rightarrow \mathbb{R}^{n_{y_i}}, f_i : \mathbb{R}^{n_{x_i}} \times \mathbb{R}^{n_{w_i}} \rightarrow \mathbb{R}^{n_{x_i}},$$

$$K_{ij} \in \mathbb{R}^{n_{y_i} \times n_{y_j}},$$

while $s_{\tau} \doteq (s_i^{\tau_i})_{i \in \mathbb{N}} \in S_{\tau} \doteq \prod_{i \in \mathbb{N}} S_i^{\tau_i}$ is the multiperiodic control process of the IC system. We denote the set of all such processes satisfying the constraints (2)-(6) with a fixed multiperiod $\tau \doteq (\tau_i)_{i \in \mathbb{N}}$ by S_{τ}^{ad} , and the corresponding problem M by M_{τ} .

The objective function (1) represents the global benefits from the multiperiodic operation of the IC system, which are determined with the help of the τ_i -averaged outputs (2) of the subsystems depicting, for example, their averaged production performance or their averaged selectivity. The constraints (3) mirror the averaged availability of the resources used for the process operation, and the technological requirements for the averaged product purity. The inventory interactions (5) perform the recycling of by-products or by-streams of some subsystems to other subsystems as their input components or energy carriers [1]. Because of the flexible interactions of the subsystems each of them can be operated with their own period utilizing advantageously its dynamic properties, which leads to the nested multiperiodic optimization encompassing the static and periodic optimization as its particular cases. The structural matrices K_i and K_{ij} determine the averaged constraint of the inventory control of the i th subsystem by the averaged outputs of the other subsystems.

3. The Multifrequency Second-Order Test for Complex Systems

Constraining the variables $x_i^{\tau_i}, w_i^{\tau_i}$ to the steady functions x_i, w_i we obtain the optimal steady-state problem for the IC systems (the problem \mathcal{S}):

$$\min_{s \in S} \left\{ J(s) \doteq \sum_{i \in \mathbb{N}} a_i(y_i) \mid y_i = h_i(x_i, w_i), b_i(y_i) = 0, c_i(y_i) \leq 0, f_i(x_i, w_i) = 0, \right. \\ \left. K_i w_i \leq \sum_{i \in \mathbb{N}} K_{ij} y_j, x_i \in X_i, w_i \in W_i, (i \in \mathbb{N}) \right\}, \quad (7)$$

where $s \doteq (s_i)_{i \in \mathbb{N}} \in S \doteq \prod_{i \in \mathbb{N}} S_i$ is the steady-state control process of the IC system with the components $s_i \doteq (x_i, w_i) \in S_i \doteq \prod_{i \in \mathbb{N}} (\mathbb{R}^{n_{x_i}} \times \mathbb{R}^{n_{w_i}})$. Let $\bar{s} \doteq (\bar{s}_i)_{i \in \mathbb{N}}$ be a locally optimal steady-state process of the IC system (the \bar{s} -process) with the components $\bar{s}_i \doteq (\bar{x}_i, \bar{w}_i)$ (the \bar{s}_i -processes).

Assumption 1: The functions h_i, f_i and a_i, b_i, c_i are twice continuously differentiable in some neighbourhoods of the points \bar{s}_i and $\bar{y}_i \doteq h_i(\bar{s}_i)$, respectively ($i \in \mathbb{N}$).

Assumption 2: The steady states \bar{x}_i are lying in the interior of their box sets, *i.e.* $\bar{x}_i \in \text{Int}(X_i)$ ($i \in \mathbb{N}$).

Let $x_i(t, s_i^{\tau_i})$ be the solution of the differential equation

$$\dot{x}_i(t) = f_i(x_i(t), w_i^{\tau_i}(t)), x_i(0) = x_i^0, t \in T_i$$

for the reduced τ_i -periodic control process of the i th subsystem

$s_i^{\tau_i} \doteq (x_i^0, w_i^{\tau_i}) \in S_i^{\tau_i} \doteq R^{n_{x_i}} \times \mathcal{L}_{\infty}^{\tau_i, n_{w_i}}$, $\|s_i^{\tau_i}\| \doteq |x_i^0| + \|w_i^{\tau_i}\|_{\infty}$. Using the affine scaling of the variables $w_i^{\tau_i}$ we convert the sets W_i into the hypercubes $[-1, 1]^{n_{w_i}}$. We write their box constraints as $p_i(w_i^{\tau_i}(t)) \leq 0$, where the functions $p_i: R^{n_{w_i}} \rightarrow R^{2n_{w_i}}$ take the form $p_i(w_i^{\tau_i}(t)) \doteq \left((-w_i^{\tau_i}(t) - 1_i)^T, (w_i^{\tau_i}(t) - 1_i)^T \right)^T$, and $1_i \doteq (1, 1, \dots, 1)^T \in R^{n_{w_i}}$.

We convert the problem M_{τ} to the following reduced form (the problem M_{τ}):

$$\min_{s_{\tau} \in S_{\tau}} \left\{ J(s_{\tau}) \doteq \sum_{i \in \mathbb{N}} a_i(y_i) \mid y_i = E_{\tau_i} \left[h_i(x_i(t, s_i^{\tau_i}), w_i^{\tau_i}(t)) \right], b_i(y_i) = 0, c_i(y_i) \leq 0, \right. \\ \left. E_{\tau_i} \left[f_i(x_i(t, s_i^{\tau_i}), w_i^{\tau_i}(t)) \right] = 0, K_i w_i \leq \sum_{j \in \mathbb{N}} K_{ij} y_j, p_i(w_i^{\tau_i}(t)) \leq 0, t \in T_{\tau_i} (i \in \mathbb{N}) \right\},$$

where $s_{\tau} \doteq (s_i^{\tau_i})_{i \in \mathbb{N}} \in S_{\tau} \doteq \prod_{i \in \mathbb{N}} S_i^{\tau_i}$ is the reduced multiperiodic control process normed as $\|s_{\tau}\| \doteq \max_{i \in \mathbb{N}} \|s_i^{\tau_i}\|$. The set of all admissible processes of the problem M_{τ} is denoted by S_{τ}^{ad} .

The \bar{s} -process induces a reduced locally optimal steady-state process $\bar{s} = (\bar{s}_i)_{i \in \mathbb{N}}$ of the problem S (the \bar{S} -process) with the components $\bar{s}_i \doteq (\bar{x}_i, \bar{w}_i)$. The problem M_{τ} is locally proper at \bar{s} iff \bar{s} is not its local minimum.

We approximate the controls $w_i^{\tau_i}$ by the trigonometric polynomials $\tilde{w}_i^{\tau_i} \in T_{\kappa_i}^{\tau_i, n_{w_i}}$ (the T-controls) defined as

$$\tilde{w}_i^{\tau_i}(t) \doteq w_i + \tilde{w}_i^{\tau_i}(t), \tilde{w}_i^{\tau_i}(t) \doteq \sum_{\kappa=1}^{\kappa_i} \left(\tilde{w}_{i0}^{\kappa} \cos \kappa \omega_i t + \tilde{w}_{i1}^{\kappa} \sin \kappa \omega_i t \right) (t \in T_{\tau_i})$$

with the coefficients $\tilde{w}_{i\sigma}^{\kappa} \in R^{n_{w_i}}$ and the operation frequency $\omega_i \doteq 2\pi/\tau_i$. We denote by $\tilde{w}_i \doteq (w_i^T, \tilde{w}_i^T)^T \in R^{n_{\tilde{w}_i}}$ ($n_{\tilde{w}_i} \doteq n_{w_i} + n_{\tilde{w}_i}$) the set of the coefficients of the T-controls

with $\tilde{w}_i \doteq \left(\left(\tilde{w}_{i0}^{\kappa} \right)_{\kappa=1}^{\kappa_i}, \left(\tilde{w}_{i1}^{\kappa} \right)_{\kappa=1}^{\kappa_i} \right)^T \in R^{n_{\tilde{w}_i}}$ ($n_{\tilde{w}_i} \doteq 2\kappa_i n_{w_i}$). We distinguish the subvectors

$w_{ik} \in R^{n_{w_{ik}}}$ ($k = 0, 1, 2$) of the vector w_i connected with its internal part $\bar{w}_{i0} \in (-1_{i0}, 1_{i0})$, and its boundary parts $\bar{w}_{i1} = 1_{i1}$ and $\bar{w}_{i2} = -1_{i2}$, where

$1_{ik} \doteq (1, 1, \dots, 1)^T \in R^{n_{w_{ik}}}$, $k = 0, 1, 2$. We fix the control w_{i0} on its optimal steady-state level \bar{w}_{i0} , while we shift the controls w_{ik} , $k = 1, 2$ to the interior of their box sets. We

impose on the subvectors w_{ik} the pure dynamic T-controls $\tilde{w}_{ik}^{\tau_i} \in T_{\kappa_i}^{\tau_i, n_{w_{ik}}}$. We set $0_{ik} \doteq (0, 0, \dots, 0)^T \in R^{n_{w_{ik}}}$, $k = 1, 2$. We write the generalized function of the box constraints as $p_i(w_i^{\tau_i}(t), d_i^{\pm}) \leq 0$, where the functions $p_i: R^{n_{w_i}} \rightarrow R^{2n_{w_i}}$ take the form

$p_i(w_i^{\tau_i}(t), d_i^{\pm}) \doteq \left((-w_i^{\tau_i}(t) - d_i^-)^T, (w_i^{\tau_i}(t) - d_i^+)^T \right)^T$ for $d_i^{\pm} \in R^{n_{w_i}}$, and

$p_i(w_i^{\tau_i}(t), d_i) \doteq p_i(w_i^{\tau_i}(t), d_i^{\pm})$ for $d_i^+ = d_i^-$.

We write the multi-trigonometric approximation \tilde{M}_{τ} of the problem M_{τ} :

$$\min_{\tilde{s}_{\tau} \in \tilde{S}_{\tau}} \left\{ J(\tilde{s}_{\tau}) \doteq \sum_{i \in \mathbb{N}} a_i(y_i) \mid y_i = E_{\tau_i} \left[h_i(x_i(t, \tilde{s}_i^{\tau_i}), \tilde{w}_i^{\tau_i}(t)) \right], b_i(y_i) = 0, c_i(y_i) \leq 0, \right. \\ \left. E_{\tau_i} \left[f_i(x_i(t, \tilde{s}_i^{\tau_i}), \tilde{w}_i^{\tau_i}(t)) \right] = 0, K_i w_i \leq \sum_{j \in \mathbb{N}} K_{ij} y_j, p_i(\tilde{w}_i^{\tau_i}) \leq 0 (i \in \mathbb{N}) \right\},$$

where the mappings $p_i : T_{\kappa_i}^{\tau_i, n_{w_i}} \rightarrow R^{n_{p_i}}$ ($n_{p_i} \doteq 2(n_{w_{i1}} + n_{w_{i2}} + r_i n_{w_i})$) determining the constraints on the T-controls are defined as

$$p_i(\tilde{w}_i^{\tau_i}) \doteq (p_{i0}^T(\tilde{w}_i^{\tau_i}), p_{i1}^T(\tilde{w}_i^{\tau_i}), \dots, p_{ir}^T(\tilde{w}_i^{\tau_i}))^T$$

with

$$p_{i0}(\tilde{w}_i^{\tau_i}) \doteq (p_i^T(w_{i1}, d_{i1}^\pm), p_i^T(w_{i2}, d_{i2}^\pm))^T, \quad d_{i1}^+ \doteq 1_{i1}, d_{i1}^- \doteq 0_{i1}, d_{i2}^+ \doteq 0_{i2}, d_{i2}^- \doteq 1_{i2},$$

$$p_{ir}(\tilde{w}_i^{\tau_i}) \doteq (p_i^T(\tilde{w}_{i0}^{\tau_i}(t_{ir}), d_{i0}), p_i^T(\tilde{w}_{i1}^{\tau_i}(t_{ir}), d_{i1}), p_i^T(\tilde{w}_{i2}^{\tau_i}(t_{ir}), d_{i2}))^T,$$

$$t_{ir} \doteq \tau_i(r-1)/r_i, \quad r \in \mathbb{R}_i \doteq \{1, 2, \dots, r_i\},$$

$$d_{i0} \doteq \rho_i(1_{i0} - |\bar{w}_{i0}|), \quad d_{i1} \doteq \rho_i(1_{i1} - w_{i1}), \quad d_{i2} \doteq \rho_i(w_{i2} - 1_{i2}),$$

and $\rho_i \doteq \cos^{-1}(\pi\kappa_i/r_i)$, and $\tilde{S}_\tau \doteq (\tilde{S}_i^{\tau_i})_{i \in \mathbb{N}} \in \tilde{S} \doteq \prod_{i \in \mathbb{N}} \tilde{S}_i^{\tau_i}$ is the reduced multi-trigonometric control process of the IC system with the components

$\tilde{S}_i^{\tau_i} \doteq (x_i^0, w_i^{\tau_i}) \in \tilde{S}_i^{\tau_i} \doteq R^{n_{x_i}} \times T_{\kappa_i}^{\tau_i, n_{w_i}}$. The set of all admissible control processes of the problem \tilde{M}_τ is denoted by \tilde{S}_τ^{ad} .

Assumption 3: The number of points r_i of the discrete time grid $\{t_{ir}\}_{r=1}^{r_i}$ is coordinated with the degree κ_i of the trigonometric polynomials $\tilde{w}_i^{\tau_i}$ such that $r_i > 2\kappa_i$.

Lemma 1. *The \bar{s} -process and the problems \tilde{M}_τ , M_τ and M_τ have the following nesting $\bar{s} \in \tilde{S}_\tau^{ad} \subset S_\tau^{ad} \subset S_\tau$, which means that the set of the reduced admissible multi-trigonometric control processes \tilde{S}_τ^{ad} contains the \bar{s} -process, and is contained in the set of the reduced admissible multiperiodic control processes S_τ^{ad} , which can be extended to the set of admissible multiperiodic control processes S_τ^{ad} .*

Proof. The \bar{s} -process satisfies the constraints p_{i0} by their definition. It also verifies the constraints p_{ir} , $r \in \mathbb{R}_i$, since its dynamic parts are nullified $\tilde{w}_{ik}^{\tau_i} = 0_{ik}$ and $d_{ik} > 0$, $k = 0, 1, 2$. Thus $\bar{s} \in \tilde{S}_\tau^{ad}$. The constraints $p_i(\tilde{w}_{i0}^{\tau_i}(t_{ir}), d_{i0}) \leq 0$ mean that $|\tilde{w}_{i0}^{\tau_i}(t_{ir})| \leq d_{i0}$ ($r \in \mathbb{R}_i$), which implies, by the uniform norm evaluation of the T-control [7] [8], the inequalities $|\tilde{w}_{i0}^{\tau_i}(t)| \leq 1_{i0} - |\bar{w}_{i0}|$ and $|\bar{w}_{i0} + \tilde{w}_{i0}^{\tau_i}(t)| \leq 1_{i0}$ ($t \in T_{\tau_i}$). The constraints $p_i(\tilde{w}_{i1}^{\tau_i}(t_{ir}), d_{i1}) \leq 0$ mean that $|\tilde{w}_{i1}^{\tau_i}(t_{ir})| \leq c_i(1_{i1} - w_{i1})$ ($r \in \mathbb{R}_i$), and imply by the same evaluation $-(1_{i1} - w_{i1}) \leq \tilde{w}_{i1}^{\tau_i}(t) \leq 1_{i1} - w_{i1}$ ($t \in T_{\tau_i}$). On the other hand the constraint $p_i(w_{i1}, d_{i1}^\pm) \leq 0$ involves $0_{i1} \leq w_{i1} \leq 1_{i1}$. Hence $-1_{i1} - w_{i1} \leq -(1_{i1} - w_{i1})$ and $-1_{i1} \leq w_{i1} + \tilde{w}_{i1}^{\tau_i}(t) \leq 1_{i1}$ ($t \in T_{\tau_i}$). Similarly the constraints $p_i(\tilde{w}_{i2}^{\tau_i}(t_{ir}), d_{i2}) \leq 0$ and $p_i(w_{i2}, d_{i2}^\pm) \leq 0$ imply $-1_{i2} \leq w_{i2} + \tilde{w}_{i2}^{\tau_i}(t) \leq 1_{i2}$ ($t \in T_{\tau_i}$). Thus $\tilde{S}_\tau^{ad} \subset S_\tau^{ad}$. The latter set can be extended to the set S_τ^{ad} . \square

Let $L(y, x, w, \mu) \doteq \sum_{i \in \mathbb{N}} L_i(y_i, x_i, w_i, \mu)$ be the L(agrang)-function for the problem S with

$$L_i(y_i, x_i, w_i, \mu) \doteq \mu_0 a_i(y_i) + \mu_{h_i}^T (h_i(x_i, w_i) - y_i) + \mu_{b_i}^T b_i(y_i) + \mu_{c_i}^T \bar{c}_i(y_i) + \mu_{f_i}^T f_i(x_i, w_i) + \mu_{K_i}^T \bar{K}_i w_i - \sum_{j \in \mathbb{N}} \mu_{K_j}^T \bar{K}_j(y_i) + \mu_{\bar{p}_i}^T \bar{p}_i(w_i),$$

where μ_0 is the multiplier connected with the performance index of the problem S , and $\mu_{h_i} \in R^{n_{h_i}}$, $\mu_{b_i} \in R^{n_{b_i}}$, $\mu_{c_i} \in R^{n_{c_i}}$, $\mu_{f_i} \in R^{n_{f_i}}$, $\mu_{K_i} \in R^{n_{K_i}}$ and $\mu_{\bar{p}_i} \in R^{n_{\bar{p}_i}}$ are the multipliers assigned to the constraints h_i , b_i , \bar{c}_i , f_i , \bar{K}_i and \bar{p}_i , and

$\mu \doteq (\mu_0, \mu_1^T, \dots, \mu_N^T)^T \in R^{n_\mu}$ ($n_\mu \doteq 1 + n_{\mu_1} + \dots + n_{\mu_N}$) is the multiplier of the problem S with $\mu_i \doteq (\mu_{h_i}^T, \mu_{b_i}^T, \mu_{c_i}^T, \mu_{f_i}^T, \mu_{K_i}^T, \mu_{\bar{p}_i}^T)^T \in R^{n_{\mu_i}}$ ($n_{\mu_i} \doteq n_{h_i} + n_{b_i} + n_{c_i} + n_{f_i} + n_{K_i} + n_{\bar{p}_i}$), and $\bar{c}_i \in R^{n_{\bar{c}_i}}$ is the active part of the constraint c_i at $\bar{y}_i \doteq h_i(\bar{x}_i, \bar{w}_i)$, $\bar{K}_i \in R^{n_{\bar{K}_i}}$ is the active part of the constraint K_i at (\bar{w}_i, \bar{y}_i) , and $\bar{p}_i \in R^{n_{\bar{p}_i}}$ is the active part of the constraint p_i at \bar{y}_i . We set

$$\bar{L}'_{i, y_i}(\mu) \doteq L'_{i, y_i}(\bar{y}_i, \bar{x}_i, \bar{w}_i), \bar{L}'_{i, x_i}(\mu) \doteq L'_{i, x_i}(\bar{y}_i, \bar{x}_i, \bar{w}_i), \bar{L}'_{i, w_i}(\mu) \doteq L'_{i, w_i}(\bar{y}_i, \bar{x}_i, \bar{w}_i).$$

We exploit the finite-dimensional optimization theory avoiding regularity conditions discussed for nonlinear programming problems in [9], and in [10] as a particular case of a variety of abstract optimization problems.

Lemma 2. *If \bar{s} is a local minimum of the problem S , then there exists a nonzero multiplier $\mu \in R^\mu$ such that the following conditions are satisfied*

$$\mu_0 \geq 0, \mu_{\bar{c}_i} \geq 0, \mu_{\bar{p}_i} \geq 0,$$

$$\bar{L}'_{i, y_i}(\mu) = 0, \bar{L}'_{i, x_i}(\mu) = 0, \bar{L}'_{i, w_i}(\mu) = 0 \quad (i \in \mathbb{N}). \tag{8}$$

Let $\nu_i \in R^{n_{\bar{p}_i}}$ be the multipliers for the active constraints \bar{p}_i , let

$$\nu \doteq (\nu_1, \nu_2, \dots, \nu_N)^T \in R^{n_\nu} \left(n_\nu \doteq \sum_{i \in \mathbb{N}} n_{\bar{p}_i} \right),$$

and let $\lambda \doteq (\mu^T, \nu^T)^T \in R^{n_\lambda}$ ($n_\lambda \doteq n_\mu + n_\nu$) be the multiplier of the problem \tilde{M}_τ . We set $x_0 \doteq (x_i^0)_{i \in \mathbb{N}}$, $\tilde{w}_\tau \doteq (\tilde{w}_i^{\tau_i})_{i \in \mathbb{N}}$, $\ell_i(\tilde{w}_i^{\tau_i}, \nu_i) \doteq \nu_i^T \bar{p}_i(\tilde{w}_i^{\tau_i})$, and we write the L -function of the problem \tilde{M}_τ :

$$L_\tau(y, x_0, \tilde{w}_\tau, \lambda) \doteq \sum_{i \in \mathbb{N}} \left(E_{\tau_i} \left[L_i(y_i, x_i(t, s_i^{\tau_i}), \tilde{w}_i^{\tau_i}(t), \mu) \right] + \ell_i(\tilde{w}_i^{\tau_i}, \nu_i) \right).$$

We abbreviate the (partial) derivatives evaluated at \bar{s} as

$$A_i \doteq a'_i(\bar{y}_i), B_i \doteq b'_i(\bar{y}_i), \bar{C}_i \doteq \bar{c}'_i(\bar{y}_i), H_{i1} \doteq h'_{i, x_i}(\bar{s}_i), H_{i2} \doteq h'_{i, w_i}(\bar{s}_i), F_{i1} \doteq f'_{i, x_i}(\bar{s}_i),$$

$$F_{i2} \doteq f'_{i, w_i}(s_i), \bar{P}_i \doteq \bar{p}'_{i, w_i}(\bar{w}_i), \bar{\ell}'_{i, w_i}(\nu_i) \doteq \ell'_{i, w_i}(\bar{w}_i, \nu_i), \bar{\ell}'_{i, \bar{w}_i}(\nu_i) \doteq \ell'_{i, \bar{w}_i}(\bar{w}_i, \nu_i),$$

$$\bar{L}'_{\tau, y_i}(\lambda) \doteq L'_{\tau, y_i}(\bar{y}, \bar{x}, \bar{w}, \lambda), \bar{L}'_{\tau, x_i^0}(\lambda) \doteq L'_{\tau, x_i^0}(\bar{y}, \bar{x}, \bar{w}, \lambda),$$

$$\bar{L}'_{\tau, w_i}(\lambda) \doteq L'_{\tau, w_i}(\bar{y}, \bar{x}, \bar{w}, \lambda), \bar{L}'_{\tau, \bar{w}_i}(\lambda) \doteq L'_{\tau, \bar{w}_i}(\bar{y}, \bar{x}, \bar{w}, \lambda).$$

Assumption 4: The matrices $j\omega_i I_{n_{x_i}} - F_{i1}$ are nonsingular for all $\omega_i \doteq 2\pi/\tau_i$ such that $\tau_i \in \mathcal{T}_i$.

This assumption eliminates the onset of free, and resonance oscillations in the sub-systems.

Lemma 3. *The s -process satisfies the FON conditions of the problem M_τ regardless if it is its local minimum or not. These conditions take for a nonzero multiplier $\lambda = (\mu^T, \nu^T)^T$ the form*

$$\mu_0 \geq 0, \mu_{\bar{c}_i} \geq 0, \mu_{\bar{p}_i} \geq 0, \nu_i \geq 0,$$

$$\bar{L}'_{i, y_i}(\mu) = 0, \bar{L}'_{i, x_i}(\mu) = 0, \bar{L}'_{i, w_i}(\mu) + \bar{\ell}'_{i, w_i}(\nu_i) = 0, \bar{\ell}'_{i, \bar{w}_i}(\nu_i) = 0 \quad (i \in \mathbb{N}). \tag{9}$$

Proof. The problem M_τ can be interpreted as the finite dimensional optimization problem with respect to the argument $(x_{i0}, \tilde{w}_i)_{i \in \mathbb{N}}$. The conditions (9) are a conse-

quence of the nullifying of the derivatives $\bar{L}'_{\tau, y_i}(\lambda), \bar{L}'_{\tau, x_i^0}(\lambda), \bar{L}'_{\tau, w_i}(\lambda)$ and $\bar{L}'_{\tau, \bar{w}_i}(\lambda)$. They are satisfied for $\nu = 0$ and $\mu \neq 0$ following from the conditions (8). \square

Thus the FON conditions of the problem \tilde{M}_τ cannot be used to discern improving multiperiodic controls. The second order necessary (SON) conditions exploiting the set \tilde{D}_τ of critical directions can be useful to this end. Because of the averaging operation it may be defined in terms of the variations of the constant components δx_i of the periodic state trajectories of the subsystems and the variations of their T-controls $\delta \tilde{w}_i^{\tau_i}$

$$\tilde{D}_\tau \doteq \left\{ s_\tau \in \tilde{S}_\tau : \sum_{i \in \mathbb{N}} (A_{i1} \delta x_i + A_{i2} \delta w_i) \leq 0, B_{i1} \delta x_i + B_{i2} \delta w_i = 0, \bar{C}_{i1} \delta x_i + \bar{C}_{i2} \delta w_i \leq 0, \right. \\ \left. F_{i1} \delta x_i + F_{i2} \delta w_i = 0, K_i \delta w_i \leq \sum_{j \in \mathbb{N}} (K_{ij1} \delta x_j + K_{ij2} \delta w_j), P_i \delta \tilde{w}_i \leq 0 \ (i \in \mathbb{N}) \right\}$$

where $A_{ik} \doteq A_i H_{ik}, B_{ik} \doteq B_i H_{ik}, \bar{C}_{ik} \doteq \bar{C}_i H_{ik}, K_{ijk} \doteq K_{ij} H_{jk}, k = 1, 2$.

Let $\Lambda \subset R^{n_\lambda} \doteq \{\lambda \in R^{n_\lambda} : |\lambda|_1 = 1\}$ be the set of the normalized multipliers satisfying the FON conditions (9) of the problem \tilde{M}_τ , let $\mathcal{T} \doteq \prod_{i \in \mathbb{N}} \mathcal{T}_i$ be the set of admissible multiperiods of the IC system, and let $\delta \tilde{w}_{ik}^\kappa \in C^{n_{w_{ik}}}$ are the subvectors of the complex vector $\delta \tilde{w}_i^\kappa \doteq \left(\delta \tilde{w}_{i0}^\kappa - j \delta \tilde{w}_{i1}^\kappa \right) \in C^{n_{w_i}}$ connected with the internal ($k = 0$) and boundary parts ($k = 1, 2$) of the vector \bar{w}_i , respectively. Let us denote the spectral transfer function for the i th subsystem by $G_i(j\kappa\omega_i) \doteq \left(j\omega_i I_{n_{x_i}} - F_{i1} \right)^{-1} F_{i2}$, and by

$$\Pi_i(\kappa\omega_i, \lambda) \doteq G_i^*(j\kappa\omega_i) \bar{L}_{i, x_i}''(\lambda) G_i(j\kappa\omega_i) + G_i^*(j\kappa\omega_i) \bar{L}_{i, x_i w_i}''(\lambda) \\ + \bar{L}_{i, w_i x_i}''(\lambda) G_i(j\kappa\omega_i) + \bar{L}_{i, w_i w_i}''(\lambda)$$

its Π -matrix.

The contradiction of the SON conditions for the problem \tilde{M}_τ yields

Theorem 1. *The problem M_τ is locally proper at the s-process if for a certain admissible multiperiod $\tau \in \mathcal{T}$ and a critical direction $\delta \bar{s}_\tau \in \tilde{D}_\tau$ the inequality*

$$\max_{\lambda \in \Lambda} \delta^2 \bar{L}_\tau(\lambda) < 0 \tag{10}$$

holds, where $\delta^2 \bar{L}_\tau(\lambda)$ is the second variation of L_τ at \bar{s} taking the form

$$\delta^2 \bar{L}_\tau(\lambda) \doteq \sum_{i \in \mathbb{N}} \left((H_{i1} \delta x_i + H_{i2} \delta w_i)^T \bar{L}_{i, y_i}''(\lambda) (H_{i1} \delta x_i + H_{i2} \delta w_i) + \delta x_i^T \bar{L}_{i, x_i}''(\lambda) \delta x_i \right. \\ \left. + 2 \delta x_i^T \bar{L}_{i, x_i w_i}''(\lambda) \delta w_i + \delta w_i^T \bar{L}_{i, w_i w_i}''(\lambda) \delta w_i + \frac{1}{2} \delta \tilde{w}_i^{\kappa*} \Pi_i(\kappa\omega_i, \lambda) \delta \tilde{w}_i^\kappa \right)$$

or in the structural version

$$\delta^2 \bar{L}_\tau(\lambda) \doteq \sum_{i \in \mathbb{N}} \left((H_{i1} \delta x_i + H_{i2} \delta w_i)^T \bar{L}_{i, y_i}''(\lambda) (H_{i1} \delta x_i + H_{i2} \delta w_i) + \delta x_i^T \bar{L}_{i, x_i}''(\lambda) \delta x_i \right. \\ \left. + 2 \delta x_i^T \bar{L}_{i, x_i w_i}''(\lambda) \delta w_i + \sum_{k=1}^2 \sum_{l=1}^2 \delta w_{ik}^T \bar{L}_{i, w_{ik} w_{il}}''(\lambda) \delta w_{il} + \frac{1}{2} \sum_{\kappa=1}^{\kappa_i} \sum_{k=0}^2 \sum_{l=0}^2 \delta \tilde{w}_{ik}^{\kappa*} \Pi_{ikl}(\kappa\omega_i, \lambda) \delta \tilde{w}_{il}^\kappa \right),$$

and $\Pi_{ikl}(\kappa\omega_i, \lambda)$ are the submatrices of the matrix $\Pi_i(\kappa\omega_i, \lambda)$ of the dimension $n_{w_{ik}} \times n_{w_{il}}$ with the upper left hand corner at $\left(\sum_{\bar{k}=0}^k n_{w_{i\bar{k}}}, \sum_{\bar{l}=0}^l n_{w_{i\bar{l}}} \right)$.

Proof. Lemma 2 shows that the finite-dimensional optimal steady-state process satisfies the FON conditions with a nonzero Lagrange multiplier without regularity conditions. Lemma 3 shows that this process satisfies also the FON conditions of the optimal multiperiodic control problem regardless if it is local minimum or not. This means that such conditions do not allow to distinguish improving multiperiodic control processes. For this reason the attention is directed to the SON conditions, which take for multi-harmonic control variations especially simple form connected with the generalized Π -test for single systems [11]. If the condition (10) is satisfied then the optimal steady-state process cannot be optimal for the multiperiodic control problem as violating its SON conditions. In a consequence an improving multiperiodic control process exists for the multiperiod exploited in (10). \square

The discussed second order test has the following distinctive features: it concerns the different (possibly incommensurate) basic operation frequencies ω_i of the particular subsystems utilizing advantageously their dynamic properties; structural notation of the pi-form distinguishes the improving influence of the variations of the internal as well as the upper and lower boundary extended controls; even for boundary steady-state extended controls an arbitrary large number of harmonics κ_i is applicable in the second order variation, which may be useful for highly nonlinear complex systems; the maximization in the condition (10) is equivalent to the linear programming problem solvable in finite number of iterations by the simplex algorithm avoiding the verification of the regularity conditions for the \bar{s} -process in the problem \tilde{M}_τ . On the other hand if the mentioned regularity condition can be verified by the MFCQ or the LICQ regularity condition then a normal multiplier $\lambda = (1, \tilde{\lambda})$ is applicable in the second order test.

4. Example

Let two continuously stirred tank reactors be coupled by the inventory interactions. In each of them the parallel chemical reactions $A_i \rightarrow B_i, A_i \rightarrow C_i$ take place, where A_i is the substrate of the i th reactor, B_i is its desired product, and C_i is its by-product ($i = 1, 2$). The i th reactor is τ_i -periodically operated, $x_{i1}(t), x_{i2}(t), x_{i3}(t)$ are its concentrations of A_i, B_i, C_i , respectively, and $x_i(t) = (x_{i1}^T(t), x_{i2}^T(t), x_{i3}^T(t))^T$ is its state, $w_{i1}(t)$ is its input concentration control, $w_{i2}(t)$ is its input intensity control, and $w_{i3}(t)$ is its inventory interaction transferring the by-product of the cooperating subsystem as the catalyst of its reactions, and $w_i(t) = (w_{i1}^T(t), w_{i2}^T(t), w_{i3}^T(t))^T$ is its extended control. Consider the following optimal control problem for the discussed system: minimize the objective function

$$\mathcal{J} \doteq y_{11} + y_{21}$$

being a scalar function of the averaged outputs

$$y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{pmatrix} = \frac{1}{\tau_i} \int_0^{\tau_i} \begin{pmatrix} h_{i1}(x_i(t), w_i(t)) \\ h_{i2}(x_i(t), w_i(t)) \\ h_{i3}(x_i(t), w_i(t)) \end{pmatrix} dt$$

with

$$h_{i1}(x_i(t), w_i(t)) \doteq c_{i1} w_{i3}^2(t) - c_{i2} w_{i2}(t) x_{i2}(t),$$

$$h_{i2}(x_i(t), w_i(t)) \doteq w_{i1}(t) w_{i2}(t),$$

$$h_{i3}(x_i(t), w_i(t)) \doteq x_{i3}(t),$$

and subject for $i = 1, 2$ to the local constraints

$$y_{i2} - 0.5 \leq 0,$$

$$\dot{x}_{i1}(t) = w_{i2}(t)(w_{i1}(t) - x_{i1}(t)) - \kappa_{i1} w_{i3}^{q_{i1}}(t) x_{i1}^2(t) / \left((0.2 + x_{i1}^2(t))^{p_{i1}} - \kappa_{i2} w_{i3}^{q_{i2}}(t) x_{i1}(t) \right),$$

$$\dot{x}_{i2}(t) = -w_{i2}(t) x_{i2}(t) + \kappa_{i1} w_{i3}^{q_{i1}}(t) x_{i1}^2(t) / \left((0.2 + x_{i1}^2(t))^{p_{i1}} \right),$$

$$x_{i1}(t) + x_{i2}(t) + x_{i3}(t) - 1 = 0,$$

$$\tau_i \in [0.1, 20], \quad w_{i1}(t) \in [0, 1], \quad t \in [0, \tau_i],$$

and to the interaction constraints

$$\frac{1}{\tau_1} \int_0^{\tau_1} w_{13}(t) dt \leq y_{23}, \quad \frac{1}{\tau_2} \int_0^{\tau_2} w_{23}(t) dt \leq y_{13}.$$

Thus $a_i(y_i) = y_{i1}$, $c_i(y_i) = y_{i2} - 0.5$, $K_i = (0, 0, 1)$, $K_{ii} = (0, 0, 0)$, $K_{ij} = (0, 0, 1)$ ($i \neq j$). The objective function is equivalent to the maximization of the summary gain from the useful products of both reactors. The cost of the interactions is included. We assume the parameters $q_{11} = 2$, $q_{12} = 0.75$, $q_{21} = 1$, $q_{22} = 0.5$, $\kappa_{11} = 4$, $\kappa_{12} = 5$, $\kappa_{21} = 5$, $\kappa_{22} = 5$, $p_{11} = 1$, $p_{21} = 0$. The optimal steady-state solution is obtained for the boundary input concentration controls and the internal other controls $\hat{w}_{11} = 1$, $\hat{w}_{12} = 0.5$, $\hat{w}_{13} = 0.59998$, $\hat{w}_{21} = 1$, $\hat{w}_{22} = 0.5$, $\hat{w}_{23} = 0.5$, $\hat{x}_{11} = 0.10772$, $\hat{x}_{12} = 0.15792$, $\hat{x}_{13} = 0.734355$, $\hat{x}_{21} = 0.175035$, $\hat{x}_{22} = 0.224985$, $\hat{x}_{23} = 0.59998$.

The variation of the optimal steady state solution $\delta w_i = (-0.05, 0, -0.1)$, $\delta x_1 = (0.00891, -0.0027925, 0.019006)$, $\delta x_2 = (0.003070, -0.022744, 0.019673)$ moves the boundary input concentration controls to the interiority of their box sets, and satisfies all the constraints of the set of critical directions:

$$\sum_{i=1}^2 (A_{i1} \delta x_i + A_{i2} \delta w_i) = -0.14089, \quad \bar{C}_{i1} \delta x_i + \bar{C}_{i2} \delta w_i = -0.05,$$

$$K_i \delta w_i = -0.1, \quad \sum_{i=1}^2 (K_{ij1} \delta x_j + K_{ij2} \delta w_j) = 0.038679.$$

The positive component of the second order test generated by the steady state variation $\bar{L}_{1, w_{13} w_{13}}'' \delta w_{13}^2 + \bar{L}_{2, w_{23} w_{23}}'' \delta w_{23}^2 = 0.16$ does not disturb the multiperiodic control problem to be proper.

The multifrequency second order test for the discussed complex system with the inventory interactions is shown on **Figure 1** and **Figure 2** for different number of harmonics.

The second order test obtained shows the diversified advantageous operation frequencies for particular subsystems $\omega_1 = 5.5$ and $\omega_2 = 2.5$ for the single harmonic and $\omega_1 = 4.5$ and $\omega_2 = 1.5$ for the five harmonics. Thus the multiharmonic test may yield more exact information concerning the proper operation periods for particular subsystems.

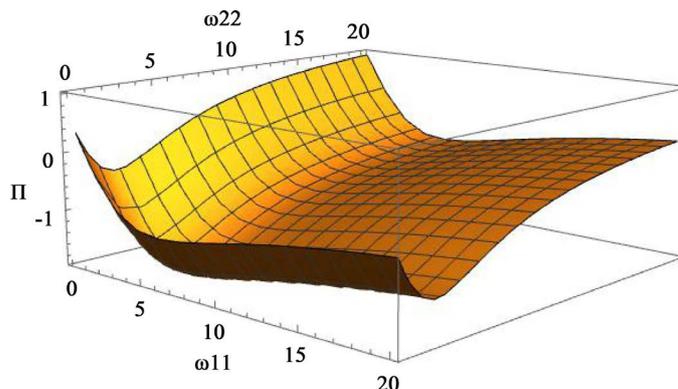


Figure 1. The single harmonic second order test for the complex system with the inventory interactions.

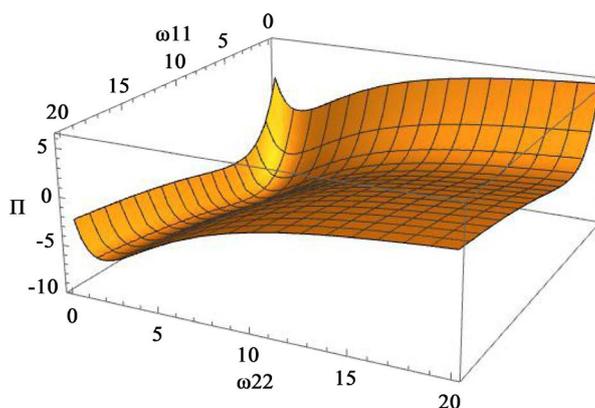


Figure 2. The five harmonics second order test for the complex system with the inventory interactions.

5. Conclusion

In this note, we formulated the optimal multiperiodic control problem for inventory constrained subsystems. It is aimed at the intensification of the productivity of complex processes. We proposed a multifrequency second-order test for complex multiperiodic systems including the boundary optimal steady-state process and an arbitrary large number of harmonics used to verify its improvement by the multiperiodic operation. We generalized the method of critical directions for single periodic systems [10] [11] to complex multiperiodic systems. We illustrated the approach proposed on the example of the multiperiodic optimization of a system of chemical reactors.

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