

# Airy, Beltrami, Maxwell, Einstein and Lanczos Potentials Revisited

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## Abstract

The purpose of this paper is to revisit the well known *potentials*, also called *stress functions*, needed in order to study the parametrizations of the stress equations, respectively provided by G.B. Airy (1863) for 2-dimensional elasticity, then by E. Beltrami (1892), J.C. Maxwell (1870) for 3-dimensional elasticity, finally by A. Einstein (1915) for 4-dimensional elasticity, both with a variational procedure introduced by C. Lanczos (1949, 1962) in order to relate potentials to Lagrange multipliers. Using the methods of *Algebraic Analysis*, namely mixing differential geometry with homological algebra and combining the *double duality test* involved with the *Spencer cohomology*, we shall be able to extend these results to an arbitrary situation with an arbitrary dimension  $n$ . We shall also explain why double duality is perfectly adapted to variational calculus with differential constraints as a way to eliminate the corresponding Lagrange multipliers. For example, the *canonical parametrization* of the stress equations is just described by the formal adjoint of the  $n^2(n^2-1)/12$  components of the linearized Riemann tensor considered as a linear second order differential operator but the minimum number of potentials needed is equal to  $n(n-1)/2$  for any *minimal parametrization*, the Einstein parametrization being “*in between*” with  $n(n+1)/2$  potentials. We provide all the above results without even using indices for writing down explicit formulas in the way it is done in any textbook today, but it could be *strictly impossible* to obtain them without using the above methods. We also revisit the *possibility* (Maxwell equations of electromagnetism) or the *impossibility* (Einstein equations of gravitation) to obtain canonical or minimal parametrizations for various equations of physics. It is nevertheless important to notice that, when  $n$  and the algorithms presented are known, most of the calculations can be achieved by using computers for the corresponding symbolic computations. Finally, though the paper is mathematically oriented as it aims providing new insights towards the mathematical foundations of general relativity, it is written in a rather self-contained way.

## Keywords

Stress Equations, Stress Functions, Elasticity Theory, Lagrange Multipliers, Formal Adjoint, Control

**Theory, General Relativity, Einstein Equations, Lanczos Potentials, Algebraic Analysis, Riemann Tensor, Weyl Tensor**

**1. Introduction**

The language of *differential modules* has been recently introduced in control theory as a way to understand in an intrinsic way the *structural properties* of systems of ordinary differential (OD) or partial differential (PD) equations (controllability, observability, identifiability, ...) [1]-[10]. A similar comment can be done for optimal control that is for variational calculus with differential constraints, and the author thanks Prof. Lars Andersson (Einstein Institute, Potsdam) for having suggested him to study the Lanczos potential within this new framework.

We start providing a few explicit examples in order to convince the reader that the corresponding computations are often becoming so tricky that nobody could achieve them or even imagine any underlying general algorithm, for example in the study of the mathematical foundations of control theory, elasticity theory or general relativity.

**EXAMPLE 1.1: OD Control Theory**

With one independent variable  $x$ , for example the time  $t$  in control theory or the curvilinear abscissa  $s$  in the study of a beam, and three unknowns  $(y^1, y^2, y^3)$ . Setting formally  $d_x y^k = y_x^k$  for  $k=1,2,3$  and so on, let us consider the system made by the two *first order* OD equations depending on a variable coefficient  $a(x)$ :

$$y_x^3 - a(x)y^2 - y_x^1 = 0, \quad y^3 - y_x^2 + y_x^1 = 0$$

In control theory, if  $a = cst$  is a constant parameter, one could bring the system to *any first order Kalman form* and check that the corresponding control system is *controllable* if and only if  $a(a-1) \neq 0$ , that is  $a \neq 0$  and  $a \neq 1$  (exercise), independently of the choice of 1 *input* and 2 *outputs* among the 3 control variables [11]. In addition to that, using the second OD equation in the form  $y^3 = y_x^2 - y_x^1$  and substituting in the first, we get the only *second order* OD equation:

$$y_{xx}^1 + y_x^1 - y_{xx}^2 + a(x)y^2 = 0$$

a result leading to a kind of “vicious circle” because the only way to test controllability is ... to bring this second order equation back to a first order system and there are a lot of possibilities. Again, *in any case*, the only critical values are  $a = 0$  and  $a = 1$ . Of course, one could dream about a direct approach providing the same result in an intrinsic way. Introducing the operator  $d = d_x$  as the (formal) derivative with respect to  $x$ , we may rewrite the last equation in the form:

$$d(d+1)y^1 = (d^2 - a)y^2$$

Replacing the operators  $d(d+1)$  and  $d^2 - a$  by the polynomials  $\chi(\chi+1)$  and  $\chi^2 - a$ , the two polynomials have a common root  $\chi = 0 \Rightarrow a = 0$  or  $\chi = -1 \Rightarrow a = 1$  and we find back the desired critical values but such a result is not intrinsic at all. However, we notice that, for example  $a = 0 \Rightarrow d((d+1)y^1 - dy^2) = 0$ . Introducing  $z' = y_x^1 + y^1 - y_x^2$ , we get  $z'_x = 0$  while  $a = 1 \Rightarrow (d+1)(dy^1 - (d-1)y^2)$  that is, setting  $z'' = y_x^1 - y^2$ , we get now  $z''_x + z'' = 0$ . Calling “*torsion element*” any scalar quantity made from the unknowns and their derivatives but satisfying at least one OD equation, we discover that such quantities do exist ... if and only if  $a = 0$  or  $a = 1$  (exercise). Of course, the existence of any torsion element breaks at once the controllability of the system but the converse is not evident at all, a result leading nevertheless to the feeling that *a control system is controllable if and only if no torsion element can be found* and such an idea can be extended “*mutatis mutandis*” to any system of PD equations [6]. However, this result could be useful if and only if there is a test for checking such a property of the system.

Now, using a variable parameter  $a(x)$ , *not a word of the preceding approach is left* but the concept of a torsion element still exists. We shall prove, at the end of the paper, that the condition  $a(a-1) \neq 0$  becomes  $\partial_x a + a^2 - a \neq 0$  and that the computations needed are quite far from the previous ones. We ask the reader familiar with classical control theory to make his mind a few minutes (or hours!) to agree with us by trying to re-

cover himself such a differential condition.

**EXAMPLE 1.2: OD Optimal Control Theory**

OD optimal control is the study of OD variational calculus with OD constraints described by OD control systems. However, while studying optimal control, the author of this paper has been surprised to discover that, *in all cases*, the OD constraints were defined by means of controllable control systems. It is only at the end of this paper that the importance of such an assumption will be explained. For the moment, we shall provide an example allowing to exhibit all the difficulties involved. For this, let  $y^1 = f^1(x), y^2 = f^2(x)$  be a solution of the following single input/single output (SISO) OD control system where  $a$  is a constant parameter:

$$y_x^1 + y^1 - y_x^2 - ay^2 = 0$$

Proceeding as before, the two polynomials replacing the respective operators are  $\chi + 1, \chi + a$  and can only have the common root  $a = 1$ . Accordingly, the system is controllable if and only if  $a \neq 1$  for any choice of input and output. Now, let us introduce the so-called “cost function” and let us look at the extremum of the integral  $\int \frac{1}{2} \left( (y^1)^2 - (y^2)^2 \right) dx$  under the previous OD constraint. It is well known that the proper way to study such a problem is to introduce a *Lagrange multiplier*  $\lambda$  and to vary the new integral:

$$\int \left[ \frac{1}{2} \left( (y^1)^2 - (y^2)^2 \right) + \lambda \left( y_x^1 + y^1 - y_x^2 - ay^2 \right) \right] dx$$

The corresponding *Euler-Lagrange* (EL) equations are:

$$\begin{cases} y^1 \rightarrow -\lambda_x + \lambda + y^1 = 0 \\ y^2 \rightarrow \lambda_x - a\lambda - y^2 = 0 \end{cases}$$

to which we must add the OD constraint when varying  $\lambda$ . Summing the two EL equations, we get  $(a - 1)\lambda = y^1 - y^2$  and *two possibilities*:

- 1)  $a = 1 \Rightarrow y^1 - y^2 = 0$  compatible with the constraint.
- 2)  $a \neq 1 \Rightarrow \lambda = (y^1 - y^2) / (a - 1)$ .

Substituting, we get:

$$\begin{cases} y_x^1 - y_x^2 - ay^1 + y^2 = 0 \\ y_x^1 - y_x^2 + y^1 - ay^2 = 0 \end{cases}$$

*This system may not be formally integrable.* Indeed, by subtraction, we get  $(a + 1)(y^1 - y^2) = 0$  and must consider the following *two possibilities*:

- $a = -1 \Rightarrow y_x^1 + y^1 - y_x^2 + y^2 = 0$
- $a \neq -1 \Rightarrow y^1 = y^2 = 0$

Summarising the results so far obtained, we discover that *the Lagrange multiplier is known if and only if the system is controllable*. Also, if  $a = -1$ , we may exhibit the parametrization  $\xi_x - \xi = y^1, \xi_x + \xi = y^2$  and the cost function becomes  $2\xi\xi_x = d_x(\xi^2)$ . Equivalently, *when the system is controllable it can be parametrized and the variational problem with constraint becomes a variational problem without any constraint which, sometimes, does not provide EL equations.* We finally understand that extending such a situation to PD variational calculus with PD constraints needs new techniques.

**EXAMPLE 1.3: Elasticity Theory**

In classical elasticity, the *stress tensor density*  $\sigma = (\sigma^{ij} = \sigma^{ji})$  existing inside an elastic body is a symmetric 2-tensor density introduced by A. Cauchy in 1822. The corresponding *Cauchy stress equations* can be written as  $\partial_r \sigma^{ir} = f^i$  where the right member describes the local density of forces applied to the body, for example gravitation. With zero second member, we study the possibility to “*parametrize*” the system of PD equations  $\partial_r \sigma^{ir} = 0$ , namely to express its general solution by means of a certain number of arbitrary functions or *potentials*, called *stress functions*. Of course, the problem is to know about the number of such functions and the order of the parametrizing operator. In what follows, the space has  $n$  local coordinates  $x = (x^i) = (x^1, \dots, x^n)$ . For

$n = 1, 2, 3$  one may introduce the Euclidean metric  $\omega = (\omega_{ij} = \omega_{ji})$  while, for  $n = 4$ , one may consider the Minkowski metric. A few definitions used thereafter will be provided later on.

- $n = 1$  : There is no possible parametrization of  $\partial_i \sigma = 0$ .
- $n = 2$  : The stress equations become  $\partial_1 \sigma^{11} + \partial_2 \sigma^{12} = 0, \partial_1 \sigma^{21} + \partial_2 \sigma^{22} = 0$ . Their second order parametrization  $\sigma^{11} = \partial_{22} \phi, \sigma^{12} = \sigma^{21} = -\partial_{12} \phi, \sigma^{22} = \partial_{11} \phi$  has been provided by George Biddell Airy (1801-1892) in 1863 [12]. It can be simply recovered in the following manner:

$$\begin{aligned} \partial_1 \sigma^{11} = \partial_2 (-\sigma^{12}) &\Rightarrow \exists \varphi, \sigma^{11} = \partial_2 \varphi, \sigma^{12} = -\partial_1 \varphi, \\ \partial_2 \sigma^{22} = \partial_1 (-\sigma^{21}) &\Rightarrow \exists \psi, \sigma^{22} = \partial_1 \psi, \sigma^{21} = -\partial_2 \psi \\ \sigma^{12} = \sigma^{21} &\Rightarrow \partial_1 \varphi = \partial_2 \psi \Rightarrow \exists \phi, \varphi = \partial_2 \phi, \psi = \partial_1 \phi \end{aligned}$$

We get the second order system:

$$\begin{cases} \sigma^{11} \equiv \partial_{22} \phi = 0 & \boxed{1 \quad 2} \\ -\sigma^{12} \equiv \partial_{12} \phi = 0 & \boxed{1 \quad \bullet} \\ \sigma^{22} \equiv \partial_{11} \phi = 0 & \boxed{1 \quad \bullet} \end{cases}$$

which is involutive with one equation of class 2, 2 equations of class 1 and it is easy to check that the 2 corresponding first order CC are just the stress equations. As we have a system with constant coefficients, we may use localization [6] [13] in order to transform the 2 PD equations into the 2 linear equations  $\chi_1 \sigma^{11} + \chi_2 \sigma^{12} = 0, \chi_1 \sigma^{21} + \chi_2 \sigma^{22} = 0$  and get

$$\sigma^{11} = -\frac{\chi_2}{\chi_1} \sigma^{12} = -\frac{(\chi_2)^2}{\chi_1 \chi_2} \sigma^{12}, \quad \sigma^{22} = -\frac{\chi_1}{\chi_2} \sigma^{12} = -\frac{(\chi_1)^2}{\chi_1 \chi_2} \sigma^{12}$$

Setting  $\sigma^{12} = -\chi_1 \chi_2 \phi$ , we finally get  $\sigma^{11} = (\chi_2)^2 \phi, \sigma^{22} = (\chi_1)^2 \phi$  and obtain the previous parametrization by delocalizing, that is replacing now  $\chi_i$  by  $\partial_i$ .

- $n = 3$  : Things become quite more delicate when we try to parametrize the 3 PD equations:

$$\partial_1 \sigma^{11} + \partial_2 \sigma^{12} + \partial_3 \sigma^{13} = 0, \quad \partial_1 \sigma^{21} + \partial_2 \sigma^{22} + \partial_3 \sigma^{23} = 0, \quad \partial_1 \sigma^{31} + \partial_2 \sigma^{32} + \partial_3 \sigma^{33} = 0$$

Of course, localization could be used similarly by dealing with the 3 linear equations:

$$\chi_1 \sigma^{11} + \chi_2 \sigma^{12} + \chi_3 \sigma^{13} = 0, \quad \chi_1 \sigma^{21} + \chi_2 \sigma^{22} + \chi_3 \sigma^{23} = 0, \quad \chi_1 \sigma^{31} + \chi_2 \sigma^{32} + \chi_3 \sigma^{33} = 0$$

having rank 3 for 6 unknowns but, even if we succeed bringing all the fractions to the same denominator as before after easy but painful calculus, there is an additional difficulty which is well hidden. Indeed, coming back to the previous Example when  $a = cst$ , say  $a = 1$ , we should get  $(d^2 + d)y^1 = (d^2 - 1)y^2 \Rightarrow \chi(\chi + 1)y^1 = (\chi + 1)(\chi - 1)y^2 \Rightarrow \chi y^1 = (\chi - 1)y^2 \Rightarrow y^1 = \frac{\chi - 1}{\chi} y^2$ . Hence, setting  $y^2 = \chi y, y^1 = (\chi - 1)y$ , we only get a

parametrization of the *first order* OD equation  $z \equiv \dot{y}^1 - \dot{y}^2 + y^2 = 0$  leading to  $\dot{z} + z = 0$ . Accordingly, localization does indeed provide a parametrization, ... if we already know there exists a possibility to parametrize the given system or if we are able to check that we have obtained such a parametrization by using involution, a way to supersede the use of Janet or Gröbner bases as was proved for the case  $n = 2$  [14]. Also, if we proceed along such a way, we should surely loose any geometric argument that could exist.

A direct computational approach has been provided by Eugenio Beltrami (1835-1900) in 1892 [15], James Clerk Maxwell (1831-1879) in 1870 [16] and Giacinto Morera (1856-1909) in 1892 [17] by introducing the 6 stress functions  $\phi_{ij} = \phi_{ji}$  through the parametrization obtained by considering:

$$\begin{aligned} \sigma^{11} &= \partial_{33} \phi_{22} + \partial_{22} \phi_{33} - 2\partial_{23} \phi_{23} \\ \sigma^{12} = \sigma^{21} &= \partial_{13} \phi_{23} + \partial_{23} \phi_{13} - \partial_{33} \phi_{12} - \partial_{12} \phi_{33} \end{aligned}$$

and the additional 4 relations obtained by using a cyclic permutation of  $(1, 2, 3)$ . The system:

$$\left\{ \begin{array}{l} \sigma^{11} \equiv \partial_{33}\phi_{22} + \partial_{22}\phi_{33} - 2\partial_{23}\phi_{23} = 0 \\ -\sigma^{12} \equiv \partial_{33}\phi_{12} + \partial_{12}\phi_{33} - \partial_{13}\phi_{23} - \partial_{23}\phi_{13} = 0 \\ \sigma^{22} \equiv \partial_{33}\phi_{11} + \partial_{11}\phi_{33} - 2\partial_{13}\phi_{13} = 0 \\ \sigma^{13} \equiv \partial_{23}\phi_{12} + \partial_{12}\phi_{23} - \partial_{22}\phi_{13} - \partial_{13}\phi_{22} = 0 \\ -\sigma^{23} \equiv \partial_{23}\phi_{11} + \partial_{11}\phi_{23} - \partial_{12}\phi_{13} - \partial_{13}\phi_{12} = 0 \\ \sigma^{33} \equiv \partial_{22}\phi_{11} + \partial_{11}\phi_{22} - 2\partial_{12}\phi_{12} = 0 \end{array} \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$$

is involutive with 3 equations of class 3, 3 equations of class 2 and no equation of class 1. The 3 CC are describing the stress equations which admit therefore a parametrization ... justifying the localization approach “*a posteriori*” but without any geometric framework [18].

Surprisingly, the Maxwell parametrization is obtained by keeping  $\phi_{11} = A, \phi_{22} = B, \phi_{33} = C$  while setting  $\phi_{12} = \phi_{23} = \phi_{31} = 0$  in order to obtain the system:

$$\left\{ \begin{array}{l} \sigma^{11} \equiv \partial_{33}B + \partial_{22}C = 0 \\ \sigma^{22} \equiv \partial_{33}A + \partial_{11}C = 0 \\ -\sigma^{23} \equiv \partial_{23}A = 0 \\ \sigma^{33} \equiv \partial_{22}A + \partial_{11}B = 0 \\ -\sigma^{13} \equiv \partial_{13}B = 0 \\ -\sigma^{12} \equiv \partial_{12}C = 0 \end{array} \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline 1 & \bullet & \bullet \\ \hline 1 & \bullet & \bullet \\ \hline \end{array}$$

However, *this system may not be involutive* and no CC can be found “*a priori*” because the coordinate system is surely not  $\delta$ -regular. Indeed, effecting the linear change of coordinates  $x^1 \rightarrow x^1 + x^3, x^2 \rightarrow x^2 + x^3, x^3 \rightarrow x^3$ , we obtain the involutive system:

$$\left\{ \begin{array}{l} \partial_{33}C + \partial_{13}C + \partial_{23}C + \partial_{12}C = 0 \\ \partial_{33}B + \partial_{13}B = 0 \\ \partial_{33}A + \partial_{23}A = 0 \\ \partial_{23}C - \partial_{13}B - \partial_{13}C - \partial_{12}C + \partial_{22}C = 0 \\ \partial_{23}A - \partial_{22}C + \partial_{13}B + 2\partial_{12}C - \partial_{11}C = 0 \\ \partial_{22}A + \partial_{22}C - 2\partial_{12}C + \partial_{11}B + \partial_{11}C = 0 \end{array} \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$$

and it is easy to check that the 3 CC obtained just amount to the desired 3 stress equations when coming back to the original system of coordinates. Again, *if there is a geometrical background, this change of local coordinates is hiding it totally*. Moreover, we notice that the stress functions kept in the procedure are just the ones on which  $\partial_{33}$  is acting. The reason for such an apparently technical choice is related to very general deep arguments in the theory of differential modules that will only be explained at the end of the paper. The Morera parametrization is obtained similarly by keeping now  $\phi_{23} = L, \phi_{13} = M, \phi_{12} = N$  while setting  $\phi_{11} = \phi_{22} = \phi_{33} = 0$ .

- $n \geq 4$ : As already explained, localization *cannot* be applied directly as we don't know if a parametrization may exist and in any case no analogy with the previous situations  $n = 1, 2, 3$  could be used. Moreover, no known differential geometric background could be used at first sight in order to provide a hint towards the solution. Now, if  $\omega$  is the Minkowski metric and  $\phi = GM/r$  is the gravitational potential, then  $\phi/c^2 \ll 1$  and a perturbation  $\Omega \in S_2T^*$  of  $\omega$  may satisfy in vacuum the 10 second order *Einstein equations* for the 10  $\Omega$ :

$$E_{ij} \equiv \omega^{rs} (d_{ij}\Omega_{rs} + d_{rs}\Omega_{ij} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) - \omega_{ij} (\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}) = 0$$

by introducing the corresponding second order *Einstein operator*  $S_2T^* \xrightarrow{\text{Einstein}} S_2T^* : \Omega \rightarrow E$  when  $n = 4$  [19]. Though it is well known that the corresponding second order *Einstein operator* is parametrizing the stress equations, the challenge of parametrizing Einstein equations has been proposed in 1970 by J. Wheeler for 1000 \$ and solved *negatively* in 1995 by the author who only received 1 \$. We shall see that, *exactly as before and*

though it is quite striking, the key ingredient will be to use the linearized Riemann tensor considered as a second order operator [6] [20]. As an even more striking fact, we shall discover that the condition  $n \geq 4$  has only to do with Spencer cohomology for the symbol of the conformal Killing equations.

**EXAMPLE 1.4: PD Control Theory**

The aim of this last example is to prove that the possibility to exhibit two different parametrizations of the stress equations which has been presented in the previous example has surely nothing to do with the proper mathematical background of elasticity theory!

For this, let us consider the (trivially involutive) inhomogeneous PD equations with two independent variables  $(x^1, x^2)$ , two unknown functions  $(\eta^1, \eta^2)$  and a second member  $\zeta$  :

$$\partial_2 \eta^1 - \partial_1 \eta^2 + x^2 \eta^2 = \zeta$$

Multiplying on the left by a test function  $\lambda$  and integrating by parts, the corresponding inhomogeneous adjoint system of PD equations is:

$$\begin{cases} \eta^1 \rightarrow -\partial_2 \lambda = \mu^1 \\ \eta^2 \rightarrow \partial_1 \lambda + x^2 \lambda = \mu^2 \end{cases}$$

Using crossed derivatives, we get  $\lambda = \partial_2 \mu^2 + \partial_1 \mu^1 + x^2 \mu^1$  and substituting, we get the two CC:

$$\begin{cases} -\partial_{22} \mu^2 - \partial_{12} \mu^1 - x^2 \partial_2 \mu^1 - 2\mu^1 = \nu^1 \\ \partial_{12} \mu^2 + \partial_{11} \mu^1 + 2x^2 \partial_1 \mu^1 + x^2 \partial_2 \mu^2 + (x^2)^2 \mu^1 - \mu^2 = \nu^2 \end{cases}$$

The corresponding generating CC for the second member  $(\nu^1, \nu^2)$  is:

$$\partial_2 \nu^2 + \partial_1 \nu^1 + x^2 \nu^1 = 0$$

Therefore  $\nu^2$  is differentially dependent on  $\nu^1$  but  $\nu^1$  is also differentially dependent on  $\nu^2$ .

Multiplying the first equation by the test function  $\xi^1$ , the second equation by the test function  $\xi^2$ , adding and integrating by parts, we get the canonical parametrization  $(\xi^1, \xi^2) \rightarrow (\eta^1, \eta^2)$  :

$$\begin{cases} \mu^2 \rightarrow -\partial_{22} \xi^1 + \partial_{12} \xi^2 - x^2 \partial_2 \xi^2 - 2\xi^2 = \eta^2 \\ \mu^1 \rightarrow -\partial_{12} \xi^1 + x^2 \partial_2 \xi^1 - \xi^1 + \partial_{11} \xi^2 - 2x^2 \partial_1 \xi^2 + (x^2)^2 \xi^2 = \eta^1 \end{cases} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & \bullet \\ \hline \end{array}$$

of the initial system with zero second member. The system (up to sign) is involutive and the kernel of this parametrization has differential rank equal to 1.

Keeping  $\xi^1 = \xi$  while setting  $\xi^2 = 0$ , we get the first minimal parametrization  $\xi \rightarrow (\eta^1, \eta^2)$  :

$$\begin{cases} -\partial_{22} \xi = \eta^2 \\ -\partial_{12} \xi + x^2 \partial_2 \xi - \xi = \eta^1 \end{cases} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & \bullet \\ \hline \end{array}$$

The system is again involutive (up to sign) and the parametrization is minimal because the kernel of this parametrization has differential rank equal to 0. With a similar comment, setting now  $\xi^1 = 0$  while keeping  $\xi^2 = \xi'$ , we get the second minimal parametrization  $\xi' \rightarrow (\eta^1, \eta^2)$  :

$$\begin{cases} \partial_{11} \xi' - 2x^2 \partial_1 \xi' + (x^2)^2 \xi' = \eta^1 \\ \partial_{12} \xi' - x^2 \partial_2 \xi' - 2\xi' = \eta^2 \end{cases}$$

**EXAMPLE 1.5: PD Optimal Control Theory**

Let us revisit briefly the foundation of n-dimensional elasticity theory as it can be found today in any textbook, restricting our study to  $n = 2$  for simplicity. If  $x = (x^1, x^2)$  is a point in the plane and  $\xi = (\xi^1(x), \xi^2(x))$  is the displacement vector, lowering the indices by means of the Euclidean metric, we may introduce the “small” deformation tensor  $\epsilon = (\epsilon_{ij} = \epsilon_{ji} = (1/2)(\partial_i \xi_j + \partial_j \xi_i))$  with  $n(n+1)/2 = 3$  (independent) components  $(\epsilon_{11}, \epsilon_{12} = \epsilon_{21}, \epsilon_{22})$ . If we study a part of a deformed body, for example a thin elastic plane sheet, by means of a variational principle, we may introduce the local density of free energy  $\varphi(\epsilon) = \varphi(\epsilon_{ij} | i \leq j) = \varphi(\epsilon_{11}, \epsilon_{12}, \epsilon_{22})$  and vary the total free energy  $\Phi = \int \varphi(\epsilon) dx$  with  $dx = dx^1 \wedge dx^2$  by introducing  $\sigma^{ij} = \partial \varphi / \partial \epsilon_{ij}$  for  $i \leq j$  in order

to obtain  $\delta\Phi = \int (\sigma^{11}\delta\epsilon_{11} + \sigma^{12}\delta\epsilon_{12} + \sigma^{22}\delta\epsilon_{22})dx$ . Accordingly, the “decision” to define the stress tensor  $\sigma$  by a symmetric matrix with  $\sigma^{12} = \sigma^{21}$  is purely artificial within such a variational principle. Indeed, the usual Cauchy device (1828) assumes that each element of a boundary surface is acted on by a surface density of force  $\sigma$  with a linear dependence  $\sigma = (\sigma^{ir}(x)n_r)$  on the outward normal unit vector  $n = (n_r)$  and does not make any assumption on the stress tensor. It is only by an equilibrium of forces and couples, namely the well known phenomenological static torsor equilibrium, that one can “prove” the symmetry of  $\sigma$ . However, even if we assume this symmetry, we now need the different summation  $\sigma^{ij}\delta\epsilon_{ij} = \sigma^{11}\delta\epsilon_{11} + 2\sigma^{12}\delta\epsilon_{12} + \sigma^{22}\delta\epsilon_{22} = \sigma^{ir}\partial_r\delta\xi_i$ . An integration by parts and a change of sign produce the integral  $\int (\partial_r\sigma^{ir})\delta\xi_i dx$  leading to the stress equations  $\partial_r\sigma^{ir} = 0$  already considered. This classical approach to elasticity theory, based on invariant theory with respect to the group of rigid motions, cannot therefore describe equilibrium of torsors by means of a variational principle where the proper torsor concept is totally lacking. It is however widely used through the technique of “finite elements” where it can also be applied to electromagnetism (EM) with similar quadratic (piezoelectricity) or cubic (photoelasticity) Lagrangian integrals. In this situation, the 4-potential  $A$  of EM is used in place of  $\xi$  while the EM field  $dA = F = (\mathbf{B}, \mathbf{E})$  is used in place of  $\epsilon$  and the Maxwell equations  $dF = 0$  are used in place of the Riemann CC for  $\epsilon$ .

However, there exists another equivalent procedure dealing with a variational calculus with constraint. Indeed, as we shall see later on, the deformation tensor is not any symmetric tensor as it must satisfy  $n^2(n^2 - 1)/12$  compatibility conditions (CC), that is only  $\partial_{22}\epsilon_{11} + \partial_{11}\epsilon_{22} - 2\partial_{12}\epsilon_{12} = 0$  when  $n = 2$ . In this case, introducing the Lagrange multiplier  $\lambda$ , we have to vary the new integral  $\int [\varphi(\epsilon) + \lambda(\partial_{22}\epsilon_{11} + \partial_{11}\epsilon_{22} - 2\partial_{12}\epsilon_{12})]dx$  for an arbitrary  $\epsilon$ . Setting  $\lambda = -\phi$ , a double integration by parts now provides the parametrization  $\sigma^{11} = \partial_{22}\phi$ ,  $\sigma^{12} = \sigma^{21} = -\partial_{12}\phi$ ,  $\sigma^{22} = \partial_{11}\phi$  of the stress equations by means of the Airy function  $\phi$  and the formal adjoint of the Riemann CC, on the condition to observe that we have in fact  $2\sigma^{12} = -2\partial_{12}\phi$  as another way to understand the deep meaning of the factor “2” in the summation. The same variational calculus with constraint may thus also be used in order to “shortcut” the introduction of the EM potential.

Finally, using the constitutive relations of the material establishing an isomorphism  $\sigma \leftrightarrow \epsilon$ , one can also introduce a local free energy  $\psi(\sigma)$  in a variational problem having now for constraint the stress equations, with the same comment as above (see [6], p. 915, for more details). The well known Minkowski constitutive relations  $(\mathbf{B}, \mathbf{E}) \leftrightarrow (\mathbf{H}, \mathbf{D})$  can be similarly used for EM.

In arbitrary dimension, the above compatibility conditions are nothing else but the linearized Riemann tensor in Riemannian geometry, a crucial mathematical tool in the theory of general relativity and a good reason for studying the work of Cornelius Lanczos (1893-1974) as it can be found in [21]-[23] or in a few modern references [24]-[31]. The starting point of Lanczos has been to take EM as a model in order to introduce a Lagrangian that should be quadratic in the Riemann tensor ( $\rho_{i,j}^k \Rightarrow \rho_{ij} = \rho_{i,rj}^r = \rho_{ji} \Rightarrow \rho = \omega^{ij}\rho_{ij}$ ) while considering it independently of its expression through the second order derivatives of a metric ( $\omega_{ij}$ ) with inverse ( $\omega^{ij}$ ) or the first order derivatives of the corresponding Christoffel symbols ( $\gamma_{ij}^k$ ). According to the previous paragraph, the corresponding variational calculus must involve PD constraints made by the Bianchi identities and the new Lagrangian to vary must therefore contain as many Lagrange multipliers as the number of Bianchi identities (care!) that can be written under the form:

$$\nabla_r \rho_{i,j}^k + \nabla_i \rho_{l,jr}^k + \nabla_j \rho_{l,ri}^k = 0 \Rightarrow \nabla_r \rho_{i,j}^r = \nabla_i \rho_{lj} - \nabla_j \rho_{li}$$

Meanwhile, Lanczos and followers have been looking for a kind of parametrization of the Bianchi identities, exactly like the Lagrange multiplier has been used as an Airy potential for the stress equations. However, we shall prove that the definition of a Riemann candidate and the answer to this question cannot be done without the knowledge of the Spencer cohomology. Moreover, we have pointed out the existence of well known couplings between elasticity and electromagnetism, namely piezoelectricity and photoelasticity, which are showing that, in the respective Lagrangians, the EM field is on equal footing with the deformation tensor and not with the Riemann tensor. This fact is showing the shift by one step that must be used in the physical interpretation of the differential sequences involved and cannot be avoided. Meanwhile, the ordinary derivatives  $\partial_i$  can be used in place of the covariant derivatives  $\nabla_i$  when dealing with the linearized framework as the Christoffel symbols vanish when Euclidean or Minkowskian metrics are used.

The next tentative of Lanczos has been to extend his approach to the Weyl tensor:

$$\tau_{l,ij}^k = \rho_{l,ij}^k - \frac{1}{(n-2)} (\delta_i^k \rho_{lj} - \delta_j^k \rho_{li} + \omega_{ij} \omega^{ks} \rho_{si} - \omega_{li} \omega^{ks} \rho_{sj}) + \frac{1}{(n-1)(n-2)} (\delta_i^k \omega_{lj} - \delta_j^k \omega_{li}) \rho$$

The main problem is now that the Spencer cohomology of the symbols of the conformal Killing equations, in particular the 2-acyclicity, will be *absolutely needed* in order to study the Vessiot structure equations providing the Weyl tensor and its relation with the Riemann tensor. It will follow that *the CC for the Weyl tensor are not first order contrary to the CC for the Riemann tensor made by the Bianchi identities*, another reason for justifying the above shift by one step.

Finally, comparing the various parametrizations already obtained in the previous examples, it seems that the procedures are similar, even when dealing with systems having variable coefficients. The purpose of the paper is to prove that, in order to obtain a general algorithm, we shall need a lot of new tools involving at the same time *commutative algebra, homological algebra, differential algebra and differential geometry* that will be recalled in the next sections. Finally, like in any good crime story, it is only at the real end of the paper that we shall be able to revisit and compare all these examples in a unique framework.

## 2) MODULE THEORY

Before entering the heart of the next section dealing with extension modules, we need a few technical definitions and results from commutative algebra [13] [32].

**DEFINITION 2.1:** A ring  $A$  is said to be *unitary* if it has a (unique) element  $1 \in A$  such that  $1a = a1 = a$ ,  $\forall a \in A$  and *commutative* if  $ab = ba, \forall a, b \in A$ . A non-zero element  $a \in A$  is called a *zero-divisor* if one can find a non-zero  $b \in A$  such that  $ab = 0$  and a ring is called an *integral domain* if it has no zero-divisor. From now on, all rings considered will be unitary integral domains as we shall deal mainly with rings of partial differential operators.

**DEFINITION 2.2:** A ring  $K$  is called a *field* if every non-zero element  $a \in K$  is a *unit*, that is one can find an element  $b \in K$  such that  $ab = 1 \in K$ .

**DEFINITION 2.3:** A *module*  $M$  over a ring  $A$  or simply an *A-module* is a set of elements  $x, y, z, \dots$  which is an abelian group for an addition  $(x, y) \rightarrow x + y$  with an action  $A \times M \rightarrow M : (a, x) \rightarrow ax$  satisfying:

- $a(x + y) = ax + ay, \forall a \in A, \forall x, y \in M$
- $a(bx) = (ab)x, \forall a, b \in A, \forall x \in M$
- $(a + b)x = ax + bx, \forall a, b \in A, \forall x \in M$
- $1x = x, \forall x \in M$

The set of modules over a ring  $A$  will be denoted by  $\text{mod}(A)$ . A module over a field is called a *vector space*.

**DEFINITION 2.4:** A map  $f : M \rightarrow N$  between two  $A$ -modules is called a *homomorphism* over  $A$  if  $f(x + y) = f(x) + f(y), \forall x, y \in M$  and  $f(ax) = af(x), \forall a \in A, \forall x \in M$ . We successively define:

- $\ker(f) = \{x \in M \mid f(x) = 0\}$
- $\text{coim}(f) = M / \ker(f)$
- $\text{im}(f) = \{y \in N \mid \exists x \in M, f(x) = y\}$
- $\text{coker}(f) = N / \text{im}(f)$

with an isomorphism  $\text{coim}(f) \simeq \text{im}(f)$  induced by  $f$ .

**DEFINITION 2.5:** We say that a chain of modules and homomorphisms is a *sequence* if the composition of two successive such homomorphisms is zero. A sequence is said to be *exact* if the kernel of each map is equal to the image of the map preceding it. An injective homomorphism is called a *monomorphism*, a surjective homomorphism is called an *epimorphism* and a bijective homomorphism is called an *isomorphism*. A short exact sequence is an exact sequence made by a monomorphism followed by an epimorphism.

**PROPOSITION 2.6:** If one has a short exact sequence:

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

then the following conditions are equivalent:

- There exists an epimorphism  $u : M \rightarrow M'$  such that  $u \circ f = \text{id}_{M'}$  (*left inverse* of  $f$ ).
- There exists a monomorphism  $v : M'' \rightarrow M$  such that  $g \circ v = \text{id}_{M''}$  (*right inverse* of  $g$ ).

**DEFINITION 2.7:** In the above situation, we say that the short exact sequence *splits*. The relation  $f \circ u + v \circ g = \text{id}_M$  provides an isomorphism  $(u, g) : M \rightarrow M' \oplus M''$  with inverse  $f + v : M' \oplus M'' \rightarrow M$ .

The short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$  cannot split over  $\mathbb{Z}$ .

For the sake of clarity, as a few results will also be valid for modules over non-commutative rings, we shall denote by  ${}_A M_B$  a *bimodule*  $M$  which is a left module for  $A$  with operation  $(a, x) \rightarrow ax$  and a right module for  $B$  with operation  $(x, b) \rightarrow xb$ . In the commutative case, lower indices are not needed. If  $M = {}_A M$  and  $N = {}_A N$  are two left  $A$ -modules, the set of  $A$ -linear maps  $f : M \rightarrow N$  will be denoted by  $\text{hom}_A(M, N)$  or simply  $\text{hom}(M, N)$  when there will be no confusion and there is a canonical isomorphism  $\text{hom}(A, M) \simeq M : f \rightarrow f(1)$  with inverse  $x \rightarrow (a \rightarrow ax)$ . When  $A$  is commutative,  $\text{hom}(M, N)$  is again an  $A$ -module for the law  $(bf)(x) = f(bx)$ . In the non-commutative case, things are much more complicate and we have:

**LEMMA 2.8:** Given  ${}_A M$  and  ${}_A N_B$ , then  $\text{hom}_A(M, N)$  becomes a right module over  $B$  for the law  $(fb)(x) = f(x)b$ . A similar result can be obtained with  ${}_A M_B$  and  ${}_A N$ , where  $\text{hom}_A(M, N)$  now becomes a left module over  $B$  for the law  $(bf)(x) = f(xb)$ .

**THEOREM 2.9:** If  $M, M', M''$  are  $A$ -modules, the sequence:

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

is exact if and only if the sequence:

$$0 \rightarrow \text{hom}(M'', N) \rightarrow \text{hom}(M, N) \rightarrow \text{hom}(M', N)$$

is exact for any  $A$ -module  $N$ .

**COROLLARY 2.10:** The short exact sequence:

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

splits if and only if the short exact sequence:

$$0 \rightarrow \text{hom}(M'', N) \rightarrow \text{hom}(M, N) \rightarrow \text{hom}(M', N) \rightarrow 0$$

is exact for any module  $N$ .

**DEFINITION 2.11:** If  $M$  is a module over a ring  $A$ , a *system of generators* of  $M$  over  $A$  is a family  $\{x_i\}_{i \in I}$  of elements of  $M$  such that any element of  $M$  can be written  $x = \sum_{i \in I} a_i x_i$  with only a finite number of nonzero  $a_i$ . An  $A$ -module is called *noetherian* if every submodule of  $M$  (and thus  $M$  itself) is finitely generated.

One has the following standard technical result:

**PROPOSITION 2.12:** In a short exact sequence of modules, the central module is noetherian if and only if the two other modules are noetherian. As a byproduct, if  $A$  is a noetherian ring and  $M$  is a finitely generated module over  $A$ , then  $M$  is noetherian.

Accordingly, if  $M$  is generated by  $\{x_1, \dots, x_r\}$ , there is an epimorphism  $A^r \rightarrow M : (1, 0, \dots, 0) \rightarrow x_1, \dots, (0, \dots, 1) \rightarrow x_r$ . The kernel of this epimorphism is thus also finitely generated, say by  $\{y_1, \dots, y_s\}$  and we therefore obtain the exact sequence  $A^s \rightarrow A^r \rightarrow M \rightarrow 0$  that can be extended inductively to the left. *Such a property will always be assumed in the sequel.*

**DEFINITION 2.13:** In this case, we say that  $M$  is *finitely presented*.

We now turn to the definition and brief study of tensor products of modules over rings that will not be necessarily commutative unless stated explicitly.

Let  $M = M_A$  be a right  $A$ -module and  $N = {}_A N$  be a left  $A$ -module. We may introduce the free  $\mathbb{Z}$ -module made by finite formal linear combinations of elements of  $M \times N$  with coefficients in  $\mathbb{Z}$ .

**DEFINITION 2.14:** The tensor product of  $M$  and  $N$  over  $A$  is the  $\mathbb{Z}$ -module  $M \otimes_A N$  obtained by quotienting the above  $\mathbb{Z}$ -module by the submodule generated by the elements of the form:

$$(x + x', y) - (x, y) - (x', y), (x, y + y') - (x, y) - (x, y'), (xa, y) - (x, ay)$$

and the image of  $(x, y)$  will be denoted by  $x \otimes y$ .

It follows from the definition that we have the relations:

$$(x + x') \otimes y = x \otimes y + x' \otimes y, x \otimes (y + y') = x \otimes y + x \otimes y', xa \otimes y = x \otimes ay$$

and there is a canonical isomorphism  $M \otimes_A A \simeq M, A \otimes_A N \simeq N$ . When  $A$  is commutative, we may use left modules only and  $M \otimes_A N$  becomes a left  $A$ -module.

**EXAMPLE 2.15:** If  $A = \mathbb{Z}, M = \mathbb{Z}/2\mathbb{Z}$  and  $N = \mathbb{Z}/3\mathbb{Z}$ , we have  $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z}) = 0$  because  $x \otimes y = 3(x \otimes y) - 2(x \otimes y) = x \otimes 3y - 2x \otimes y = 0 - 0 = 0$ .

We present the technique of *localization* in order to introduce rings and modules of fractions.

**Definition 2.16:** A subset  $S$  of a ring  $A$  is said to be *multiplicatively closed* if  $\forall s, t \in S \Rightarrow st \in S$  and  $1 \in S$ . By a *left ring of fractions* or *left localization* of a noncommutative ring  $A$  with respect to a multiplicatively closed subset  $S$  of  $A$ , we mean a ring denoted by  $S^{-1}A$  with a monomorphism  $A \rightarrow S^{-1}A : a \rightarrow 1^{-1}a$  or simply  $a$  such that:

- 1)  $s$  is invertible in  $S^{-1}A$ , with inverse  $s^{-1}1$  or simply  $s^{-1}, \forall s \in S$ .
- 2) Each element of  $S^{-1}A$  or *fraction* has the form  $s^{-1}a$  for some  $s \in S, a \in A$ .

We have to distinguish carefully  $s^{-1}a$  from  $as^{-1}$  and we recover the standard notation  $a/s$  of the commutative case when two fractions  $a/s$  and  $b/t$  can be reduced to the same denominator  $st = ts$ . The following proposition is essential for constructing localizations.

**Proposition 2.17:** If there exists a left localization of  $A$  with respect to  $S$ , then we must have  $Sa \cap As \neq \emptyset, \forall a \in A, \forall s \in S$ . A set  $S$  satisfying this condition is called a *left Ore set*.

*Proof:* As  $S^{-1}A$  must be a ring, the element  $as^{-1}$  in  $S^{-1}A$  must be of the form  $t^{-1}b$  for some  $t \in S, b \in A$ . Accordingly,  $as^{-1} = t^{-1}b \Rightarrow ta = bs$  with  $t \in S, b \in A$ .

Q.E.D.

**Lemma 2.18:** If  $S$  is a left Ore set in a ring  $A$ , then  $As \cap At \cap S \neq \emptyset, \forall s, t \in S$  and two fractions can be brought to the same denominator.

*Proof:* From the left Ore condition, we can find  $u \in S$  and  $a \in A$  such that  $us = at \in S$ . More generally, we can find  $u, v \in A$  such that  $us = vt \in S$  and we successively get:

$$(us)^{-1}(ua) = s^{-1}u^{-1}ua = s^{-1}a, \quad (vt)^{-1}(vb) = t^{-1}v^{-1}vb = t^{-1}b$$

so that the two fractions  $s^{-1}a$  and  $t^{-1}b$  can be brought to the same denominator  $us = vt$ .

Q.E.D.

Let us now define an equivalence relation on  $S \times A$  by saying that  $(s, a) \sim (t, b)$  if one can find  $u, v \in S$  such that  $us = vt \in S$  and  $ua = vb$ . Such a relation is clearly reflexive and symmetric, thus we only need to prove that it is transitive. So let  $(s_1, a_1) \sim (s_2, a_2)$  and  $(s_2, a_2) \sim (s_3, a_3)$ . Then we can find  $u_1, u_2 \in A$  such that  $u_1s_1 = u_2s_2 \in S$  and  $u_1a_1 = u_2a_2$ . Also we can find  $v_2, v_3 \in A$  such that  $v_2s_2 = v_3s_3 \in S$  and  $v_2a_2 = v_3a_3$ . Now, from the Ore condition, one can find  $w_1, w_3 \in A$  such that  $w_1u_1s_1 = w_3v_3s_3 \in S$  and thus  $w_1u_2s_2 = w_3v_2s_2 \in S$ , that is to say  $(w_1u_2 - w_3v_2)s_2 = 0$ . As  $A$  is an integral domain, we have  $w_1u_2 - w_3v_2 = 0 \Rightarrow w_1u_2 = w_3v_2 \Rightarrow w_1u_1a_1 = w_1u_2a_2 = w_3v_2a_2 = w_3v_3a_3$  as wished. We finally define  $S^{-1}A$  to be the quotient of  $S \times A$  by the above equivalence relation with  $\theta : A \rightarrow S^{-1}A : a \rightarrow 1^{-1}a$ . The sum  $(s, a) + (t, b)$  will be defined to be  $(us = vt, ua + vb)$  and the product  $(s, a) \times (t, b)$  will be defined to be  $(s^{-1}a)(t^{-1}b) = s^{-1}(at^{-1})b = s^{-1}u^{-1}cb = (us)^{-1}(cb)$  whenever  $ua = ct$ .

A similar approach can be used in order to define and construct modules of fractions whenever  $S$  satisfies the two conditions of the last proposition. For this we need a preliminary lemma:

**LEMMA 2.19:** If  $S$  is a left Ore set in a ring  $A$  and  $M$  is a left module over  $A$ , the set:

$$t_s(M) = \{x \in M \mid \exists s \in S, sx = 0\}$$

is a submodule of  $M$  called the *S-torsion submodule* of  $M$ .

*Proof:* If  $x, y \in t_s(M)$ , we may find  $s, t \in S$  such that  $sx = 0, ty = 0$ . Now, we can find  $u, v \in A$  such that  $us = vt \in S$  and we successively get  $us(x + y) = usx + vty = 0 \Rightarrow x + y \in t_s(M)$ . Also,  $\forall a \in A$ , using the Ore condition for  $S$ , we can find  $b \in A, t \in S$  such that  $ta = bs$  and we get  $tax = bsx = 0 \Rightarrow ax \in t_s(M)$ .

Q.E.D.

**DEFINITION 2.20:** By a *left module of fractions* or *left localization* of  $M$  with respect to  $S$ , we mean a left module  $S^{-1}M$  over  $S^{-1}A$  both with a homomorphism  $\theta = \theta_S : M \rightarrow S^{-1}M : x \rightarrow 1^{-1}x = s^{-1}sx$  such that:

- 1) Each element of  $S^{-1}M$  has the form  $s^{-1}x$  for  $s \in S, x \in M$ .
- 2)  $\ker(\theta_S) = t_s(M)$ .

In order to construct  $S^{-1}M$ , we shall define an equivalence relation on  $S \times M$  by saying that  $(s, x) \sim (t, y)$  if there exists  $u, v \in A$  such that  $us = vt \in S$  and  $ux = vy$ . The main property of localization is expressed by the following theorem:

**Theorem 2.21:** If one has an exact sequence:

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

then one also has the exact sequence:

$$S^{-1}M' \xrightarrow{s^{-1}f} S^{-1}M \xrightarrow{s^{-1}g} S^{-1}M''$$

where  $S^{-1}f(s^{-1}x) = s^{-1}f(x)$ .

As a link between tensor product and localization, we notice that the multiplication map  $S^{-1}A \times M \rightarrow S^{-1}M$  given by  $(s^{-1}a, x) \rightarrow s^{-1}ax$  induces an isomorphism  $S^{-1}A \otimes_A M \rightarrow S^{-1}M$  of modules over  $S^{-1}A$  when  $S^{-1}A$  is considered as a right module over  $A$  with  $(s^{-1}a)b = s^{-1}ab$  and  $M$  as a left module over  $A$ . In particular, when  $A$  is a commutative integral domain and  $S = A - \{0\}$ , the field  $K = Q(A) = S^{-1}A$  is called the field of fractions of  $A$  and we have the short exact sequence:

$$0 \rightarrow A \rightarrow K \rightarrow K/A \rightarrow 0$$

If now  $M$  is a left  $A$ -module, we may tensor this sequence by  $M$  on the right with  $A \otimes M = M$  but we do not get in general an exact sequence. The defect of exactness *on the left* is nothing else but the *torsion submodule*  $t(M) = \{x \in M \mid \exists 0 \neq s \in A, sx = 0\} \subseteq M$  and we have the long exact sequence:

$$0 \rightarrow t(M) \rightarrow M \rightarrow K \otimes_A M \rightarrow K/A \otimes_A M \rightarrow 0$$

as we may describe the central map as follows:

$$x \rightarrow 1 \otimes x = s^{-1}s \otimes x = s^{-1} \otimes sx, \quad \forall 0 \neq s \in A$$

As we saw in the Introduction, such a result allows to understand why controllability has to do with localization which is introduced implicitly through the *transfer matrix* in control theory. In particular, a module  $M$  is said to be a *torsion module* if  $t(M) = M$  and a *torsion-free module* if  $t(M) = 0$ .

**DEFINITION 2.22:** A module in  $\text{mod}(A)$  is called a *free module* if it has a *basis*, that is a system of generators linearly independent over  $A$ . When a module  $F$  is free, the number of generators in a basis, and thus in any basis, is called the *rank* of  $F$  over  $A$  and is denoted by  $\text{rank}_A(F)$ . In particular, if  $F$  is free of finite rank  $r$ , then  $F \simeq A^r$ . More generally, a module  $P$  is said to be *projective* if there exists another (projective) module  $Q$  such that  $P \oplus Q = F$  and any short exact sequence splits if it ends with a projective module (see [6], p. 638-644) for a formal test).

If  $M$  is any module over a ring  $A$  and  $F$  is a maximum free submodule of  $M$ , then  $M/F = T$  is a torsion module. Indeed, if  $x \in M, x \notin F$ , then one can find  $a \in A$  such that  $ax \in F$  because, otherwise,  $F \subset \{F, x\}$  should be free submodules of  $M$  with a strict inclusion. In that case, the *rank* of  $M$  is by definition the rank of  $F$  over  $A$ . When  $A$  is commutative, one has:

**LEMMA 2.23:**  $\text{rk}_A(M) = \dim_K(K \otimes_A M)$ .

*Proof:* Taking the tensor product by  $K$  over  $A$  of the short exact sequence  $0 \rightarrow F \rightarrow M \rightarrow T \rightarrow 0$ , we get an isomorphism  $K \otimes_A F \simeq K \otimes_A M$  because  $K \otimes_A T = 0$  (exercise) and the lemma follows from the definition of the rank.

Q.E.D.

**PROPOSITION 2.24:** (*additivity property of the rank*) If  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  is a short exact sequence of modules over a ring  $A$ , then we have  $\text{rk}_A(M) = \text{rk}_A(M') + \text{rk}_A(M'')$ .

*Proof:* Let us consider the following diagram with exact left/right columns and central row:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & F' & \rightarrow & F' \oplus F'' & \rightarrow & F'' & \rightarrow 0 \\
 & \downarrow i' & & \downarrow i & & \downarrow i'' & \\
 0 \rightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \rightarrow 0 \\
 & \downarrow p' & & \downarrow p & & \downarrow p'' & \\
 0 \rightarrow & T' & \rightarrow & T & \rightarrow & T'' & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & 
 \end{array}$$

where  $F'(F'')$  is a maximum free submodule of  $M'(M'')$  and  $T' = M'/F'(T'' = M''/F'')$  is a torsion module. Pulling back by  $g$  the image under  $i''$  of a basis of  $F''$ , we may obtain by linearity a map  $\sigma: F'' \rightarrow M$  and we define  $i = f \circ i' \circ \pi' + \sigma \circ \pi''$  where  $\pi': F' \oplus F'' \rightarrow F'$  and  $\pi'': F' \oplus F'' \rightarrow F''$  are the canonical projections on each factor of the direct sum. We have  $i|_{F'} = f \circ i'$  and  $g \circ i = g \circ \sigma \circ \pi'' = i'' \circ \pi''$ . Hence, the diagram is commutative and thus exact with  $rk_A(F' \oplus F'') = rk_A(F') + rk_A(F'')$  trivially. Finally, if  $T'$  and  $T''$  are torsion modules, it is easy to check that  $T$  is a torsion module too and  $F' \oplus F''$  is thus a maximum free submodule of  $M$ .

Q.E.D.

**DEFINITION 2.25:** If  $f: M \rightarrow N$  is any morphism, the *rank* of  $f$  will be defined to be  $rk_A(f) = rk_A(im(f))$ .

We provide a few additional properties of the rank that will be used in the sequel. For this we shall set  $M^* = hom_A(M, A)$  and, for any morphism  $f: M \rightarrow N$  we shall denote by  $f^*: N^* \rightarrow M^*$  the corresponding morphism which is such that  $f^*(h) = h \circ f, \forall h \in hom_A(N, A)$ .

**PROPOSITION 2.26:** When  $A$  is a commutative integral domain and  $M$  is a finitely presented module over  $A$ , then  $rk_A(M) = rk_A(M^*)$ .

*Proof:* Applying  $hom_A(\bullet, A)$  to the short exact sequence in the proof of the preceding lemma while taking into account  $T^* = 0$ , we get a monomorphism  $0 \rightarrow M^* \rightarrow F^*$  and obtain therefore  $rk_A(M^*) \leq rk_A(F^*)$ . However, as  $F \simeq A^r$  with  $r < \infty$  because  $M$  is finitely generated, we get  $F^* \simeq A^r$  too because  $A^* = A$ . It follows that  $rk_A(M^*) \leq rk_A(F^*) = rk_A(F) = rk_A(M)$  and thus  $rk_A(M^*) \leq rk_A(M)$ .

Now, if  $F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$  is a finite presentation of  $M$ , applying  $hom_A(\bullet, A)$  to this presentation, we get the ker/coker exact sequence:

$$0 \leftarrow N \leftarrow F_1^* \xleftarrow{d_1^*} F_0^* \leftarrow M^* \leftarrow 0$$

Applying  $hom_A(\bullet, A)$  to this sequence while taking into account the isomorphisms  $F_0^{**} \simeq F_0, F_1^{**} \simeq F_1$ , we get the ker/coker exact sequence:

$$0 \rightarrow N^* \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$$

Counting the ranks, we obtain:

$$rk_A(N) - rk_A(M^*) = rk_A(F_1^*) - rk_A(F_0^*) = rk_A(F_1) - rk_A(F_0) = rk_A(N^*) - rk_A(M)$$

and thus:

$$(rk_A(M) - rk_A(M^*)) + (rk_A(N) - rk_A(N^*)) = 0$$

As both two numbers in this sum are non-negative, they must be zero and we finally get  $rk_A(M) = rk_A(M^*), rk_A(N) = rk_A(N^*)$ .

Q.E.D.

**COROLLARY 2.27:** Under the condition of the proposition, we have  $rk_A(f) = rk_A(f^*)$ .

*Proof:* Introducing the *ker / coker* exact sequence:

$$0 \rightarrow K \rightarrow M \xrightarrow{f} N \rightarrow Q \rightarrow 0$$

we have:  $rk_A(f) + rk_A(Q) = rk_A(N)$ . Applying  $hom_A(\bullet, A)$  and taking into account Theorem 2.9, we have the exact sequence:

$$0 \rightarrow Q^* \rightarrow N^* \xrightarrow{f^*} M^*$$

and thus:  $rk_A(f^*) + rk_A(Q^*) = rk_A(N^*)$ . Using the preceding proposition, we get  $rk_A(Q) = rk_A(Q^*)$  and  $rk_A(N) = rk_A(N^*)$ , that is to say  $rk_A(f) = rk_A(f^*)$ .

Q.E.D.

### 3) HOMOLOGICAL ALGEBRA

We need a few definitions and results from homological algebra [13] [33] [34] and start recalling the well known Cramer's rule for linear systems through the exactness of the ker/coker sequence for modules when  $f: M \rightarrow N$  is a linear map (homomorphism):

$$0 \rightarrow \ker(f) \rightarrow M \xrightarrow{f} N \rightarrow \operatorname{coker}(f) \rightarrow 0$$

In the case of vector spaces over a field  $K$ , we successively have  $\operatorname{rk}(f) = \dim(\operatorname{im}(f))$ ,  $\dim(\ker(f)) = \dim(M) - \operatorname{rk}(f)$ ,  $\dim(\operatorname{coker}(f)) = \dim(N) - \operatorname{rk}(f) = \dim(N) - \dim(M) + \dim(\ker(f))$  of compatibility conditions, and obtain by subtraction:

$$\dim(\ker(f)) - \dim(M) + \dim(N) - \dim(\operatorname{coker}(f)) = 0$$

In the case of modules, we may replace the dimension by the rank and obtain the same relations because of the additive property of the rank. We may also define *cohomology theory* as follows:

**DEFINITION 3.1:** If one has a sequence  $L \xrightarrow{f} M \xrightarrow{g} N$ , that is if  $g \circ f = 0$ , then one may introduce the submodules  $\operatorname{coboundary} = B = \operatorname{im}(f) \subseteq \ker(g) = \operatorname{cocycle} = Z \subseteq M$  and define the cohomology at  $M$  to be the quotient  $H = Z/B$ .

We now introduce the *extension modules* in an elementary manner, using the standard notation  $\operatorname{hom}_A(M, A) = M^*$ . Using a *free resolution* of an  $A$ -module  $M$ , that is to say a long exact sequence:

$$\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$$

where  $F_0, F_1, \dots$  are free modules, namely modules isomorphic to powers of  $A$  and  $M = \operatorname{coker}(d_1) = F_0 / \operatorname{im}(d_1)$ . We may *take out*  $M$  and obtain the *deleted sequence*:

$$\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

which is of course no longer exact. We may apply the functor  $\operatorname{hom}_A(\bullet, A)$  and obtain the sequence:

$$\dots \xleftarrow{d_2^*} F_1^* \xleftarrow{d_1^*} F_0^* \leftarrow 0$$

in order to state:

**DEFINITION 3.2:** We set:

$$\begin{aligned} \operatorname{ext}^0(M) &= \operatorname{ext}_A^0(M, A) = \ker(d_1^*) = M^*, \\ \operatorname{ext}^i(M) &= \operatorname{ext}_A^i(M, A) = \ker(d_{i+1}^*) / \operatorname{im}(d_i^*), \forall i \geq 1 \end{aligned}$$

The extension modules have the following three main properties [6] [13] [33] [34]:

**PROPOSITION 3.3:** The extension modules do not depend on the resolution of  $M$  chosen.

**PROPOSITION 3.4:** If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of  $A$ -modules, then we have the following *connecting long exact sequence*:

$$0 \rightarrow M''^* \rightarrow M^* \rightarrow M'^* \rightarrow \operatorname{ext}^1(M'') \rightarrow \operatorname{ext}^1(M) \rightarrow \operatorname{ext}^1(M') \rightarrow \operatorname{ext}^2(M'') \rightarrow \operatorname{ext}^2(M) \rightarrow \dots$$

of extension modules. Moreover  $\operatorname{ext}^i(P) = 0, \forall i \geq 1$  whenever  $P$  is a projective module.

**PROPOSITION 3.5:**  $\operatorname{ext}^i(M)$  is a torsion module,  $\forall i \geq 1$ .

*Proof:* Having in mind that  $B_i = \operatorname{im}(d_i^*)$  and  $Z_i = \ker(d_{i+1}^*)$ , we obtain  $\operatorname{rk}(B_i) = \operatorname{rk}(d_i^*) = \operatorname{rk}(d_i)$  and  $\operatorname{rk}(Z_i) = \operatorname{rk}(F_i^*) - \operatorname{rk}(d_{i+1}^*) = \operatorname{rk}(F_i) - \operatorname{rk}(d_{i+1})$ . However, we started from a resolution, that is an exact sequence in which  $\operatorname{rk}(d_i) + \operatorname{rk}(d_{i+1}) = \operatorname{rk}(F_i)$ . It follows that  $\operatorname{rk}(B_i) = \operatorname{rk}(Z_i)$  and thus  $\operatorname{rk}(H_i) = \operatorname{rk}(Z_i) - \operatorname{rk}(B_i) = 0$ , that is to say  $\operatorname{ext}^i(M)$  is a torsion module for  $i \geq 1, \forall M \in \operatorname{mod}(A)$ .

Q.E.D.

The next theorem and its corollary constitute the main results that will be used for applications through a classification of modules [1] [2] [6]-[8] [34] [35]:

**THEOREM 3.6:** The following long exact sequence:

$$0 \rightarrow \operatorname{ext}^1(N) \rightarrow M \xrightarrow{\epsilon} M^{**} \rightarrow \operatorname{ext}^2(N) \rightarrow 0$$

is isomorphic to the *ker/coker* long exact sequence for the central morphism  $\epsilon$  which is defined by  $\epsilon(x)(f) = f(x), \forall x \in M, \forall f \in M^*$ .

*Proof:* Introducing  $K = \operatorname{im}(d_1^*)$ , we may obtain two short exact sequences, a left one starting with  $K$  and a right one finishing with  $K$  as follows:

$$\begin{array}{ccccccc}
 0 & \leftarrow & N & \leftarrow & F_1^* & \xleftarrow{d_1^*} & F_0^* & \leftarrow & M^* & \leftarrow & 0 \\
 & & & & & \swarrow & & \searrow & & & \\
 & & & & & & K & & & & \\
 & & & & & \swarrow & & \searrow & & & \\
 & & & & 0 & & & & 0 & & 
 \end{array}$$

Using the two corresponding long exact connecting sequences, we get  $ext^1(K) \simeq ext^2(N)$  from the one starting with  $N^*$  which is also providing the left exact column of the next diagram and the exact central row of this diagram from the one starting with  $K^*$ . The Theorem is finally obtained by a chase proving that the full diagram is commutative and exact:

$$\begin{array}{ccccccc}
 & & F_1 & = & F_1 & & \\
 & & \downarrow & & \downarrow d_1 & & \\
 0 & \rightarrow & K^* & \xrightarrow{d_1} & F_0 & \rightarrow & M^{**} \rightarrow ext^2(N) \rightarrow 0 \\
 & & \downarrow & & \downarrow & \nearrow & \\
 0 & \rightarrow & ext^1(N) & \rightarrow & M & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Q.E.D.

**COROLLARY 3.7:**  $t(M) = ext^1(N) = ker(\epsilon)$ .

*Proof:* As  $ext^1(N) \subseteq M$  is a torsion module, we have therefore  $ext^1(N) \subseteq t(M)$ . Now, if  $x \in t(M)$ , we may find  $0 \neq a \in A$  such that  $ax = 0$  and  $\epsilon(x)(f) = f(x) \Rightarrow f(ax) = af(x) = 0 \Rightarrow f(x) = 0$  because  $A$  is an integral domain, that is  $t(M) \subseteq ker(\epsilon) = ext^1(N)$  and thus  $t(M) = ext^1(N) = ker(\epsilon)$ .

Q.E.D.

**DEFINITION 3.8:** A module  $M$  will be called *torsion-free* if  $ext^1(N) = t(M) = 0$  and *reflexive* if  $ext^1(N) = 0, ext^2(N) = 0$ .

Despite all these results, a major difficulty still remains. Indeed, we have  $M = coker(d_1) = {}_A M$  as a left module over  $A$  but, using the bimodule structure of  $A = {}_A A_A$  and Lemma 2.13, it follows that  $M^* = ker(d_1^*) = M_A^*$  is a right module over  $A$  and thus  $N = coker(d_1^*) = N_A$  is also a right module over  $A$ . However, as we shall see, all the differential modules used through applications will be left modules over the ring of differential operators and it will therefore not be possible to use dual sequences as we did without being able to “pass from left to right and vice-versa”. For this purpose we now need many delicate results from differential geometry, in particular a way to deal with the *formal adjoint* of an operator as we did many times in the Introduction.

**4) SYSTEM THEORY**

If  $E$  is a vector bundle over the base manifold  $X$  with projection  $\pi$  and local coordinates  $(x, y) = (x^i, y^k)$  projecting onto  $x = (x^i)$  for  $i = 1, \dots, n$  and  $k = 1, \dots, m$ , identifying a map with its graph, a (local) section  $f : U \subset X \rightarrow E$  is such that  $\pi \circ f = id$  on  $U$  and we write  $y^k = f^k(x)$  or simply  $y = f(x)$ . For any change of local coordinates  $(x, y) \rightarrow (\bar{x} = \varphi(x), \bar{y} = A(x)y)$  on  $E$ , the change of section is  $y = f(x) \rightarrow \bar{y} = \bar{f}(\bar{x})$  such that  $\bar{f}^l(\varphi(x)) \equiv A_k^l(x) f^k(x)$ . The new vector bundle  $E^*$  obtained by changing the *transition matrix*  $A$  to its inverse  $A^{-1}$  is called the *dual vector bundle* of  $E$ . Differentiating with respect to  $x^i$  and using new coordinates  $y_i^k$  in place of  $\partial_i f^k(x)$ , we obtain  $\bar{y}_r^l \partial_i \varphi^r(x) = A_k^l(x) y_i^k + \partial_i A_k^l(x) y^k$ . Introducing a multi-index  $\mu = (\mu_1, \dots, \mu_n)$  with length  $|\mu| = \mu_1 + \dots + \mu_n$  and prolonging the procedure up to order  $q$ , we may construct in this way, by patching coordinates, a vector bundle  $J_q(E)$  over  $X$ , called the *jet bundle of order  $q$*  with local coordinates  $(x, y_q) = (x^i, y_\mu^k)$  with  $0 \leq |\mu| \leq q$  and  $y_0^k = y^k$ . We have therefore epimorphisms  $\pi_q^{q+r} : J_{q+r}(E) \rightarrow J_q(E), \forall q, r \geq 0$ . For a later use, we shall set  $\mu + 1_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$  and define the operator  $j_q : E \rightarrow J_q(E) : f \rightarrow j_q(f)$  on sections by the local formula  $j_q(f) : (x) \rightarrow (\partial_\mu f^k(x) | 0 \leq |\mu| \leq q, k = 1, \dots, m)$ . Moreover, a jet coordinate  $y_\mu^k$  is said to be of *class  $i$*  if  $\mu_1 = \dots = \mu_{i-1} = 0, \mu_i \neq 0$ . We finally introduce the *Spencer operator*  $D : J_{q+1}(E) \rightarrow T^* \otimes J_q(E) : f_{q+1} \rightarrow j_1(f_q) - f_{q+1}$  with

$$(Df_{q+1})_{\mu,i}^k = \partial_i f_{\mu}^k - f_{\mu+1}^k.$$

**DEFINITION 4.1:** A system of PD equations of order  $q$  on  $E$  is a vector subbundle  $R_q \subset J_q(E)$  locally defined by a constant rank system of linear equations for the jets of order  $q$  of the form  $a_k^{\mu}(x)y_{\mu}^k = 0$ . Its *first prolongation*  $R_{q+1} \subset J_{q+1}(E)$  will be defined by the equations  $a_k^{\mu}(x)y_{\mu}^k = 0, a_k^{\mu}(x)y_{\mu+1}^k + \partial_i a_k^{\mu}(x)y_{\mu}^k = 0$  which may not provide a system of constant rank as can easily be seen for  $xy_x - y = 0 \Rightarrow xy_{xx} = 0$  where the rank drops at  $x = 0$ .

The next definition of *formal integrability* (FI) will be crucial for our purpose.

**DEFINITION 4.2:** A system  $R_q$  is said to be *formally integrable* if the  $R_{q+r}$  are vector bundles  $\forall r \geq 0$  (regularity condition) and no new equation of order  $q+r$  can be obtained by prolonging the given PD equations more than  $r$  times,  $\forall r \geq 0$  or, equivalently, we have induced epimorphisms  $\pi_{q+r}^{q+r+1} : R_{q+r+1} \rightarrow R_{q+r}$ ,  $\forall r \geq 0$  allowing to compute “*step by step*” formal power series solutions.

A formal test has been first sketched by C. Riquier in 1910 [36], then improved by M. Janet in 1920 [20] [37] and by E. Cartan in 1945 [38], finally rediscovered in 1965, totally independently, by B. Buchberger who introduced Gröbner bases, using the name of his thesis advisor. However all these tentatives have been largely superseded and achieved in an intrinsic way, again totally independently of the previous approaches, by D.C. Spencer in 1965 [20] [39] [40].

**DEFINITION 4.3:** The family  $g_{q+r}$  of vector spaces over  $X$  defined by the purely linear equations  $a_k^{\mu}(x)y_{\mu+\nu}^k = 0$  for  $|\mu| = q, |\nu| = r$  is called the *symbol* at order  $q+r$  and only depends on  $g_q$ .

The following procedure, where one may have to change linearly the independent variables if necessary, is the key towards the next definition which is intrinsic even though it must be checked in a particular coordinate system called  $\delta$ -regular (see [6] [20] and [39] for more details):

- *Equations of class  $n$ :* Solve the maximum number  $\beta_q^n$  of equations with respect to the jets of order  $q$  and class  $n$ . Then call  $(x^1, \dots, x^n)$  *multiplicative variables*.

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- *Equations of class  $i$ :* Solve the maximum number of *remaining* equations with respect to the jets of order  $q$  and class  $i$ . Then call  $(x^1, \dots, x^i)$  *multiplicative variables* and  $(x^{i+1}, \dots, x^n)$  *non-multiplicative variables*.

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- *Remaining equations of order  $\leq q-1$ :* Call  $(x^1, \dots, x^n)$  *non-multiplicative variables*.

**DEFINITION 4.4:** The above multiplicative and non-multiplicative variables can be visualized respectively by integers and dots in the corresponding *Janet board*. A system of PD equations is said to be *involutive* if its first prolongation can be achieved by prolonging its equations only with respect to the corresponding multiplicative variables. The following numbers are called *characters*:

$$\alpha_q^i = m(q+n-i-1)! / ((q-1)!(n-i)!) - \beta_q^i, \quad \forall 1 \leq i \leq n \Rightarrow \alpha_q^1 \geq \dots \geq \alpha_q^n$$

For an involutive system,  $(y^{\beta_q^{n+1}}, \dots, y^m)$  can be given arbitrarily.

For an involutive system of order  $q$  in the above *solved form*, we shall use to denote by  $y_{pri}$  the *principal jet coordinates*, namely the leading terms of the solved equations in the sense of involution. Accordingly, any formal derivative of a principal jet coordinate is again a principal jet coordinate. The remaining jet coordinates will be called *parametric jet coordinates* and denoted by  $y_{par}$ . Now, the symbol of  $J_q$  is the zero symbol and is thus trivially involutive at any order  $q$ . Accordingly, if we introduce the *multiplicative variables*  $x^1, \dots, x^i$  for the parametric jets of order  $q$  and class  $i$ , the formal derivative or a parametric jet of strict order  $q$  and class  $i$  by one of its multiplicative variables is uniquely obtained and cannot be a principal jet of order  $q+1$  which is coming from a uniquely defined principal jet of order  $q$  and class  $i$ .

**PROPOSITION 4.5:** Using the Janet board and the definition of involutivity, we get:

$$\dim(g_{q+r}) = \sum_{i=1}^n \frac{(r+i-1)!}{r!(i-1)!} \alpha_q^i \Rightarrow \dim(R_{q+r}) = \dim(R_{q-1}) + \sum_{i=1}^n \frac{(r+i)!}{r!i!} \alpha_q^i$$

Let  $T$  be the tangent vector bundle of vector fields on  $X$ ,  $T^*$  be the cotangent vector bundle of 1-forms on  $X$  and  $\wedge^s T^*$  be the vector bundle of  $s$ -forms on  $X$  with usual bases  $\{dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_s}\}$  where we have set

$I = (i_1 < \dots < i_s)$ . Also, let  $S_q T^*$  be the vector bundle of symmetric q-covariant tensors. Moreover, if  $\xi, \eta \in T$  are two vector fields on  $X$ , we may define their *bracket*  $[\xi, \eta] \in T$  by the local formula  $([\xi, \eta])^i(x) = \xi^r(x) \partial_r \eta^i(x) - \eta^s(x) \partial_s \xi^i(x)$  leading to the *Jacobi identity*  $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0, \forall \xi, \eta, \zeta \in T$ . We may finally introduce the *exterior derivative*  $d : \wedge^r T^* \rightarrow \wedge^{r+1} T^* : \omega = \omega_i dx^i \rightarrow d\omega = \partial_i \omega_j dx^i \wedge dx^j$  with  $d^2 = d \circ d \equiv 0$  in the *Poincaré sequence*:

$$\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} \dots \xrightarrow{d} \wedge^n T^* \rightarrow 0$$

In a purely algebraic setting, one has [20] [39] [40]:

**PROPOSITION 4.6:** There exists a map  $\delta : \wedge^s T^* \otimes S_{q+1} T^* \otimes E \rightarrow \wedge^{s+1} T^* \otimes S_q T^* \otimes E$  which restricts to  $\delta : \wedge^s T^* \otimes g_{q+1} \rightarrow \wedge^{s+1} T^* \otimes g_q$  and  $\delta^2 = \delta \circ \delta = 0$ .

*Proof:* Let us introduce the family of s-forms  $\omega = \{\omega_\mu^k = v_{\mu,l}^k dx^l\}$  and set  $(\delta\omega)_\mu^k = dx^i \wedge \omega_{\mu+i}^k$ . We obtain at once  $(\delta^2\omega)_\mu^k = dx^i \wedge dx^j \wedge \omega_{\mu+i+j}^k = 0$ .

Q.E.D.

The kernel of each  $\delta$  in the first case is equal to the image of the preceding  $\delta$  but this may no longer be true in the restricted case and we set (see [39], p. 85-88 for more details):

**DEFINITION 4.7:** We denote by  $B_{q+r}^s(g_q) \subseteq Z_{q+r}^s(g_q)$  and  $H_{q+r}^s(g_q) = Z_{q+r}^s(g_q) / B_{q+r}^s(g_q)$  respectively the coboundary space, cocycle space and cohomology space at  $\wedge^s T^* \otimes g_{q+r}$  of the restricted  $\delta$ -sequence which only depend on  $g_q$  and may not be vector bundles. The symbol  $g_q$  is said to be *s-acyclic* if  $H_{q+r}^1 = \dots = H_{q+r}^s = 0, \forall r \geq 0$ , *involutive* if it is n-acyclic and *finite type* if  $g_{q+r} = 0$  becomes trivially involutive for r large enough. For a later use, we notice that a symbol  $g_q$  is involutive and of finite type if and only if  $g_q = 0$ . Finally,  $S_q T^* \otimes E$  is involutive  $\forall q \geq 0$  if we set  $S_0 T^* \otimes E = E$ .

**FI CRITERION 4.8:** If  $\pi_q^{q+1} : R_{q+1} \rightarrow R_q$  is an epimorphism of vector bundles and  $g_q$  is 2-acyclic (involutive), then  $R_q$  is formally integrable (involutive).

**EXAMPLE 4.9:** The system  $R_2$  defined by the three PD equations

$$y_{33} = 0, \quad y_{23} - y_{11} = 0, \quad y_{22} = 0$$

is homogeneous and thus automatically formally integrable but  $g_2$  is not involutive though finite type because  $g_4 = 0$ . Elementary computations of ranks of matrices show that the  $\delta$ -map:

$$0 \rightarrow \wedge^2 T^* \otimes g_3 \xrightarrow{\delta} \wedge^3 T^* \otimes g_2 \rightarrow 0$$

is a  $3 \times 3$  isomorphism and thus  $g_3$  is 2-acyclic with  $\dim(g_3) = 1$ , a *crucial intrinsic* property totally absent from any “old” work and quite more easy to handle than its Koszul dual.

The main use of involution is to construct differential sequences that are made up by successive *compatibility conditions* (CC) of order one. In particular, when  $R_q$  is involutive, the differential operator  $\mathcal{D} : E \xrightarrow{j_q} J_q(E) \xrightarrow{\Phi} J_q(E) / R_q = F_0$  of order  $q$  with space of solutions  $\Theta \subset E$  is said to be *involutive* and one has the canonical *linear Janet sequence* ([31], p. 144):

$$0 \rightarrow \Theta \rightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \dots \xrightarrow{\mathcal{D}_n} F_n \rightarrow 0$$

where each other operator is first order involutive and generates the CC of the preceding one with the *Janet bundles*  $F_r = \wedge^r T^* \otimes J_q(E) / (\wedge^r T^* \otimes R_q + \delta(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E))$ . As the Janet sequence can be “cut at any place”, that is can also be constructed anew from any intermediate operator, *the numbering of the Janet bundles has nothing to do with that of the Poincaré sequence for the exterior derivative*, contrary to what many physicists still believe ( $n = 3$  with  $\mathcal{D} = \text{div}$  provides the simplest example). Moreover, the fiber dimension of the Janet bundles can be computed at once inductively from the board of multiplicative and non-multiplicative variables that can be exhibited for  $\mathcal{D}$  by working out the board for  $\mathcal{D}_1$  and so on. For this, the number of rows of this new board is the number of dots appearing in the initial board while the number  $nb(i)$  of dots in the column  $i$  just indicates the number of CC of class  $i$  for  $i = 1, \dots, n$  with  $nb(i) < nb(j), \forall i < j$ . When  $R_q$  is not involutive but formally integrable and the  $r$ -prolongation of its symbol  $g_q$  becomes 2-acyclic, it is known that the generating CC are of order  $r + 1$  (see [39], Example 6, p. 120 and previous Example).

**DEFINITION 4.10:** More generally, a differential sequence is said to be *formally exact* if each operator generates the CC of the operator preceding it.

**EXAMPLE 4.11:** ([41], §38, p 40, is providing the first intuition of formal integrability) The second order

system  $y_{11} = 0, y_{13} - y_2 = 0$  is neither formally integrable nor involutive. Indeed, we get  $d_3 y_{11} - d_1 (y_{13} - y_2) = y_{12}$  and  $d_3 y_{12} - d_2 (y_{13} - y_2) = y_{22}$ , that is to say *each first and second* prolongation does bring a new second order PD equation. Considering the new system  $y_{22} = 0, y_{12} = 0, y_{13} - y_2 = 0, y_{11} = 0$ , the question is to decide whether this system is involutive or not. In such a simple situation, as there is no PD equation of class 3, the evident permutation of coordinates  $(1, 2, 3) \rightarrow (3, 2, 1)$  provides the following involutive second order system with one equation of class 3, 2 equations of class 2 and 1 equation of class 1:

$$\left\{ \begin{array}{l} \Phi^4 \equiv y_{33} = 0 \\ \Phi^3 \equiv y_{23} = 0 \\ \Phi^2 \equiv y_{22} = 0 \\ \Phi^1 \equiv y_{13} - y_2 = 0 \end{array} \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline 1 & \bullet & \bullet \\ \hline \end{array}$$

We have  $\alpha_3^3 = 0, \alpha_2^2 = 0, \alpha_1^1 = 2$  and the corresponding CC system is easily seen to be the following involutive first order system:

$$\left\{ \begin{array}{l} \Psi^4 \equiv d_3 \Phi^3 - d_2 \Phi^4 = 0 \\ \Psi^3 \equiv d_3 \Phi^2 - d_2 \Phi^3 = 0 \\ \Psi^2 \equiv d_3 \Phi^1 - d_1 \Phi^4 + \Phi^3 = 0 \\ \Psi^1 \equiv d_2 \Phi^1 - d_1 \Phi^3 + \Phi^2 = 0 \end{array} \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$$

The final CC system is the involutive first order system:

$$\left\{ \Omega \equiv d_3 \Psi^1 - d_2 \Psi^2 + d_1 \Psi^4 - \Psi^3 = 0 \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

We get therefore the (formally exact) Janet sequence:

$$0 \rightarrow \Theta \rightarrow 1 \rightarrow 4 \rightarrow 4 \rightarrow 1 \rightarrow 0$$

However, keeping only  $\Phi^1$  and  $\Phi^4$  while using the fact that  $d_{33}$  commutes with  $d_{13} - d_2$ , we get the formally exact sequence  $0 \rightarrow \Theta \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 0$  which is *not* a Janet sequence. We finally check that each  $\Phi^1, \Phi^2, \Phi^3$  is separately differentially dependent on  $\Phi^4$  because we have successively  $d_3 \Phi^3 - d_2 \Phi^4 = 0, d_{33} \Phi^2 - d_{22} \Phi^4 = 0, d_{33} \Phi^1 - d_{13} \Phi^4 + d_2 \Phi^4 = 0$ .

Finally, we may extend the restriction  $D : R_{q+1} \rightarrow T^* \otimes R_q$  of the Spencer operator to:

$$D : \wedge^r T^* \otimes R_{q+1} \rightarrow \wedge^{r+1} T^* \otimes R_q : \alpha \otimes f_{q+1} \rightarrow d\alpha \otimes f_q + (-1)^r \alpha \wedge Df_{q+1} \Rightarrow D^2 = D \circ D \equiv 0$$

in order to construct the *first Spencer sequence* which is another resolution of  $\Theta$  because the kernel of the first  $D$  is such that  $f_{q+1} \in R_{q+1}, Df_{q+1} = 0 \Leftrightarrow f_{q+1} = j_{q+1}(f), f \in \Theta$  when  $q$  is large enough.

### 5) DIFFERENTIAL MODULES

Let  $K$  be a *differential field*, that is a field containing  $\mathbb{Q}$  with  $n$  commuting *derivations*  $\{\partial_1, \dots, \partial_n\}$  with  $\partial_i \partial_j = \partial_j \partial_i = \partial_{ij}, \forall i, j = 1, \dots, n$  such that  $\partial_i(a+b) = \partial_i a + \partial_i b, \partial_i(ab) = (\partial_i a)b + a\partial_i b, \forall a, b \in K$  and  $\partial_i(1/a) = -(1/a^2)\partial_i a, \forall a \in K$ . Using an implicit summation on multi-indices, we may introduce the (noncommutative) *ring of differential operators*  $D = K[d_1, \dots, d_n] = K[d]$  with elements  $P = a^\mu d_\mu$  such that  $|\mu| < \infty$  and  $d_i a = ad_i + \partial_i a$ . The highest value of  $|\mu|$  with  $a^\mu \neq 0$  is called the *order* of the operator  $P$  and the ring  $D$  with multiplication  $(P, Q) \rightarrow P \circ Q = PQ$  is filtered by the order  $q$  of the operators. We have the *filtration*  $0 \subset K = D_0 \subset D_1 \subset \dots \subset D_q \subset \dots \subset D_\infty = D$ . Moreover, it is clear that  $D$ , as an algebra, is generated by  $K = D_0$  and  $T = D_1/D_0$  with  $D_1 = K \oplus T$  if we identify an element  $\xi = \xi^i d_i \in T$  with the vector field  $\xi = \xi^i(x)\partial_i$  of differential geometry, but with  $\xi^i \in K$  now. It follows that  $D = {}_D D_D$  is a *bimodule* over itself, being at the same time a left  $D$ -module  ${}_D D$  by the composition  $P \rightarrow QP$  and a right  $D$ -module  $D_D$  by the composition  $P \rightarrow PQ$  with  $D_r D_s = D_{r+s}, \forall r, s \geq 0$ .

If we introduce *differential indeterminates*  $y = (y^1, \dots, y^m)$ , we may extend  $d_i y_\mu^k = y_{\mu+1_i}^k$  to  $\Phi^\tau \equiv a_k^{\tau\mu} y_\mu^k \xrightarrow{d_i} d_i \Phi^\tau \equiv a_k^{\tau\mu} y_{\mu+1_i}^k + \partial_i a_k^{\tau\mu} y_\mu^k$  for  $\tau = 1, \dots, p$ . Therefore, setting  $Dy^1 + \dots + Dy^m = Dy = D^m$  and calling  $I = D\Phi \subset Dy$  the *differential module of equations*, we obtain by residue the *differential module* or *D-module*  $M = Dy/D\Phi$ , denoting the residue of  $y_\mu^k$  by  $\bar{y}_\mu^k$  when there can be a confusion. Introducing the two free differential modules  $F_0 = D^{m_0}, F_1 = D^{m_1}$ , we obtain equivalently the *free presentation*  $F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$  of order  $q$  when  $m_0 = m, m_1 = p$  and  $d_1 = D$ . We shall moreover assume that  $\mathcal{D}$  provides a

strict morphism, namely that the corresponding system  $R_q$  is formally integrable. It follows that  $M$  can be endowed with a *quotient filtration* obtained from that of  $D^m$  which is defined by the order of the jet coordinates  $y_q$  in  $D_q y$ . We have therefore the *inductive limit*  $0 = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \dots \subseteq M_q \subseteq \dots \subseteq M_\infty = M$  with  $d_i M_q \subseteq M_{q+1}$  but it is important to notice that  $D_r D_q = D_{q+r} \Rightarrow D_r M_q = M_{q+r}, \forall q, r \geq 0 \Rightarrow M = DM_q, \forall q \geq 0$  in this particular case. It also follows from Noetherian arguments and involution that  $D_r I_q = I_{q+r}, \forall r \geq 0$  though we have in general only  $D_r I_s \subseteq I_{r+s}, \forall r \geq 0, \forall s < q$ . As  $K \subset D$ , we may introduce the *forgetful functor*  $\text{for} : \text{mod}(D) \rightarrow \text{mod}(K) : {}_D M \rightarrow {}_K M$ .

More generally, introducing the successive CC as in the preceding section while changing slightly the numbering of the respective operators, we may finally obtain the *free resolution* of  $M$ , namely the exact sequence  $\dots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$ . In actual practice, *one must never forget that  $\mathcal{D} = \Phi \circ j_q$  acts on the left on column vectors in the operator case and on the right on row vectors in the module case*. Also, with a slight abuse of language, when  $\mathcal{D} = \Phi \circ j_q$  is involutive as in Section 2 and thus  $R_q = \ker(\Phi)$  is involutive, one should say that  $M$  has an *involutive presentation* of order  $q$  or that  $M_q$  is *involutive*.

**DEFINITION 5.1:** Setting  $P = a^\mu d_\mu \in D \xleftarrow{ad} ad(P) = (-1)^{|\mu|} d_\mu a^\mu \in D$ , we have  $ad(ad(P)) = P$  and  $ad(PQ) = ad(Q)ad(P), \forall P, Q \in D$ . Such a definition can be extended to any matrix of operators by using the transposed matrix of adjoint operators and we get:

$$\langle \lambda, \mathcal{D}\xi \rangle = \langle ad(\mathcal{D})\lambda, \xi \rangle + \text{div}(\dots)$$

from integration by part, where  $\lambda$  is a row vector of test functions and  $\langle \rangle$  the usual contraction. We quote the useful formulas  $[ad(\xi), ad(\eta)] = ad(\xi)ad(\eta) - ad(\eta)ad(\xi) = -ad([\xi, \eta]), \forall \xi, \eta \in T$  (*care about the minus sign*) and  $rk_D(\mathcal{D}) = rk_D(ad(\mathcal{D}))$  as in ([32], p. 610-612).

**REMARK 5.2:** As can be seen from the examples of the Introduction, when  $\mathcal{D}$  is involutive, then  $ad(\mathcal{D})$  may not be involutive. In the differential framework, we may set  $rk_D(\mathcal{D}) = m - \alpha_q^n = \beta_q^n$ . Comparing to similar concepts used in *differential algebra*, this number is just the maximum number of differentially independent equations to be found in the differential module  $I$  of equations. Indeed, pointing out that differential indeterminates in differential algebra are nothing else than jet coordinates in differential geometry and using standard notations, we have  $K\{y\} = \lim_{q \rightarrow \infty} K[y_q]$ . In that case, the differential ideal  $I$  automatically generates a prime differential ideal  $\mathfrak{p} \subset K\{y\}$  providing a *differential extension*  $L/K$  with  $L = \mathcal{Q}(K\{y\}/\mathfrak{p})$  and *differential transcendence degree*  $\text{diffird}(L/K) = \alpha_q^n$ , a result explaining the notations [39]. Now, from the dimension formulas of  $R_{q+r}$ , we obtain at once  $rk_D(M) = \alpha_q^n$  and thus  $rk_D(\mathcal{D}) = m - rk_D(M)$  in a coherent way with any free presentation of  $M$  starting with  $\mathcal{D}$ . However,  $\mathcal{D}$  acts on the left in differential geometry but on the right in the theory of differential modules. For an operator of order zero, we recognize the fact that the rank of a matrix is equal to the rank of the transposed matrix.

**PROPOSITION 5.3:** If  $f \in \text{aut}(X)$  is a local diffeomorphisms on  $X$ , we may set  $x = f^{-1}(y) = g(y)$  and we have the *identity*:

$$\frac{\partial}{\partial y^k} \left( \frac{1}{\Delta(g(y))} \partial_i f^k(g(y)) \right) \equiv 0 \Rightarrow \frac{\partial}{\partial y^k} \left( \frac{1}{\Delta} \frac{\partial f^k}{\partial x^i} \mathcal{A}^i \right) = \frac{1}{\Delta} \frac{\partial f^k}{\partial x^i} \frac{\partial \mathcal{A}^i}{\partial y^k} = \frac{1}{\Delta} \partial_i \mathcal{A}^i$$

and the adjoint of the well defined intrinsic operator  $\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* = T^* : A \rightarrow \partial_i A$  is (*minus*) the well defined intrinsic operator  $\wedge^n T^* \xleftarrow{d} \wedge^n T^* \otimes T = \wedge^{n-1} T^* : \partial_i \mathcal{A}^i \leftarrow \mathcal{A}^i$ . Accordingly, if we have an operator  $E \xrightarrow{\mathcal{D}} F$ , we obtain the *formal adjoint* operator  $\wedge^n T^* \otimes E^* \xleftarrow{ad(\mathcal{D})} \wedge^n T^* \otimes F^*$ .

Now, with operational notations, let us consider the two differential sequences:

$$\begin{array}{c} \xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta \\ \nu \xleftarrow{ad(\mathcal{D})} \mu \xleftarrow{ad(\mathcal{D}_1)} \lambda \end{array}$$

where  $\mathcal{D}_1$  generates all the CC of  $\mathcal{D}$ . Then  $\mathcal{D}_1 \circ \mathcal{D} \equiv 0 \Leftrightarrow ad(\mathcal{D}) \circ ad(\mathcal{D}_1) \equiv 0$  but  $ad(\mathcal{D})$  may not generate all the CC of  $ad(\mathcal{D}_1)$ . Passing to the module framework, we just recognize the definition of  $\text{ext}_D^1(M)$ . Now, exactly like we defined the differential module  $M$  from  $\mathcal{D}$ , let us define the differential module  $N$  from  $ad(\mathcal{D})$ . Then  $\text{ext}_D^1(N) = t(M)$  does not depend on the presentation of  $M$ .

Having in mind that  $D$  is a  $K$ -algebra, that  $K$  is a left  $D$ -module with the standard action  $(D, K) \rightarrow K : (P, a)$

$\rightarrow P(a):(d_i, a) \rightarrow \partial_i a$  and that  $D$  is a bimodule over itself, we have only two possible constructions leading to the following two definitions:

**DEFINITION 5.4:** We may define the *inverse system*  $R = \text{hom}_K(M, K)$  of  $M$  and introduce  $R_q = \text{hom}_K(M_q, K)$  as the *inverse system of order  $q$* .

**DEFINITION 5.5:** We may define the right differential module  $M^* = \text{hom}_D(M, D)$ .

The first definition is leading to the *inverse systems* introduced by Macaulay in [41] (see [43] for details). As for the second, we have (see [1], p. 21, [6], p. 483-495, [42], [43] for details).

**THEOREM 5.6:** When  $M$  and  $N$  are left  $D$ -modules, then  $\text{hom}_K(M, N)$  and  $M \otimes_K N$  are left  $D$ -modules. In particular  $R = \text{hom}_K(M, K)$  is also a left  $D$ -module for the *Spencer operator*. Moreover, the structures of left  $D$ -modules existing therefore on  $M \otimes_K N$  and  $\text{hom}_K(N, L)$  are now coherent with the *adjoint isomorphism* for  $\text{mod}(D)$ :

$$\varphi: \text{hom}_D(M \otimes_K N, L) \xrightarrow{=} \text{hom}_D(M, \text{hom}_K(N, L)), \forall L, M, N \in \text{mod}(D)$$

*Proof:* For any  $f \in \text{hom}_K(M, N)$ , let us define:

$$(af)(m) = af(m) = f(am) \quad \forall a \in K, \forall m \in M$$

$$(\xi f)(m) = \xi f(m) - f(\xi m) \quad \forall \xi = \xi^i d_i \in T, \forall m \in M$$

It is easy to check that  $\xi a = a\xi + \xi(a)$  in the operator sense and that  $\xi\eta - \eta\xi = [\xi, \eta]$  is the standard bracket of vector fields. We have in particular with  $d$  in place of any  $d_i$ :

$$\begin{aligned} ((da)f)(m) &= (d(af))(m) = d(af(m)) - af(dm) \\ &= (\partial a)f(m) + ad(f(m)) - af(dm) \\ &= (a(df))(m) + (\partial a)f(m) \\ &= ((ad + \partial a)f)(m) \end{aligned}$$

For any  $m \otimes n \in M \otimes_K N$  with arbitrary  $m \in M$  and  $n \in N$ , we may then define:

$$a(m \otimes n) = am \otimes n = m \otimes an \in M \otimes_K N$$

$$\xi(m \otimes n) = \xi m \otimes n + m \otimes \xi n \in M \otimes_K N$$

and conclude similarly with:

$$\begin{aligned} (da)(m \otimes n) &= d(a(m \otimes n)) = d(am \otimes n) \\ &= d(am) \otimes n + am \otimes dn \\ &= (\partial a)m \otimes n + a(dm) \otimes n + am \otimes dn \\ &= (ad + \partial a)(m \otimes n) \end{aligned}$$

Using  $K$  in place of  $N$ , we finally get  $(d_i f)_\mu^k = (d_i f)(y_\mu^k) = \partial_i f_\mu^k - f_{\mu+1_i}^k$  that is we recognize exactly the *Spencer operator* with now  $Df = dx^i \otimes d_i f$  and thus:

$$\begin{aligned} (d_i(d_j f))_\mu^k &= \partial_{ij} f_\mu^k - (\partial_i f_{\mu+1_j}^k + \partial_j f_{\mu+1_i}^k) + f_{\mu+1_i+1_j}^k \\ \Rightarrow d_i(d_j f) &= d_j(d_i f) = d_{ij} f \end{aligned}$$

In fact,  $R$  is the *projective limit* of  $\pi_q^{q+r}: R_{q+r} \rightarrow R_q$  in a coherent way with jet theory [18] [19].

The next result is entrelacing the two left structures that we have just provided through the formula  $(g(m))(n) = f(m \otimes n) \in N$  defining the map  $\varphi$  whenever  $f \in \text{hom}_D(M \otimes_K N, L)$  is given and  $\varphi(f) = g$ . Using any  $\xi \in T$ , we get successively in  $L$ :

$$\begin{aligned}
 (\xi(g(m)))(n) &= \xi((g(m))(n)) - (g(m))(\xi n) \\
 &= \xi(f(m \otimes n)) - f(m \otimes \xi n) \\
 &= f(\xi(m \otimes n)) - f(m \otimes \xi n) \\
 &= f(\xi m \otimes n + m \otimes \xi n) - f(m \otimes \xi n) \\
 &= f(\xi m \otimes n) = (g(\xi m))(n)
 \end{aligned}$$

and thus  $\xi(g(m)) = g(\xi m), \forall m \in M$  or simply  $\xi \circ g = g \circ \xi$ .

Q.E.D.

**COROLLARY 5.7:** If  $M$  and  $N$  are right  $D$ -modules, then  $\text{hom}_K(M, N)$  is a left  $D$ -module. Moreover, if  $M$  is a left  $D$ -module and  $N$  is a right  $D$ -module, then  $M \otimes_K N$  is a right  $D$ -module.

*Proof:* If  $M$  and  $N$  are right  $D$ -modules, we just need to set  $(\xi f)(m) = f(m\xi) - f(m)\xi, \forall \xi \in T, \forall m \in M$  and conclude as before. Similarly, if  $M$  is a left  $D$ -module and  $N$  is a right  $D$ -module, we just need to set  $(m \otimes n)\xi = m \otimes n\xi - \xi m \otimes n$ .

Q.E.D.

**REMARK 5.8:** When  $M = {}_D M \in \text{mod}(D)$  and  $N = N_D$ , then  $\text{hom}_K(N, M)$  cannot be endowed with any left or right differential structure. When  $M = M_D$  and  $N = N_D$ , then  $M \otimes_K N$  cannot be endowed with any left or right differential structure (see [1], p. 24 for more details).

As  $D = {}_D D_D$  is a bimodule, then  $M^* = \text{hom}_D(M, D)$  is a right  $D$ -module according to Lemma 2.13 and the module  $N$  defined by the ker/coker sequence  $0 \leftarrow N \leftarrow F_1^* \xleftarrow{D^*} F_0^* \leftarrow M^* \leftarrow 0$  is thus a right module  $N_D$ .

**COROLLARY 5.9:** We have the *side changing* procedure  $N_D \rightarrow N = {}_D N = \text{hom}_K(\wedge^n T^*, N_D)$  with inverse  $M = M \rightarrow M_D = \wedge^n T^* \otimes_K M$  whenever  $M, N \in \text{mod}(D)$ .

*Proof:* According to the above Theorem, we just need to prove that  $\wedge^n T^*$  has a natural right module structure over  $D$ . For this, if  $\alpha = adx^1 \wedge \dots \wedge dx^n \in T^*$  is a volume form with coefficient  $a \in K$ , we may set  $\alpha \cdot P = ad(P)(a)dx^1 \wedge \dots \wedge dx^n$  when  $P \in D$ . As  $D$  is generated by  $K$  and  $T$ , we just need to check that the above formula has an intrinsic meaning for any  $\xi = \xi^i d_i \in T$ . In that case, we check at once:

$$\alpha \cdot \xi = -\partial_i(a\xi^i)dx^1 \wedge \dots \wedge dx^n = -\mathcal{L}(\xi)\alpha$$

by introducing the Lie derivative of  $\alpha$  with respect to  $\xi$ , along the intrinsic formula  $\mathcal{L}(\xi) = i(\xi)d + di(\xi)$  where  $i(\cdot)$  is the interior multiplication and  $d$  is the exterior derivative of exterior forms. According to well known properties of the Lie derivative, we get:

$$\begin{aligned}
 \alpha \cdot (a\xi) &= (\alpha \cdot \xi) \cdot a - \alpha \cdot \xi(a), \\
 \alpha \cdot (\xi\eta - \eta\xi) &= -[\mathcal{L}(\xi), \mathcal{L}(\eta)]\alpha = -\mathcal{L}([\xi, \eta])\alpha = \alpha \cdot [\xi, \eta].
 \end{aligned}$$

Q.E.D.

Collecting all the results so far obtained, if a differential operator  $\mathcal{D}$  is given in the framework of differential geometry, we may keep the same notation  $\mathcal{D}$  in the framework of differential modules which are *left* modules over the ring  $D$  of linear differential operators and apply duality, provided we use the notation  $\mathcal{D}^*$  and deal with *right* differential modules or use the notation  $\text{ad}(\mathcal{D})$  and deal again with *left* differential modules by using the *left*  $\leftrightarrow$  *right* conversion procedure.

**DEFINITION 5.10:** If a differential operator  $\xi \xrightarrow{\mathcal{D}} \eta$  is given, a *direct problem* is to find (generating) *compatibility conditions* (CC) as an operator  $\eta \xrightarrow{\mathcal{D}_1} \zeta$  such that  $\mathcal{D}\xi = \eta \Rightarrow \mathcal{D}_1\eta = 0$ . Conversely, given  $\eta \xrightarrow{\mathcal{D}_1} \zeta$ , the *inverse problem* will be to look for  $\xi \xrightarrow{\mathcal{D}} \eta$  such that  $\mathcal{D}_1$  generates the CC of  $\mathcal{D}$  and we shall say that  $\mathcal{D}_1$  is *parametrized* by  $\mathcal{D}$  if such an operator  $\mathcal{D}$  is existing.

**REMARK 5.11:** Of course, solving the direct problem (Janet, Spencer) is *necessary* for solving the inverse problem. However, though the direct problem always has a solution, the inverse problem may not have a solution at all and the case of the Einstein operator is one of the best non-trivial PD counterexamples (compare [10] to [34]). It is rather striking to discover that, in the case of OD operators, it took almost 50 years to understand that the possibility to solve the inverse problem was equivalent to the controllability of the corresponding control system (compare [11] to [34]).

As  $ad(ad(P)) = P, \forall P \in D$ , any operator is the adjoint of a certain operator and we get:

**FORMAL TEST 5.12:** The *double duality test* needed in order to check whether  $t(M) = 0$  or not and to find out a parametrization if  $t(M) = 0$  has 5 steps which are drawn in the following diagram where  $ad(\mathcal{D})$  generates the CC of  $ad(\mathcal{D}_1)$  and  $\mathcal{D}'_1$  generates the CC of  $\mathcal{D}$ :

$$\begin{array}{ccccccc}
 & & & & \zeta' & 5 & \\
 & & & & \nearrow^{\mathcal{D}'_1} & & \\
 4 & \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta & 1 \\
 & & & & & & \\
 3 & \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda & 2
 \end{array}$$

**THEOREM 5.13:**  $\mathcal{D}_1$  parametrized by  $\mathcal{D} \Leftrightarrow \mathcal{D}_1 = \mathcal{D}'_1 \Leftrightarrow t(M) = 0 \Leftrightarrow ext^1(N) = 0$ .

**REMARK 5.14:** When an operator  $\mathcal{D}_1$  can be parametrized by an operator  $\mathcal{D}$ , we may ask whether or not  $\mathcal{D}$  can be parametrized again by an operator  $\mathcal{D}_{-1}$  and so on. A good comparison can be made with hunting rifles as a few among them, called double rifles, are equipped with a double trigger mechanism, allowing to shoot again once one has already shot. In a mathematical manner, the question is to know whether the differential module defined by  $\mathcal{D}_1$  is torsion-free or reflexive. The main difficulty is that these intrinsic properties highly depend on the choice of the parametrizing operator. The simplest example is provided by the Poincaré sequence for  $n = 3$  made by the successive *grad, curl, div* operators. Indeed, any student knows that *curl* is parametrizing *div* and that *grad* is parametrizing *curl*. However, we may parametrize  $\partial_1 \eta^1 + \partial_2 \eta^2 + \partial_3 \eta^3 = 0$  by choosing  $\partial_3 \xi^1 = \eta^1, \partial_3 \xi^2 = \eta^2, -\partial_1 \xi^1 - \partial_2 \xi^2 = \eta^3$  with 2 potentials  $(\xi^1, \xi^2)$  only instead of the usual 3 potentials  $(\xi^1, \xi^2, \xi^3)$  and cannot proceed ahead as before. Other important examples will be provided in the next section dealing with applications, in particular the one involving Einstein equations when  $n = 4$ . This comment points out the reason for using the extension modules.

It remains to study a delicate question on which all the examples of the Introduction were focussing. Indeed, if a parametrization of a given system of OD or PD equations is possible, that is, equivalently, if the corresponding differential module is torsion-free, it appears that different parametrizations may exist with quite different numbers of potentials needed. Accordingly, it should be useful to know about the possibility to have upper and lower bounds for these numbers when  $n > 1$ , particularly in elasticity theory, because when  $n = 1$ , an OD module  $M$  with  $t(M) = 0$  being *automatically* isomorphic to a free module  $E$ , the number of potentials needed is equal to  $rk_D(M) = rk_D(E)$ . We shall use the language of differential modules in order to improve and apply a few results already presented in ([7], Theorem 7+ Appendix).

**THEOREM 5.15:** Let  $F_1 \xrightarrow{\mathcal{D}_1} F_0 \rightarrow M \rightarrow 0$  be a finite free presentation of the differential module  $M = coker(\mathcal{D}_1)$  and assume we already know that  $t(M) = 0$  by using the formal test. Accordingly, we have obtained the exact sequence  $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{\mathcal{D}} E$  of free differential modules where  $\mathcal{D}$  is the parametrizing operator. Then, there exists other parametrizations  $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{\mathcal{D}'} E'$  called *minimal parametrizations* and such that  $coker(\mathcal{D}')$  is a torsion module.

*Proof:* We first explain the reason for using the word “minimal”. Indeed, we have  $rk_D(M) = rk_D(F_0) - rk_D(im(\mathcal{D}_1)) = rk_D(\mathcal{D}) \leq rk_D(E)$  but also  $rk_D(M) = rk_D(\mathcal{D}') = rk_D(E')$  and thus  $rk_D(E') = rk_D(M) \leq rk_D(E)$  as a way to get a lower bound for the number of potentials but not to get a differential geometric framework.

While applying the formal test in the operator language,  $ad(\mathcal{D})$  is describing the (generating) CC of  $ad(\mathcal{D}_1)$  and we shall denote by  $ad(\mathcal{D}_{-1})$  the (generating) CC of  $ad(\mathcal{D})$  as we did in Example 1.3. In the module framework, going on with left differential modules, when  $F$  is a free left module, we shall denote by  $\tilde{F}$  the corresponding *converted* left differential module of the right differential module  $F^*$ . The reader not familiar with duality may look at the situations met in electromagnetism and elasticity in ([6], p. 492-495). If  $L = coker(ad(\mathcal{D}_{-1})) = im(ad(\mathcal{D})) \subset \tilde{F}_0$  and  $\tilde{E}'$  is the largest free differential submodule of  $L$  ( $D^3$  in Example 1.3,  $D$  in Example 1.4), then  $T = L/\tilde{E}'$  is a torsion module and we have the following commutative and exact diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \rightarrow & \tilde{E}' & = & \tilde{E}' & \rightarrow & 0 \\
 \downarrow & & \downarrow & \swarrow & \downarrow & & \\
 \tilde{E}_{-1} & \xrightarrow{ad(\mathcal{D}_{-1})} & \tilde{E} & \rightarrow & L & \rightarrow & 0 \\
 \parallel & & \downarrow & & \downarrow & & \\
 \tilde{E}_{-1} & \rightarrow & \tilde{E}'' & \rightarrow & T & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & & 
 \end{array}$$

where the central vertical monomorphism  $\tilde{E}' \rightarrow \tilde{E}$  is obtained by pulling a basis of  $\tilde{E}'$  back to  $\tilde{E}$  as we did in the diagram of Proposition 2.24. Coming back to the operators  $ad(\mathcal{D})$  and  $ad(\mathcal{D}_1)$ , we get the following commutative and exact diagram allowing to define  $ad(\mathcal{D}')$  by composition:

$$\begin{array}{ccccccc}
 \tilde{F}_1 & \xleftarrow{ad(\mathcal{D}_1)} & \tilde{F}_0 & \xleftarrow{ad(\mathcal{D})} & \tilde{E} & & \\
 & & \swarrow & & \swarrow & & \\
 & & \parallel & & L & & \uparrow \\
 & & \swarrow & & \swarrow & & \\
 \tilde{F}_0 & & \xleftarrow{ad(\mathcal{D}')} & & \tilde{E}' & \leftarrow & 0 \\
 & & & & \uparrow & \swarrow & \\
 & & & & 0 & & 0
 \end{array}$$

We have:

$$ad(\mathcal{D}_1) \circ ad(\mathcal{D}') \equiv 0 \Rightarrow \mathcal{D}' \circ \mathcal{D}_1 \equiv 0 \Rightarrow \ker(D) = \text{im}(\mathcal{D}_1) \subseteq \ker(\mathcal{D}') \subset F_0$$

and obtain by duality the following commutative and exact diagram where  $M' = \text{coim}(\mathcal{D}')$ :

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \nearrow \\
 & & 0 & & M & & \\
 & & \downarrow & & \nearrow \downarrow \searrow & & \\
 0 \rightarrow & \ker(\mathcal{D}) & \rightarrow & F_0 & \xrightarrow{p} & E & \\
 & & & & \downarrow & & 0 \\
 & & & & \downarrow & \nearrow & \\
 & & \downarrow & \parallel & M' & & \downarrow \\
 & & & & \nearrow \downarrow \searrow & & \\
 0 \rightarrow & \ker(\mathcal{D}') & \rightarrow & F_0 & \xrightarrow{p'} & E' & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

However, though the upper sequence  $F_1 \xrightarrow{\mathcal{D}_1} F_0 \rightarrow M \rightarrow 0$  is exact by definition because  $M = \text{coim}(\mathcal{D}) = \text{coker}(\mathcal{D}_1)$ , the lower induced sequence  $F_1 \xrightarrow{\mathcal{D}_1} F_0 \rightarrow M' \rightarrow 0$  may not be exact. With  $rk_D = rk$  for simplicity,  $t(M) = 0$  and the induced epimorphism  $M \rightarrow M' \rightarrow 0$ , we obtain:

$$rk(M) = rk(\mathcal{D}) = rk(ad(\mathcal{D})) = rk(\tilde{F}_0) - rk(ad(\mathcal{D}_1)) = rk(F_0) - rk(\mathcal{D}_1)$$

$$\begin{aligned}
 rk(M') &= rk(\mathcal{D}') = rk(ad(\mathcal{D}')) = rk(\tilde{E}') = rk(L) = rk(\tilde{F}_0) - rk(ad(\mathcal{D}_1)) = rk(F_0) - rk(\mathcal{D}_1) \\
 &\Rightarrow rk(M) = rk(M') \Rightarrow rk(ker(M \rightarrow M')) = 0 \Rightarrow ker(M \rightarrow M') \subseteq t(M) \subseteq (M) \\
 &\Rightarrow ker(M \rightarrow M') = 0 \Rightarrow M \simeq M'
 \end{aligned}$$

Accordingly,  $\mathcal{D}'$  is a minimal parametrization of  $\mathcal{D}_1$  contrary to  $\mathcal{D}$  in general and we invite the reader to repeat the proof by using operators and their adjoints as in the formal test.

Q.E.D.

**6) APPLICATIONS**

**EXAMPLE 6.1: OD Control Theory Revisited**

The following result is well known and can be found in any textbook of algebra [6] [13] [32]:

**PROPOSITION 6.2:** If  $A$  is a principal ideal domain, that is if any ideal in  $A$  is generated by a single element, then any torsion-free module over  $A$  is free and thus projective.

As this is the case of the ring  $D = K[d_x] = K[d]$  when  $n = 1$ , we obtain the following corollary of the preceding parametrizing Theorem, allowing to extend the Kalman test of controllability to PD systems with variable coefficients as we did in the Introduction (see [6]-[9] [11] for details).

**COROLLARY 6.3:** If  $\mathcal{D}_1$  is surjective, then  $ad(\mathcal{D}_1)$  is injective if and only if  $M$  is projective.

*Proof:* As  $\mathcal{D}_1$  is surjective, replacing  $M$  by  $P$ , we have the following short exact sequence:

$$0 \rightarrow F_1 \xrightarrow{\mathcal{D}_1} F_0 \rightarrow P \rightarrow 0$$

As  $P$  is projective, this short exact sequence splits with  $F_0 \simeq P \oplus F_1$  [6] [13] [32]. Using Proposition 2.6, we can construct a right inverse operator  $\mathcal{P}_1$  of  $\mathcal{D}_1$  with now  $\mathcal{D}_1 \circ \mathcal{P}_1 = id_{F_1}$  for the corresponding morphisms. Applying duality and Corollary 2.10, we get the short exact sequence:

$$0 \leftarrow F_1^* \xleftarrow{\mathcal{D}_1^*} F_0^* \leftarrow P^* \leftarrow 0$$

It follows that  $\mathcal{D}_1^*$  is surjective and the adjoint operator  $ad(\mathcal{D}_1)$  is injective because  $N = 0$ .

Conversely, if  $ad(\mathcal{D}_1)$  is injective, there exists a left inverse  $ad(\mathcal{P}_1)$  of  $ad(\mathcal{D}_1)$  providing a right inverse  $\mathcal{P}_1$  of  $\mathcal{D}_1$  (care). We may thus use again Corollary 2.10 because  $F_0^{**} \simeq F_0$  and  $F_1^{**} \simeq F_1$ . Meanwhile, we have proved that, if  $n = 1$  and  $t(M) = 0$ , it is always possible to find an injective parametrization but Example 1.4 is showing that this result is no longer true when  $n > 1$ .

Q.E.D.

Multiplying the control system of Example 1.1 by a test function  $\lambda$  and integrating by parts, the kernel of the operator thus obtained is defined by the OD equations:

$$\lambda_{xx} - \lambda_x = 0, \quad \lambda_{xx} - a\lambda = 0 \Rightarrow \lambda_x - a\lambda = 0 \Rightarrow (\partial_x a + a^2 - a)\lambda = 0$$

The formal adjoint of the operator defining the control system is thus injective if and only if we have  $\partial_x a + a^2 - a\lambda \neq 0$ , a result absolutely not evident at first sight but explaining why we used the same notation for a test function and for a Lagrange multiplier.

**EXAMPLE 6.4: Elasticity Theory Revisited**

The Killing operator  $\mathcal{D}: T \rightarrow S_2 T^*$  is a defined by  $\xi \in T \rightarrow \mathcal{D}\xi = \mathcal{L}(\xi)\omega = \Omega = 2\epsilon \in S_2 T^*$  with  $\omega_j \partial_i \xi^r + \omega_r \partial_j \xi^r + \xi^r \partial_r \omega_{ij} = \Omega_{ij} = 2\epsilon_{ij}$  where  $\xi$  is the displacement vector,  $\mathcal{L}(\xi)\omega$  is the Lie derivative of  $\omega$  with respect to  $\xi$  and  $\epsilon$  is the infinitesimal deformation tensor of textbooks. It is a Lie operator because its solutions  $\Theta \subset T$  satisfy  $[\Theta, \Theta] \subset \Theta$ . The corresponding first order Killing system  $R_1 \subset J_1(T)$  is not involutive because its symbol  $g_1 \subset T^* \otimes T$  is finite type with first prolongation  $g_2 = 0$  and thus  $rk(\mathcal{D}) = n$ . Accordingly, as  $\omega$  is a flat constant metric, the second order CC are described by an operator  $\mathcal{D}_1$  coming from the linearization of the Riemann tensor obtained in a standard way by setting  $\omega \rightarrow \omega + t\Omega$  with a small parameter  $t \ll 1$ , dividing by  $t$  and taking the limit when  $t \rightarrow 0$ . Finally, raising the index by means of the metric, the adjoint operator  $\wedge^n T^* \otimes T^* \xleftarrow{ad(\mathcal{D})} \wedge^n T^* \otimes S_2 T$  is defined by the intrinsic stress equations  $\nabla_r \sigma^{ir} \equiv \partial_r \sigma^{ir} + \gamma_{rs}^i \sigma^{rs} = f^i$  where  $\nabla$  is the covariant derivative and  $\gamma$  the Christoffel symbols ([6], p. 494, [44], p. 236).

- Airy parametrization of the stress equations when  $n = 2$  gives  $rk(E') = rk(E) = 1$  and we have thus 1 potential only. By duality, working out the corresponding adjoint operators, we obtain the two formally exact



the successive prolongations  $\rho_r(\Phi): J_{q+r}E \rightarrow J_r(F_0)$  defined by  $d_v\Phi^r = z_v^r$  for  $0 \leq |v| \leq r$  have kernel  $R_{q+r}$ . The symbol morphism  $\sigma_r(\Phi): S_{q+r}T^* \otimes E \rightarrow S_rT^* \otimes F_0$  with kernel  $g_{q+r}$  is induced by the projection of  $\rho_r(\Phi)$  onto  $\rho_{r-1}(\Phi)$  (see [39], p. 256 or [41], p. 233 for details). If we use such a procedure for a first order system with no zero or first order CC, we have  $q=1, E=T, F_0=J_1(T)/R_1$ . The Killing system  $R_1$  is formally integrable ( $R_2$  involutive) if and only if  $\omega$  has constant Riemannian curvature:

$$\rho_{1,ij}^k = c(\delta_i^k \omega_{ij} - \delta_j^k \omega_{ii})$$

with  $c=0$  when  $\omega$  is the flat Minkowski metric [12] [20] [39]. In general, we may apply the Spencer  $\delta$ -map to the top row obtained with  $r=2$  in order to get the commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & g_3 & \rightarrow & S_3T^* \otimes T & \rightarrow & S_2T^* \otimes F_0 & \rightarrow F_1 \rightarrow 0 \\ & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\ 0 \rightarrow & T^* \otimes g_2 & \rightarrow & T^* \otimes S_2T^* \otimes T & \rightarrow & T^* \otimes T^* \otimes F_0 & \rightarrow 0 \\ & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\ 0 \rightarrow & \wedge^2 T^* \otimes g_1 & \rightarrow & \underline{\wedge^2 T^* \otimes T^* \otimes T} & \rightarrow & \wedge^2 T^* \otimes F_0 & \rightarrow 0 \\ & \downarrow \delta & & \downarrow \delta & & \downarrow & \\ 0 \rightarrow & \wedge^3 T^* \otimes T & = & \wedge^3 T^* \otimes T & \rightarrow & 0 & \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

with exact rows and exact columns but the first that may not be exact at  $\wedge^2 T^* \otimes g_1$ . We shall denote by  $B^2(g_1)$  the coboundary as the image of the central  $\delta$ , by  $Z^2(g_1)$  the cocycle as the kernel of the lower  $\delta$  and by  $H^2(g_1) = Z^2(g_1)/B^2(g_1)$  the Spencer  $\delta$ -cohomology at  $\wedge^2 T^* \otimes g_1$ .

In the classical Killing system,  $g_1 \subset T^* \otimes T$  is defined by  $\omega_{ij}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r = 0 \Rightarrow \xi_r^r = 0, g_2 = 0, g_3 = 0$ . Applying the previous diagram, we discover that the Riemann tensor  $(\rho_{i,ij}^k) \subset \wedge^2 T^* \otimes T^* \otimes T$  is a section of

the bundle  $Riemann = F_1 = H^2(g_1) = Z^2(g_1)$  with  $dim(Riemann) = (n^2(n+1)^2/4) - (n^2(n+1)(n+2)/6) = (n^2(n-1)^2/4) - (n^2(n-1)(n-2)/6) = n^2(n^2-1)/12$  by using the top row or the left column. We obtain at once the two properties of the (linearized) Riemann tensor through the chase involved, namely  $(R_{i,ij}^k) \in \wedge^2 T^* \otimes T^* \otimes T$  is killed by both  $\delta$  and  $\sigma_0(\Phi)$ . However, we have no indices for  $F_1$  and cannot therefore exhibit the Ricci tensor or the Einstein tensor of general relativity by means of the usual contraction or trace. We recall briefly their standard definitions by stating  $R_{ij} = R_{ji} = R_{i,rj}^r \Rightarrow R = \omega^{ij}R_{ij} \Rightarrow E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij}R$ .

Similarly, going one step further, we get the (linearized) Bianchi identities with

$Bianchi = F_2 = H^3(g_1) = Z^3(g_1) \Rightarrow dim(Bianchi) = dim(\wedge^4 T^* \otimes T) - dim(\wedge^3 T^* \otimes g_1) = n^2(n^2-1)(n-2)/24$  as in ([46], p. 168-171). This approach is relating for the first time the concept of Riemann tensor candidate, introduced by Lanczos and others, to the Spencer  $\delta$ -cohomology of the Killing symbols.

Counting the differential ranks is now easy because  $R_1$  is formally integrable with finite type symbol and thus  $R_2$  is involutive with symbol  $g_2 = 0$ . We get:

$$\begin{aligned} rk(Killing) &= rk(Cauchy) = n \\ \Rightarrow rk(Riemann) &= dim(S_2T^*) - n = (n(n+1)/2) - n = n(n-1)/2 \\ rk(Bianchi) &= (n^2(n^2-1)/12) - (n(n-1)/2) = n(n-1)(n-2)(n+3)/12 \end{aligned}$$

that is  $rk(Bianchi) = 3$  when  $n=3$  and  $rk(Bianchi) = 14 = 20 - 6$  when  $n=4$ . Collecting all the results, we obtain that the canonical parametrization needs  $n^2(n^2-1)/12$  potentials while any minimal parametrization only needs  $n(n-1)/2$  potentials. The Einstein parametrization is thus "in between" because

$$n(n-1)/2 < n(n+1)/2 < n^2(n^2-1)/12, \forall n \geq 4.$$

The *conformal Killing system*  $\hat{R}_1 \subset J_1(T)$  is defined by eliminating the function  $A(x)$  in the system  $\mathcal{L}(\xi)\omega = A(x)\omega$ . It is also a *Lie operator*  $\hat{D}$  with solutions  $\hat{\Theta} \subset T$  satisfying  $[\hat{\Theta}, \hat{\Theta}] \subset \hat{\Theta}$ . Its symbol  $\hat{g}_1$  is defined by the linear equations  $\omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r - \frac{2}{n}\omega_{ij}\xi_r^r = 0$  which do not depend on any conformal factor and is finite type when  $n \geq 3$  because  $g_3 = 0$  but  $\hat{g}_2$  is now 2-acyclic only when  $n \geq 4$  and 3-acyclic only when  $n \geq 5$  [20] [46]-[48]. It is known that  $\hat{R}_2$  and thus  $\hat{R}_1$  too (by a chase) are formally integrable if and only if  $\omega$  has zero Weyl tensor:

$$\tau_{i,ij}^k \equiv \rho_{i,ij}^k - \frac{1}{(n-2)}(\delta_i^k \rho_{lj} - \delta_j^k \rho_{li} + \omega_{ij}\omega^{ks}\rho_{si} - \omega_{li}\omega^{ks}\rho_{sj}) + \frac{1}{(n-1)(n-2)}(\delta_i^k \omega_{lj} - \delta_j^k \omega_{li})\rho = 0$$

We may use the formula  $id_M - f \circ u = v \circ g$  of Proposition 2.6 in the *split short exact sequence* induced by the inclusions  $R_1 \subset \hat{R}_1 \Rightarrow g_1 \subset \hat{g}_1$ :

$$0 \rightarrow Ricci \rightarrow Riemann \rightarrow Weyl \rightarrow 0$$

according to the Vessiot structure equations, in particular if  $\omega$  has constant Riemannian curvature and thus  $\rho_{ij} = \rho_{i,rj}^r = c(n-1)\omega_{ij} \Rightarrow \rho = \omega^{ij}\rho_{ij} = cn(n-1)$  [20] [39] [45]-[47]. Using the same diagrams as before, we get  $Weyl = \hat{F}_1 = H^2(\hat{g}_1) \neq Z^2(\hat{g}_1)$  for defining any *Weyl tensor candidate*. As a byproduct, the linearized Weyl operator is of order 2 with a symbol  $\hat{h}_2 \subset S_2T^* \otimes \hat{F}_0$  which is not 2-acyclic by applying the  $\delta$ -map to the short exact sequence:

$$0 \rightarrow \hat{g}_{3+r} \rightarrow S_{3+r}T^* \otimes T \xrightarrow{\sigma_{2+r}(\Phi)} \hat{h}_{2+r} \rightarrow 0$$

and chasing through the commutative diagram thus obtained with  $r=0,1,2$ . As  $\hat{h}_3$  becomes 2-acyclic after one prolongation of  $\hat{h}_2$  only, it follows that *the generating CC for the Weyl operator are of order 2*, a result showing that the so-called Bianchi identities for the Weyl tensor are not CC in the strict sense of the definition as they do not involve only the Weyl tensor. Of course, these results could not have been discovered by Lanczos and followers because the formal theory of Lie pseudogroups and the Vessiot structure equations are still not acknowledged today.

For this reason, we provide a few hints in order to explain why the Vessiot structure equations *sometimes contain a few constants, sometimes none at all* as we just saw (see [39] [49] and [50] for more details). Indeed, isometries preserve the metric  $\omega = (\omega_{ij} = \omega_{ji}) \in S_2T^*$  while conformal isometries preserve the symmetric tensor density  $\hat{\omega} = \left( \hat{\omega}_{ij} = \omega_{ij} / \left( |\det(\omega)|^{\frac{1}{n}} \right) \right)$ . The respective variations are related by the similitude formula

$$\hat{\Omega}_{ij} \sim \Omega_{ij} - \frac{1}{n}\omega_{ij}\omega^{rs}\Omega_{rs}$$

which only depends on  $\omega$  and not on a conformal factor. It follows that  $F_0 = S_2T^*$

and that  $\hat{F}_0$  may be identified with the sub-bundle  $\left\{ \hat{\Omega} \in S_2T^* \mid \omega^{ij}\hat{\Omega}_{ij} = 0 \right\}$  with the above well defined epimorphism  $F_0 \rightarrow \hat{F}_0$  induced by the inclusion  $R_1 \subset \hat{R}_1$ . We set [39] [49] [50]:

**DEFINITION 6.5:** We say that a vector bundle  $F$  is associated with a Lie operator  $\mathcal{D}$  if, for any solution  $\xi \in \Theta \subset T$  of  $\mathcal{D}$ , there exists a first order operator  $\mathcal{L}(\xi): F \rightarrow F$  called *Lie derivative* with respect to  $\xi$  and such that:

- 1)  $\mathcal{L}(\xi + \eta) = \mathcal{L}(\xi) + \mathcal{L}(\eta) \quad \forall \xi, \eta \in \Theta$
- 2)  $[\mathcal{L}(\xi), \mathcal{L}(\eta)] = \mathcal{L}([\xi, \eta]) \quad \forall \xi, \eta \in \Theta$
- 3)  $\mathcal{L}(\xi)(f\eta) = f\mathcal{L}(\xi)\eta + (\xi \cdot f)\eta \quad \forall \xi \in \Theta, \forall f \in C^\infty(X), \forall \eta \in F$
- 4) If  $E$  and  $F$  are two such associated vector bundles, then:

$$\mathcal{L}(\xi)(\eta \otimes \zeta) = \mathcal{L}(\xi)\eta \otimes \zeta + \eta \otimes \mathcal{L}(\xi)\zeta, \quad \forall \xi \in \Theta, \forall \eta \in E, \forall \zeta \in F$$

In such a case, we may introduce  $\Upsilon = \Upsilon(F) = \{ \eta \in F \mid \mathcal{L}(\xi)\eta = 0, \forall \xi \in \Theta \subset T \}$ .

**PROPOSITION 6.6:** Using capital letters for linearized objects, we have:

- 1)  $\Upsilon(T) = C(\Theta) = \{ \eta \in T \mid [\xi, \eta] = 0, \forall \xi \in \Theta \} = \text{centralizer of } \Theta \text{ in } T.$

- 2)  $Y_0 = Y(F_0) = Y(S_2T^*) = \{\Omega = A\omega \in S_2T^* \mid A = cst\}$ .
- 3)  $Y_1 = Y(F)_1 = \{R_{i,j}^k = C(\delta_i^k \omega_j - \delta_j^k \omega_i) \in F_1 \mid C = cst\}$ .
- 4)  $\hat{Y}_1 = Y(\hat{F}_1) = 0 \Rightarrow Y(Ricci) = Y(Riemann)$ .

5) The Lie derivative commutes with the Janet operators  $\mathcal{D}, \mathcal{D}_1, \dots, \mathcal{D}_n$ .  
 We have in particular  $\mathcal{D}_1 : Y_0 \rightarrow Y_1 : A \rightarrow C = -cA$  (care to sign).

*Proof:* Two (nondegenerate) metrics  $\omega, \bar{\omega} \in S_2T^*$  give the same Killing system  $R_i$  if and only if  $\bar{\omega} = a\omega$  with the multiplicative group parameter  $a = cst$ . Therefore, if  $R_i$  is FI, then the two metrics have respective constant curvatures  $c$  and  $\bar{c} = c/a$ . Setting  $a = 1 + tA + \dots \Rightarrow \bar{c} = c + tC + \dots$  while linearizing these finite transformations with  $t \ll 1$  gives  $C = -cA$  when  $t \rightarrow 0$ .

Q.E.D.

However, we have yet not proved the most difficult result that could not be obtained without homological algebra and the next example will explain this additional difficulty.

**EXAMPLE 6.7:** With  $\partial_{22}\xi = \eta^2, \partial_{12}\xi = \eta^1$  for  $\mathcal{D}$ , we get  $\partial_1\eta^2 - \partial_2\eta^1 = \zeta$  for  $\mathcal{D}_1$ . Then  $ad(\mathcal{D}_1)$  is defined by  $\mu^2 = -\partial_1\lambda, \mu^1 = \partial_2\lambda$  while  $ad(\mathcal{D})$  is defined by  $\nu = \partial_{12}\mu^1 + \partial_{22}\mu^2$  but the CC of  $ad(\mathcal{D}_1)$  are generated by  $\nu' = \partial_1\mu^1 + \partial_2\mu^2$ . Using operators, we have the differential sequences:

$$\begin{array}{ccccc} \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta \\ & & & & \\ & & ad(\mathcal{D}) & & ad(\mathcal{D}_1) \\ \nu & \leftarrow & \mu & \leftarrow & \lambda \end{array}$$

where the upper sequence is formally exact at  $\eta$  but the lower sequence is not formally exact at  $\mu$ .

Passing to the module framework, we obtain the sequences:

$$\begin{array}{ccccccc} D & \xrightarrow{\mathcal{D}_1} & D^2 & \xrightarrow{\mathcal{D}} & D & \rightarrow & M \rightarrow 0 \\ & & & & & & \\ & & ad(\mathcal{D}_1) & & ad(\mathcal{D}) & & \\ D & \leftarrow & D^2 & \leftarrow & D & & \end{array}$$

where the lower sequence is not exact at  $D^2$ .

Therefore, we have to prove that the extension modules vanish, that is  $ad(\mathcal{D})$  generates the CC of  $ad(\mathcal{D}_1)$  and, conversely, that  $\mathcal{D}_1$  generates the CC of  $\mathcal{D}$ . We also remind the reader that it has not been easy to exhibit the CC of the Maxwell or Morera parametrizations when  $n = 3$  and that a direct checking for  $n = 4$  should be strictly impossible. It has been proved by L. P. Eisenhart in 1926 [49] that the solution space  $\Theta$  of the Killing system has  $n(n+1)/2$  infinitesimal generators  $\{\theta_\tau\}$  linearly independent over the constants if and only if  $\omega$  had constant Riemannian curvature, namely zero in our case. As we have a Lie group of transformations preserving the metric, the three theorems of Sophus Lie assert that  $[\theta_\rho, \theta_\sigma] = c^\tau_{\rho\sigma} \theta_\tau$  where the structure constants  $c$  define a Lie algebra  $\mathcal{G}$ . We have therefore  $\xi \in \Theta \Leftrightarrow \xi = \lambda^\tau \theta_\tau$  with  $\lambda^\tau = cst$ . Hence, we may replace locally the Killing system by the system  $\partial_i \lambda^\tau = 0$ , getting therefore the differential sequence:

$$0 \rightarrow \Theta \rightarrow \wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \dots \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \rightarrow 0$$

which is the tensor product of the Poincaré sequence by  $\mathcal{G}$ . Finally, it follows from Proposition 3.3 that the extension modules do not depend on the resolution used and thus vanish because the Poincaré sequence is self adjoint (up to sign), that is  $ad(d)$  generates the CC of  $ad(d)$  at any position, exactly like  $d$  generates the CC of  $d$  at any position. This (difficult) result explains why the differential modules we have met were torsion-free or even reflexive. We invite the reader to compare with the situation of the Maxwell equations in electro-magnetism (see [6], p. 492-494 for more details). However, we have explained in [6] [45]-[47] [51] why neither the Janet sequence nor the Poincaré sequence can be used in physics and must be replaced by the Spencer sequence which is another resolution of  $\Theta$  [39] [40] [46].

**EXAMPLE 6.8: PD Control Theory Revisited**

Comparing with the Theorem allowing to construct a minimal parametrization, we started with  $\mathcal{D}_1\eta = \zeta$  and computed  $ad(\mathcal{D}_1)\lambda = \mu$  with generating CC  $ad(\mathcal{D})\mu = \nu$ , obtaining therefore finally the generating CC  $ad(\mathcal{D}_{-1})\nu = 0$ , that is  $\partial_2\nu^2 + \partial_1\nu^1 + x^2\nu^1 = 0$ . In that case,  $rk(L) = 1$  in the diagram providing the minimal parametrization. This result explains why we had two potentials  $(\xi^1, \xi^2)$  in the canonical parametrization and

only one, namely  $(\xi^1 = \xi, 0)$  or  $(0, \xi^2 = \xi')$ , in the minimal parametrizations but it is not possible to imagine the underlying procedure.

**EXAMPLE 6.9: OD/PD Optimal Control Revisited**

Using the notations of the Formal Test 5.12, let us assume that the two differential sequences:

$$\begin{array}{ccccc} \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta \\ & & \xleftarrow{ad(\mathcal{D})} & & \xleftarrow{ad(\mathcal{D}_1)} \\ & & \nu & & \mu & & \lambda \end{array}$$

are formally exact, that is  $\mathcal{D}_1$  generates the CC of  $\mathcal{D}$  and  $ad(\mathcal{D})$  generates the CC of  $ad(\mathcal{D}_1)$ , namely  $\xi$  is a potential for  $\mathcal{D}_1$  and  $\lambda$  is a potential for  $ad(\mathcal{D})$ . We may consider a variational problem for a cost function  $\varphi(\eta)$  under the linear OD or PD constraint described by  $\mathcal{D}_1\eta = 0$ .

- Introducing convenient Lagrange multipliers  $\lambda$  while setting  $dx = dx^1 \wedge \dots \wedge dx^n$  for simplicity, we must vary the integral:

$$\Phi = \int [\varphi(\eta) + \lambda \mathcal{D}_1\eta] dx \Rightarrow \delta\Phi = \int [(\partial\varphi(\eta)/\partial\eta)\delta\eta + \lambda \mathcal{D}_1\delta\eta] dx$$

Integrating by parts, we obtain the EL equations:

$$\partial\varphi(\eta)/\partial\eta + ad(\mathcal{D}_1)\lambda = 0$$

to which we have to add the constraint  $\mathcal{D}_1\eta = 0$  obtained by varying  $\lambda$ . If  $ad(\mathcal{D}_1)$  is an injective operator, in particular if  $\mathcal{D}_1$  is formally surjective (no CC) while  $n = 1$  and  $M$  is torsion-free or  $n \geq 1$  and  $M$  is projective, then one can obtain  $\lambda$  explicitly and eliminate it by substitution ([7]). Otherwise, using the CC  $ad(\mathcal{D})$  of  $ad(\mathcal{D}_1)$  in order to eliminate  $\lambda$ , we have to study the formal integrability of the combined system:

$$ad(\mathcal{D})\partial\varphi(\eta)/\partial\eta = 0, \quad \mathcal{D}_1\eta = 0$$

which may be a difficult task as we already saw through the examples of the Introduction.

- We may also transform the given variational problem with constraint into a variational problem without any constraint if and only if the differential constraint can be parametrized. Using the parametrization of  $\mathcal{D}_1$  by  $\mathcal{D}$ , we may vary the integral:

$$\Phi = \int \varphi(\mathcal{D}\xi) dx \Rightarrow \delta\Phi = \int (\partial\varphi(\eta)/\partial\eta) \mathcal{D}\delta\xi dx$$

whenever  $\eta = \mathcal{D}\xi$  and integrate by parts for arbitrary  $\delta\xi$  in order to obtain the EL equations:

$$ad(\mathcal{D})\partial\varphi(\eta)/\partial\eta = 0, \quad \eta = \mathcal{D}\xi$$

in a coherent way with the previous approach and the Poincaré duality *geometry*  $\leftrightarrow$  *physics*.

As a byproduct, if the field equations  $\mathcal{D}_1\eta = 0$  can be parametrized by a potential  $\xi$  through the formula  $\mathcal{D}\xi = \eta$ , then the induction equations  $ad(\mathcal{D})\mu = \nu$  can be obtained by duality in a coherent way with the double duality test, ... on the condition to know what sequence must be used.

However, we have already proved in [45]-[47] [51] [52] that the Cauchy stress equations must be replaced by the Cosserat couple-stress equations and that the Janet sequence (only used in this paper) must be thus replaced by the Spencer sequence. Accordingly, it becomes clear that the work of Lanczos and followers has been based on a double confusion between fields and inductions on one side, but also between the Janet sequence and the Spencer sequence on the other side.

**FUNDAMENTAL RESULT 6.10:** The Janet and Spencer sequences for any Lie operator of finite type are formally exact by construction, both with their corresponding adjoint sequences. Lanczos has been trying to parametrize  $ad(\mathcal{D}_1)$  by  $ad(\mathcal{D}_2)$  when  $\mathcal{D}_1$  parametrizes  $\mathcal{D}_2$ . On the contrary, we have proved that one must parametrize  $ad(\mathcal{D})$  by  $ad(\mathcal{D}_1)$  when  $\mathcal{D}$  parametrizes  $\mathcal{D}_1$  as in the famous infinitesimal equivalence problem ([20], p. 332-336), with a shift by one step. This is also the only way which is coherent with the corresponding non-linear sequences and the finite equivalence problem [39] [46] [47] [50] [52] [53].

## 2. Conclusion

The effective usefulness of the double duality test seems absolutely magical in actual practice but the reader may

not forget about the amount of mathematics needed from different domains. Unhappily, in our opinion based on a long experience in dealing with applications, the most difficult part is concerned with formal integrability and involution needed in order to compute the various differential ranks involved. However, the above methods, though largely superseding the pioneering approaches of Janet and Cartan, are still not known in mechanics and in mathematical physics, particularly in general relativity or even in control theory despite many tentatives done twenty years ago. We hope that this paper will help to improve this situation in a near future, in particular when dealing with *partial differential optimal control*, which is with variational calculus with OD or PD constraints along the way that has been initiated by Lanczos for eliminating the corresponding Lagrange multipliers or using them as potentials while studying the mathematical foundations of general relativity.

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