Scientific Research Publishing

Applied Mathematics, 2022, 13, 949-959
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Mathematical Analysis of Two Approaches for Optimal Parameter Estimates to Modeling Time Dependent Properties of Viscoelastic Materials

Irina Viktorova, Sofya Alekseeva, Muhammed Kose

Mechanical Engineering Department, School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, USA
Email: iviktor@clemson.edu

How to cite this paper: Viktorova, I., Alekseeva, S. and Kose, M. (2022) Mathematical Analysis of Two Approaches for Optimal Parameter Estimates to Modeling Time Dependent Properties of Viscoelastic Materials. *Applied Mathematics*, **13**, 949-959.
https://doi.org/10.4236/am.2022.1312059

Received: November 5, 2022
Accepted: December 26, 2022
Published: December 29, 2022

Abstract

Mathematical models for phenomena in the physical sciences are typically parameter-dependent, and the estimation of parameters that optimally model the trends suggested by experimental observation depends on how model-observation discrepancies are quantified. Commonly used parameter estimation techniques based on least-squares minimization of the model-observation discrepancies assume that the discrepancies are quantified with the L^2-norm applied to a discrepancy function. While techniques based on such an assumption work well for many applications, other applications are better suited for least-squared minimization approaches that are based on other norm or inner-product induced topologies. Motivated by an application in the material sciences, the new alternative least-squares approach is defined and an insightful analytical comparison with a baseline least-squares approach is provided.

Keywords

Laplace Transform, Viscoelastic Composite, Norm Space, Inner Product Space, Least Squares Minimization, Optimal Parameter Estimation

1. Introduction

In this paper, we assume that X is the space of all continuous functions $f:[0,\infty)\to\mathbb{R}$ having a Laplace transform $F:H\to\mathbb{C}$ with $H:=\{s\in\mathbb{C}:\Re(s)>0\}$.

Parameters $p\in P\subseteq\mathbb{R}^n$ associates with a time-domain model $m(p,\cdot):[0,\infty)\to\mathbb{R}$ are considered optimal insofar as they yield a minimal model-observation discrepancy $\varepsilon:[0,\infty)\to\mathbb{R}$ defined by $\varepsilon(t):=m(p,t)-r(t)$, where function $r:[0,\infty)\to\mathbb{R}$ is obtained as a regression to a set of time-dependent observations. The model-observation discrepancy ε is assumed to be function-valued, so the phrase "minimal discrepancy" only has meaning when ε is understood to be a member of some norm-induced topology $(X,\|\cdot\|)$. Having specified the norm-induced topology to which ε belongs, the optimal parameters are then computed as an optimal solution p^* to the least squares problem (LSP)

$$\min_{p\in P}\|\varepsilon(p,\cdot)\|^2 \tag{1}$$

Two norms on X are considered in formulating the LSP (1).

The first norm, the *baseline norm*, is denoted by $\|\cdot\|_{T,\gamma}$, while the second norm, the *alternative norm*, is denoted by $\|\cdot\|_{S,s}$. (The norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ on X are defined in Section 2.) The use of the baseline norm $\|\cdot\|_{T,\gamma}$ in (1) yields a variant of a commonly used LSP for computing optimal model parameters, while the alternative norm $\|\cdot\|_{S,s}$ is motivated by the elegant closed-form expressions for certain models $m(p,\cdot)$ undertaking the Laplace transform. This is particularly true for certain creep models associated with viscoelastic materials [1]-[7].

While the use of the alternative norm $\|\cdot\|_{S,s}$ in LSP (1) has been successfully applied for computing optimal parameter estimates in [5], a theoretical foundation and justification for the use of the alternative form $\|\cdot\|_{S,s}$ in LSP (1) is in need of further development. Refining the developments began in [8] [9], this paper addresses the above need in Section 2, where 1) two inner products $\langle\cdot,\cdot\rangle_{T,\gamma}:X\times X\to\mathbb{C}$ and $\langle\cdot,\cdot\rangle_{S,s}:X\times X\to\mathbb{C}$ are defined over X and verified with respect to the inner product properties; 2) the norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ are induced from the respective inner products $\langle\cdot,\cdot\rangle_{T,\gamma}$ and $\langle\cdot,\cdot\rangle_{S,s}$; 3) from the inner product properties, a bounding relationship is established between the norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$; and 4) insight is obtained from the bounding relationship into how the parameter solutions $p\in P$ to LSP (1) $\|\cdot\|=\|\cdot\|_{T,\gamma}$ relate to the parameter solutions $p\in P$ to LSP (1) $\|\cdot\|=\|\cdot\|_{S,s}$. The first three contributions represent a substantial refinement and streamlining of the developments in [8] [9], thus paving the way for the fourth contribution which, furthermore, builds on the developments in [8] [9].

The remainder of the paper is organized as follows. From the developments in Section 2, a more simple and improved implementation of a previous application [5] becomes evident, and this is presented in Section 3. Computational setup and results are presented and discussed briefly in this same section. Lastly Section 4 concludes this paper and provides comments on future work.

2. Definition and Analysis of the Norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$

The two norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ are induced, respectively, by the following two inner products $\langle\cdot,\cdot\rangle_{T,\gamma}:X\times X\to\mathbb{C}$ and $\langle\cdot,\cdot\rangle_{S,s}:X\times X\to\mathbb{C}$ defined in the following manner for each pair $f,g\in X$ and parameters $\gamma>0$ and $s\in H$:

$$\langle f,g\rangle_{T,\gamma} := \int_0^\infty f(t)\overline{g(t)}\mathrm{e}^{-\gamma t}\mathrm{d}t \tag{2}$$

$$\langle f,g\rangle_{S,s} := \left(\int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t\right)\overline{\left(\int_0^\infty g(t)\mathrm{e}^{-st}\mathrm{d}t\right)} \tag{3}$$

It is now shown that (2) and (3) are, in fact, inner products.

Proposition 2.1. *The mappings* $\langle\cdot,\cdot\rangle_{T,\gamma}$ *given by* (2) *and* $\langle\cdot,\cdot\rangle_{S,s}$ *given by* (3) *are defined for all* $f,g\in X$ *and are furthermore inner products over X.*

Proof: Because *X* contains the continuous functions $f:[0,\infty)\to\mathbb{R}$ having a Laplace transform, the inner product $\langle\cdot,\cdot\rangle_{S,s}$ is defined for all $f,g\in X$. Also, the function *fg* defined by multiplying $f\in X$ and $g\in X$ is continuous and of exponential order [10] (follows from the same properties of *f* and *g*), and so the Laplace transform $\mathcal{L}\{fg\}$ exists, and (2) is simply the Laplace transform $\mathcal{L}\{fg\}$ evaluated at $s=\gamma$. Thus, the inner product $\langle\cdot,\cdot\rangle_{T,\gamma}$ is also defined for all $f,g\in X$.

Recall that, for any vector space *V*, an inner product $\langle\cdot,\cdot\rangle:V\times V\to\mathbb{C}$ satisfies the following rules for each $u,v,w\in V$ and $\lambda\in\mathbb{C}$ (e.g., see [11]):

I1: $\langle u,v\rangle=\overline{\langle v,u\rangle}$

I2: $\langle\lambda u,v\rangle=\lambda\langle u,v\rangle$

I3: $\langle u,v+w\rangle=\langle u,v\rangle+\langle u,w\rangle$

I4: $\langle u,v\rangle\ge 0$, and $\langle u,u\rangle=0\Rightarrow u=0$

Property I1 follows readily for $\langle\cdot,\cdot\rangle_{T,\gamma}$ by noting that *f* and *g* are real-valued functions $\mathrm{e}^{-\gamma t}$ is real-values, and so the integrand is real-valued. For $\langle\cdot,\cdot\rangle_{S,s}$, Property I1 follows from (3) by computing

$$\begin{aligned}\langle f,g\rangle_{S,s} &= \left(\int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t\right)\overline{\left(\int_0^\infty g(t)\mathrm{e}^{-st}\mathrm{d}t\right)}\\ &= F(s)\overline{G(s)}\\ &= G(s)\overline{F(s)}\\ &= \overline{\langle g,f\rangle}_{S,s}\end{aligned}$$

where *F*(*s*) and *G*(*s*) denote the Laplace transform of *f* and *g*, respectively.

Properties I2 and I3 follow easily for both $\langle\cdot,\cdot\rangle_{T,\gamma}$ and $\langle\cdot,\cdot\rangle_{S,s}$ from elementary properties of integrals.

Property I4 applies to $\langle\cdot,\cdot\rangle_{T,\gamma}$ because: 1) for each $f\in X$, the integrand of $\langle f,f\rangle_{T,\gamma}$ is always nonnegative; and 2) if $f\not\equiv 0$, then by the continuity of *f* over $[0,\infty)$, there exist $t_0\in[0,\infty)$, $\epsilon>0$, and $\delta>0$ over which $f(T)\ge\delta$ for all $T\in\left[t_0-\frac{\epsilon}{2},t_0+\frac{\epsilon}{2}\right]$. Thus, for each $\gamma>0$, we have

$\langle f,f\rangle_{T,\gamma}\ge\epsilon\delta^2\mathrm{e}^{-\gamma(t_0+\epsilon)}>0$ if $f\not\equiv 0$. From this, the implication

$\langle f,f\rangle_{T,\gamma}=0\Rightarrow f\equiv 0$ follows.

To show that property I4 applies to $\langle\cdot,\cdot\rangle_{S,s}$, first note that $\langle f,f\rangle_{S,s}\ge 0$ for all $f\in X$ follows from the definition (3), and so it remains to show that $\langle f,f\rangle_{S,s}=0\Rightarrow f=0$. This latter claim holds under application of Lerch's theorem (see, e.g., [11] [12]) to the setting where *f* is continuous. Namely, if

$\langle f,f\rangle_{S,s}=0$ (so that $F(s)\equiv 0$), then $\int_0^a f(t)\mathrm{d}t=0$ for all $a>0$. The assumed continuity of *f* on $[0,\infty)$ and the Fundamental Theorem of Calculus imply that $f\equiv 0$. Thus, I4 holds for $\langle\cdot,\cdot\rangle_{S,s}$. Hence, it has been shown that $\langle\cdot,\cdot\rangle_{T,\gamma}$ and $\langle\cdot,\cdot\rangle_{S,s}$ are both inner products over *X*.

One possible relationship between two different norms $\|\cdot\|_a$ and $\|\cdot\|_b$ called equivalence is now explored. The *equivalence* of two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ is characterized by the existence of $0<\ell\le u<\infty$ such that

$$\ell\|f\|_a\le\|f\|_b\le u\|f\|_a \quad \text{for all} \quad f\in X \tag{4}$$

(See, e.g., [11].) Using the inner-product structures defined on *X*, the Cauchy-Schwartz inequality can be used to show a bounding relationship of the form $\|f\|_{S,s}\le u\|f\|_{T,\gamma}$ for all $f\in X$, $s\in H$, and $\gamma<R(s)$ via the computation

$$\|f\|_{S,s}^2=\left|\int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t\right|^2=\left|\left\langle f(t),\mathrm{e}^{-(s-\gamma)t}\right\rangle_{T,\gamma}\right|^2\le\langle f,f\rangle_{T,\gamma}\left\langle \mathrm{e}^{-(s-\gamma)t},\mathrm{e}^{-(s-\gamma)t}\right\rangle_{T,\gamma} \tag{5}$$

$$=\left(\int_0^\infty|f(t)|^2\mathrm{e}^{-\gamma t}\mathrm{d}t\right)\left(\int_0^\infty \mathrm{e}^{-(s+\bar{s}-\gamma)}\mathrm{d}t\right)\Rightarrow\|f\|_{S,s}^2\le u\|f\|_{T,\gamma}^2 \tag{6}$$

where $u=\int_0^\infty \mathrm{e}^{-(s+\bar{s}-\gamma)}\mathrm{d}t=\dfrac{1}{s+\bar{s}-\gamma}$.

Whereas the upper bound coefficient *u* is established in (6), the lower bound coefficient $\ell>0$ necessary to establish the equivalence (4) for each fixed $s\in H$ and $0<\gamma<R(s)$ is shown not to exist through two counterexamples:

Counterexample 1: Let *f* be of the form $f(t)=\mathrm{e}^{-\omega t},\omega>0$. Then $\|f\|_{T,\gamma}=\sqrt{\dfrac{1}{2\omega+\gamma}}$ and $\|f\|_{S,s}=\dfrac{1}{|\omega+s|}$. So $\ell\le\dfrac{\|f\|_{S,s}}{\|f\|_{T,\gamma}}=\sqrt{\dfrac{2\omega+\gamma}{|\omega+s|^2}}$. Both $\lim_{\omega\to\infty}\|f\|_{T,\gamma}=0$ and $\lim_{\omega\to\infty}\|f\|_{S,s}=0$. Furthermore, since $\lim_{\omega\to\infty}\sqrt{\dfrac{2\omega+\gamma}{|\omega+s|^2}}=0$, there is no $\ell>0$ serving as a lower bound coefficient.

Counterexample 2: Let *f* be of the form $f(t)=\sin(\omega t)$, $\omega>0$. Then $\|f\|_{T,\gamma}=\sqrt{\dfrac{1}{2}\left(\dfrac{1}{\gamma}-\dfrac{\gamma}{\gamma^2+4\omega^2}\right)}$ and $\|f\|_{S,s}=\dfrac{\omega}{|s^2+\omega^2|}$. Now $\lim_{\omega\to\infty}\|f\|_{T,\gamma}=\sqrt{\dfrac{1}{2\gamma}}>0$ and $\lim_{\omega\to\infty}\|f\|_{S,s}=0$. Thus, $\lim_{\omega\to\infty}\dfrac{\|f\|_{S,s}}{\|f\|_{T,\gamma}}=0$, and so there is no lower bound $\ell>0$ on $\dfrac{\|f\|_{S,s}}{\|f\|_{T,\gamma}}$.

The lack of a lower bound coefficient $\ell>0$ is also depicted in **Figure 1** and **Figure 2** for the same two counterexamples. Thus, it is established that due to the lack of the lower bound coefficient $\ell>0$, the norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ over *X* are *not* equivalent.

The bounding relationship (6) between the norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ is also described via inclusion relationships between sublevel sets. The sublevel set $L_{\|\cdot\|}(f,P,\delta)$ is defined by

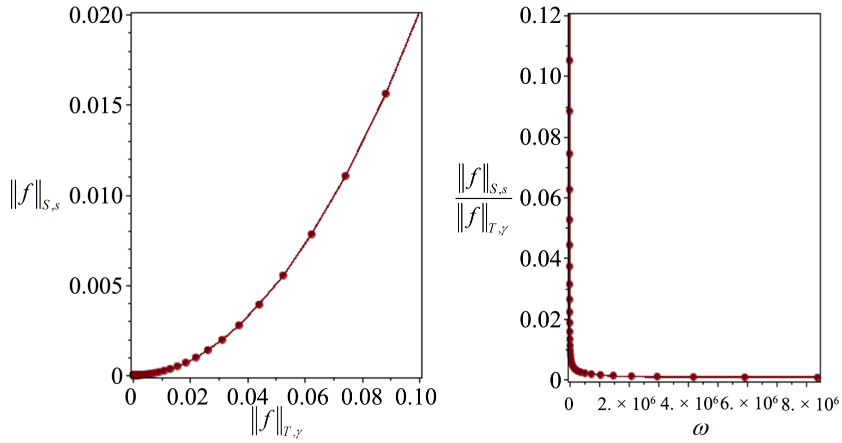

Figure 1. Illustrating the lack of a lower bound coefficient $\ell > 0$ for the norms $\|\cdot\|_{T,\gamma}$ ($\gamma = 0.05$) and $\|\cdot\|_{S,s}$ ($s = 0.1(1+i)$) with $f = e^{-\omega t}, \omega > 0$. For both plots, each point corresponds to the use of a single value of ω, where $\omega = 2^{-2+0.5k}, k = 1, \cdots, 50$.

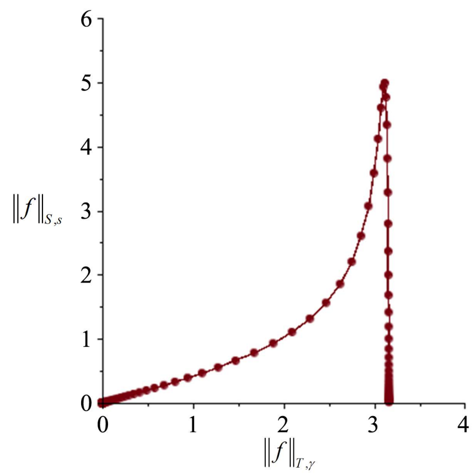

Figure 2. Plot of points $\left(\|f\|_{T,\gamma}, \|f\|_{S,s}\right)$ with $\gamma = 0.05$, $s = 0.1(1+i)$, $f(t) = \sin(\omega t)$, and frequency parameter $\omega > 0$ varying from $\omega = 2^{-19}$ to $\omega = 2^{5}$. The plotted points approaching the origin along the plotted curve correspond to ω values approaching zero, while the plotted points proceeding away from the origin along the same plotted curve correspond to ω values approaching infinity.

$$L_{\|\cdot\|}(f, P, \delta) := \left\{ p \in P : \left\| f(p, \cdot) \right\| \le \delta \right\}$$

for each *f*, *P*, and $\delta > 0$. By the existence of the bounding coefficient $u, 0 < u < \infty$, in (6), we have the inclusion

$$L_{\|\cdot\|}\left(f, P, \frac{1}{u}\delta \right) \subseteq L_{\|\cdot\|_{S,s}}(f, P, \delta) \tag{7}$$

The sublevel set inclusion (7) provides a sense in which the norm $\|\cdot\|_{S,s}$ penalizes model-observation discrepancy more leniently than the norm $\|\cdot\|_{T,\gamma}$. This leniency is observed, for example, in the plot of Figure 2 where the increasing frequency of $f(t) = \sin(\omega t)$ due to $\omega \to \infty$ leads to $\|f\|_{S,s} \to 0$

while $\|\cdot\|_{T,\gamma}\ f \to \sqrt{\frac{1}{2\gamma}} > 0$.

For application purposes, the preference between the norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ in formulating the LSP (1) depends on 1) the desired degree of leniency in penalizing imperfect model-observation fit due to the use of parameter $p \in P$; and 2) the ease and accuracy of evaluating the norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$. Next, in Section 3, the material science application of solving LSP (1) motivating the contributions of this paper is revisited where the use of each of the two norms $\|\cdot\|_{T,\gamma}$ and $\|\cdot\|_{S,s}$ is evaluated in terms of the above two preference criteria.

3. Application for Modeling Time Dependent Properties of Viscoelastic Materials

A time-dependent model $m(p,\cdot)$ for modeling creep of viscoelastic materials under an applied stress load is given by

$$m(p,t) := \frac{\sigma}{E}\left[1+\lambda\sum_{n=0}^{\infty}\frac{(-\beta)^n t^{(1-\alpha)(n+1)}}{\Gamma\left[(1-\alpha)(n+1)+1\right]}\right] \tag{8}$$

where the stress level σ and Young's modulus E are determined experimentally, and the material-specific kernel parameters (α,β,λ) satisfy

$$(\alpha,\beta,\lambda) \in \left\{(\alpha,\beta,\lambda) : 0<\alpha<1, \beta\in\mathbb{R}, \lambda\in\mathbb{R}\right\}$$

(See [1] [3] [5] for details.) The parameter α can be found from the first term of the infinite series expansion in (8) [3]. Thus, only the model parameters β and λ need to be determined as an optimal solution $p=(\beta,\lambda)$ to problem (1) with $P=\left\{p : p=(\beta,\lambda), \beta\in\mathbb{R}, \lambda\in\mathbb{R}\right\}$.

The regression function $r:[0,\infty)\to\mathbb{R}$ is fit to observations based on experiments performed for three types of composites with nanofillers [5]:

1) Pure polyamide (PA).

2) Polyamide with ultra-dispersed diamonds (PA + UDD).

3) Polyamide with carbon nanotube fillers (PA + CNT).

For each material, the tests with the corresponding three loading levels $\sigma_{0.3}$, $\sigma_{0.4}$, and $\sigma_{0.5}$ are performed, where the subscript of σ indicates that the stress applied to the materials is 30%, 40%, and 50%, respectively, of the ultimate stress, which was taken equivalent to the yielding stress of each of the tested materials. Using these experimental data, the regression functions $r(t)$ used for each data set take the form

$$r(t) = c_0 + c_1 e^{-0.1t} + c_2 e^{-0.5t} + c_1 e^{-0.02t} \tag{9}$$

where the coefficients $c_i, i=0,1,2,3$ are estimated for each data set using standard linear regression techniques. The resulting regression functions and the material-specific vales for $\sigma_{0.3}$, $\sigma_{0.4}$, and $\sigma_{0.5}$ are given in **Table 1**.

For each computation, the norm $\|\cdot\|_{T,\gamma}$ parameter $\gamma=0.005$ and the norm $\|\cdot\|_{S,s}$ parameter $s=0.01(1+i)$ are used; furthermore, the experimentally determined parameters α, E, and $\sigma=\sigma_i, i=0.3,0.4,0.5$ associated with $m(p,t)$ are provided in **Table 2**.

Table 1. Regression functions obtained from the creep experiments.

loading level	$r(t)$
	PA
$\sigma_{0.3}$	$25.3626 - 23.5786e^{-0.1t} + 23.8311e^{-0.05t} - 18.0708e^{-0.02t}$
$\sigma_{0.4}$	$35.1104 - 40.6847e^{-0.1t} + 47.1203e^{-0.05t} - 32.7179e^{-0.02t}$
$\sigma_{0.5}$	$45.6491 - 46.1334e^{-0.1t} + 56.2102e^{-0.05t} - 43.5065e^{-0.02t}$
	PA + UDD
$\sigma_{0.3}$	$25.3102 - 26.9200e^{-0.1t} + 32.8715e^{-0.05t} - 24.0642e^{-0.02t}$
$\sigma_{0.4}$	$33.0484 - 31.1413e^{-0.1t} + 33.8989e^{-0.05t} - 26.9547e^{-0.02t}$
$\sigma_{0.5}$	$41.1932 - 40.9358e^{-0.1t} + 46.6029e^{-0.05t} - 35.6518e^{-0.02t}$
	PA + CNT
$\sigma_{0.3}$	$21.6266 - 22.9993e^{-0.1t} + 26.2085e^{-0.05t} - 19.1740e^{-0.02t}$
$\sigma_{0.4}$	$28.5471 - 33.5503e^{-0.1t} + 36.0275e^{-0.05t} - 24.5412e^{-0.02t}$
$\sigma_{0.5}$	$36.5119 - 40.9524e^{-0.1t} + 43.0930e^{-0.05t} - 30.5410e^{-0.02t}$

Table 2. Setup parameters.

material	γ	s	α	$\sigma_{0.3}$	$\sigma_{0.4}$	$\sigma_{0.5}$	E
PA	0.005	0.01 (1 + *i*)	0.83	16.20	21.60	27.00	955
PA + UDD	0.01	0.01 (1 + *i*)	0.83	15.90	21.20	26.50	1008
PA + CNT	0.01	0.01 (1 + *i*)	0.83	18.72	24.96	31.20	1320

The optimal parameters $p^* = (\beta^*, \lambda^*)$ are computed as optimal solutions to LSP (1) using the baseline norm $\|\cdot\| = \|\cdot\|_{T,\gamma}$ and the alternative norm $\|\cdot\| = \|\cdot\|_{S,s}$. These computations are performed with Maple™ [13]. The computed parameter estimates are presented in **Table 3** and the resulting wellness-of-fit between the parameterized models and experimental observations are illustrated in **Figure 3**.

As observed earlier [1] [3], the model $m(p,t)$ has an elegant simplification under its Laplace transformation

$$M(p,s) := \mathcal{L}\{m(p,t)\} = \frac{\sigma}{E}\frac{1}{s}\left[1 + \frac{\lambda}{s^{1-\alpha} + \beta}\right] \tag{10}$$

Furthermore, each function r with the form (9) has a closed-form Laplace transform denoted by $R(s)$. Thus, for each s satisfying $\Re(s) > 0$, problem (1) takes the following elegant form when $\|\cdot\| = \|\cdot\|_{S,s}$:

$$\min_{p \in P} \|M(p,s) - R(s)\|_2^2 \tag{11}$$

Solving the LSP (11) is computationally more accurate and less expensive than solving the corresponding LSP (1) with $\|\cdot\| = \|\cdot\|_{T,\gamma}$. This is consistent with the

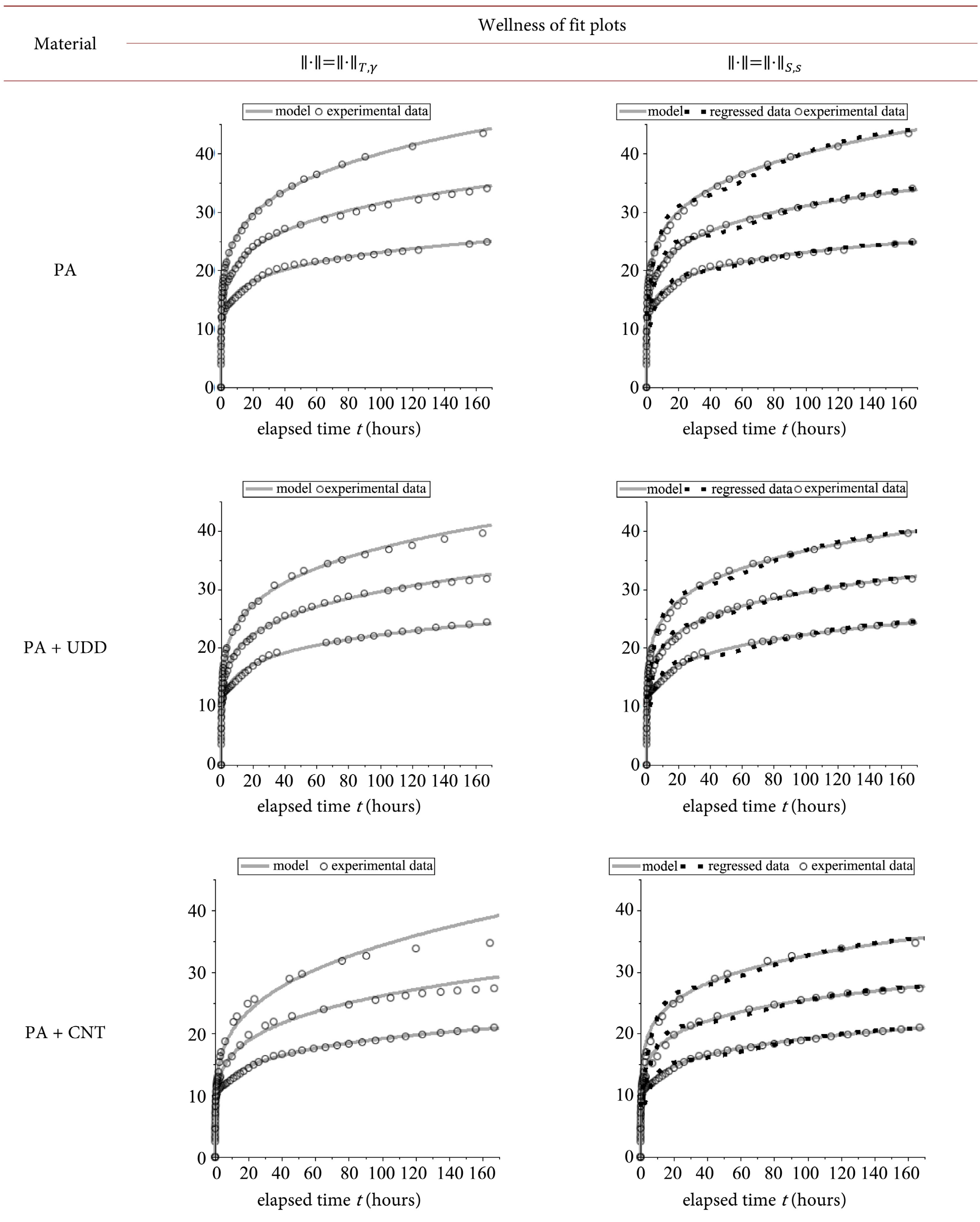

Figure 3. Wellness of fit plots using optimal parameter $p^* = (\beta^*, \lambda^*)$ solutions to problem (1) with $\|\cdot\| = \|\cdot\|_{T,\gamma}$ (left) and $\|\cdot\| = \|\cdot\|_{S,s}$ (right). Plots are given based on nine experimental data sets corresponding to three materials each with three loading levels.

Table 3. Optimal parameter estimates.

material	load	β		λ	
		$\lVert\cdot\rVert_{T,\gamma}$	$\lVert\cdot\rVert_{S,s}$	$\lVert\cdot\rVert_{T,\gamma}$	$\lVert\cdot\rVert_{S,s}$
PA	$\sigma_{0.3}$	0.061	0.083	654.621	683.217
	$\sigma_{0.4}$	−0.011	0.015	570.776	599.720
	$\sigma_{0.5}$	−0.050	−0.027	530.258	561.334
PA + UDD	$\sigma_{0.3}$	0.020	0.002	623.307	598.596
	$\sigma_{0.4}$	−0.025	0.011	561.117	608.101
	$\sigma_{0.5}$	−0.047	0.009	530.750	600.000
PA + CNT	$\sigma_{0.3}$	0.011	0.012	585.827	588.197
	$\sigma_{0.4}$	−0.075	0.034	481.834	613.805
	$\sigma_{0.5}$	−0.126	0.022	433.514	614.225

motivation and observation seen in earlier works [1] [3] [14] associated with the use of Laplace transform-based approaches to estimating the optimal model parameters.

4. Conclusions

This paper contributes a mathematical foundation for the comparison between time domain least squares parameter estimation problems formulated using the norm $\lVert\cdot\rVert_{T,\gamma}$ and Laplace domain least squares parameter estimation problems introduced in [1] [3], applied in [5] [8], and formulated using the alternative norm $\lVert\cdot\rVert_{S,s}$ as defined in Section 2. A relationship between the norms $\lVert\cdot\rVert_{T,\gamma}$ and $\lVert\cdot\rVert_{S,s}$ is analyzed in terms of norm equivalence, and in exploring this equivalence, the existence of the necessary upper bound coefficient $u, 0<u<\infty$ was shown to exist in Section 2 using the two inner product structures (2) and (3) defined on X. However, the non-existence of the corresponding lower bound coefficient $\ell, 0<\ell<u$, is demonstrated through two counterexamples. From the bounding relationship (6), inclusion relationships (7) of sublevel sets follow that provides a sense in which the norm $\lVert\cdot\rVert_{S,s}$ penalizes certain types of model-observation deviation more leniently than the norm $\lVert\cdot\rVert_{T,\gamma}$.

The plots of **Figure 3** suggest that the solutions $p^*=\left(\beta^*,\lambda^*\right)$ to LSP (1) with $\lVert\cdot\rVert=\lVert\cdot\rVert_{S,s}$ yield improved model-observation fit over the corresponding solutions with $\lVert\cdot\rVert=\lVert\cdot\rVert_{T,\gamma}$. In addition to the computational advantages associated with solving (11), the improvement is also attributed to the relatively lenient (in a sense derived from the inclusion relationships (7)) penalization of certain types of model-observation by $\lVert\cdot\rVert_{S,s}$ as compared with $\lVert\cdot\rVert_{T,\gamma}$. If the types of model-observation deviations that are penalized leniently are subjectively negligible to the model user, then the computation of the optimal solutions $\left(\beta^*, \lambda^*\right)$ to LSP (1) with $\|\cdot\| = \|\cdot\|_{S,s}$ is more flexible, and this results in subjectively improved model-observation fit as compared with the fit obtained with the use of the norm $\|\cdot\| = \|\cdot\|_{T,\gamma}$.

Acknowledgements

The authors thank Mr. Brian Dandurand of Argon National Laboratory, Chicago, IL. For valuable insights, discussions, and computations provide.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Rabotnov, Y.N. (1969) Creep Problems in Structural Members. *North-Holland Series in Applied Mathematics and Mechanics*, **7**, 803 p.

[2] Suvorova, Y.V., Sorina, T.G., Viktorova, I.V. and Mikhailov, V.V. (1980) Effect of the Loading Rate on the Character of Fracture of Carbon-Fiber-Reinforced Plastics. *Mechanics of Composite Materials*, **16**, 847-851. https://doi.org/10.1007/BF00610184

[3] Suvorova, Y.V. (2004) On the Nonlinear Hereditary Type Equation by Yu. N. Rabotnov and Its Applications. *Mechanics of Solids*, **1**, 174-181.

[4] Selivanov, M. and Chornoivan, Y. (2012) Computational Optimization of Characteristics for Composites of Viscoelastic Components. *Journal of Engineering Mathematics*, **74**, 91-100. https://doi.org/10.1007/s10665-011-9477-1

[5] Viktorova, I., Dandurand, B., Alekseeva, S. and Fronya, M. (2012) The Modeling of Creep for Polymer-Based Nanocomposites Using an Alternative Nonlinear Optimization Approach. *Mechanics of Composite Materials*, **48**, 1-14. https://doi.org/10.1007/s11029-013-9313-y

[6] Wu, Z. (2020) A New Parameter Heteromorphic Elliptic: Properties and Applications. *World Journal of Engineering and Technology*, **8**, 642-657. https://doi.org/10.4236/wjet.2020.84045

[7] Titelman, L. (2021) Generalized Parameters of Porous Materials as Similarity Numbers. *Advances in Materials Physics and Chemistry*, **11**, 177-201. https://doi.org/10.4236/ampc.2021.1111017

[8] Dandurand, B., Viktorova, I. and Alekseeva, S. (2013) A Comparison of the Time-Domain and Laplace-Domain Least Squares Parameter Estimation for Modeling Properties of Viscoelastic Materials. *Problems of Machine Building and Automatization*, **3**, 106-111.

[9] Dandurand, B. (2013) Mathematical Optimization for Engineering Design Problems. Ph.D. Thesis, Clemson University, Clemson.

[10] Churchill, R.V. (1958) Operational Mathematics. 2nd Edition, McGraw-Hill Book Company, Inc., New York.

[11] Ponnusamy, S. (2002) Foundations of Functional Analysis. Alpha Science.

[12] Weisstein, E. (2002) Lerch's Theorem. MathWorld, A Wolfram Web Resource. http://mathworld.wolfram.com/LerchsTheorem.html

[13] Maple 17. Maplesoft, a Division of Waterloo Maple Inc., Waterloo, Ontario.

[14] Dandurand, B., Viktorova, I., Alekseeva, S. and Goodson, S. (2011) Nonlinear Modeling and Optimization of Parameters for Viscoelastic Composites and Nanocomposites. *Problems of Machine Building and Automatization*, **3**, 51-57.

The List of the Variables Used in This Paper

p: parameters in time-domain model

$m(p,\cdot)$: model equation

$\varepsilon(t)$: strain

$r(t)$: regression function

$(X,\|\cdot\|)$: norm induced topology

X: space of all condition functions of real variables

F: Laplace transformation

$\|\cdot\|_{T,\gamma}$: baseline norm in real domain

$\|\cdot\|_{S,s}$: alternative norm in Laplace complex domain

V: vector space

u, v, w: vectors

λ: constant

s: complex variable

t: real variable

f, g: real valued functions

$F(s)$, $G(s)$: Laplace transforms of f and g functions

L: lower bound coefficient

ω: real parameter > 0

γ: complex valued parameter

δ: small real number

σ: stress level

E: Young's modulus

α, β, λ: material specific kernel parameters

Γ: Gamma function

c_i: regression function coefficients