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Evaluate All the Order of Every Element in the Higher Order of Group for Addition and Multiplication Composition

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Abstract

This paper aims at treating a study on the order of every element for addition and multiplication composition in the higher order of groups for different algebraic structures as groups; order of a group and order of element of a group in real numbers. Here we discuss the higher order of groups in different types of order which will give us practical knowledge to see the applications of the addition and multiplication composition. If G is a finite group, n is a positive integer and $a \in G$, then the order of the products na. When G is a finite group, every element must have finite order. However, the converse is false: there are infinite groups where each element has finite order. For example, in the group of all roots of unity in $\mathbb{C}^{\times}$ each element has finite order. Finally, we find out the order of every element of a group in different types of higher order of group.

Keywords

Order of Element of a Group, Addition Composition, Multiplication Composition, Modulo

1. Introduction

A group is a particular type of an algebraic system. Here, we propose to study the groups of the order of an element of a group, order of group and the integral powers of an element of a group etc. There are a number of common conventions regarding group notation. The group notation is o or *. We will frequently omit the symbol for the group operation but we will also often write the operation as · or + when it represents multiplication or addition in a ring, and write 1 or 0 for the corresponding identity elements respectively. It's addition +, multiplication × or (.) is used as binary operation. If the group operation is denoted as a multiplication, then an element $a \in G$ is said to be order *n* if *n* is the least positive integer such that $a^n = e$ or $O(a) \leq n$ *i.e.*, if $a^n = e$ and $a^r \neq e$ $\forall r \in N$ s.t. $r < n$. The order of *a* is denoted by $O(a)$. If $a^n \neq e$ for any $n \in N$, then *a* is said to be of zero order or infinite order. It is currently a feature of finite group theory that many theorems on finite groups [1] can only be proved by reducing them to a check that some property holds for all finite simple groups. Then the classification of finite simple groups (see *i.e.* [2] [3] [4]) comes into play and one has to be able to handle the three different families of simple groups with appropriate techniques. A further extension of Thompson's characterization was obtained much later in [5], using the full force of the classification of finite simple groups. Thus, each element of a finite group appears exactly once in each row of the table. It follows that each row (respectively column) is a rearrangement of the elements of the group. Next, we discuss the extension of the associative property to products with any number of factors. More specifically, we will prove the so-called generalized associative law which states that in a set with associative operation, a product of factors is unchanged regardless of how parentheses are inserted as long as the factors and their order of appearance in the product are unchanged, as we use additison and multiplication related theorem of group of different orders. As a result we find out the order of group of higher order group. But here we discuss the order of group of higher order group as 50. Finally, we find out the order of every element of a group in different types of the higher order groups ([6]-[15]).

2. Characterization of the Order of an Element of a Group

We begin this section related to definition with the following characterization of the order of an element of a group

Definition-1 (Multiplication Composition) ([16] [17]):

Let *e* be the identity element in (*G*,). An element $a \in G$ is said to be order *n* if $n \in Z^+$ such that $a^n = e$ or $O(a) \leq n$. *i.e.*, if $a^n = e$ and $a^r \neq e$ $\forall r \in N$ s.t. $r < n$. The order of *a* is denoted by $O(a)$. If $a^n \neq e$ for any $n \in N$, then *a* is said to be of zero order or infinite order.

Definition-1 (Addition Composition) ([18] [19]):

Let *e* be the identity element in (*G*, +). An element $a \in G$ is said to be order *n* if $n \in Z^+$ such that $na = e$ or $O(a) \leq n$. *i.e.*, if $na = e$ and $ar \neq e$ $\forall r \in N$ s.t. $0 < r < n$. The order of *a* is denoted by $O(a)$. If $na \neq e$ for any $n \in N$, then *a* is said to be of zero order or infinite order.

Example:

1) In multiplicative group Q_0, $O(a) = 1$, $O(-1) = 2$ and $O(a) = \infty$

$\forall a \in Q_0$ s.t. $a \neq \pm 1$.

2) In $(Z, +)$, $O(0) = 1$ and all other elements have infinite order.

Torsion free group [20]**:** A group G is called a torsion free group if the identity element e is the only element of finite order.

Example: $(Z, +)$, $(Q, +)$, $(R, +)$.

Torsion group or periodic group [21]**:** A group G is said to be a torsion group or periodic group if every element of G is of finite order.

Example: The multiplication group is $\{1, -1, i, -i\}$.

Mixed Group [22]**:** A group G is said to be a mixed group if $\exists$ at least two elements $a, b \in G$ s.t.

1) $o(a)$ is finite, $a \neq e$ 2) $o(b) = \infty$.

Example: The multiplicative group $(Q_0, .)$, where $Q_0 \neq 0$ is a mixed group.

For $o(-1) = 2$, $e \neq -1$, $0(a) = \infty$ if $a \neq \pm 1$ and $a \in Q_0$.

3. Properties of the Order of an Element of a Group

We begin this section of the following theorem related properties of the order of an element of a group.

3.1. Theorem [23]

Show that the order of every element of a finite group is finite.

Proof: Let G be a finite group with multiplication composition.

Let $a \in G$ be an arbitrary element.

Now we will prove that $O(a)$ is finite.

By closure property, all the elements $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, $\cdots$ etc. belong to G

i.e. $a, a^2, a^3, a^4, a^5, a^6, a^7, \ldots$ etc. belong to G.

But all these elements are not distinct. Since G is finite.

Let e be the identity in G, then $a^0 = e$.

Let us suppose that:

$$a^m = a^n \text{ where } m > n$$

$$\Rightarrow a^m a^{-n} = a^n a^{-n} = a^0 = e$$

$$\Rightarrow a^{m-n} = e \Rightarrow a^p = e, \text{ where } p = m - n > 0, \text{ as } m > n$$

also m and n are finite and hence p is a finite positive integer.

Now p is a positive integer s.t. $a^p = e$.

This proves that:

$$o(a) \leq p = \text{finite number}$$

$$i.e.\ o(a) \leq \text{a finite number} \Rightarrow o(a) \text{ is finite}$$

Remark: The order of any element of a finite group can never exceed the order of the group.

3.2. Theorem [24]

If an element a of a group G is of order n, then $a^m = e$ if and only if n is a divisor of m, *i.e.* $m = nq$.

Proof: Let $a \in G$ be arbitrary element, s.t. $o(a) = n$ and so $a^n = e$. Suppose that $a^m = e$. Now we will prove that n is a divisor of m:

$$o(a) = n,\ a^m = e \Rightarrow o(a) \le m \Rightarrow n \le m$$

if $n = m$, then clearly n is a divisor of m,
if $m > n$, then by division algorithm,

$$\exists\ q, r \in Z \text{ s.t. } m = nq + r, \text{ where } 0 \le r < n \tag{1}$$

now $a^m = a^{nq+r} = (a^n)^q a^r = e^q a^r = ea^r = a^r \Rightarrow a^m = a^r \Rightarrow e = a^r \Rightarrow a^r = e$
hence

$$a^r = e, \text{ where } 0 \le r < n \tag{2}$$

if $0 < r < n$, then $a^r = e$ is not possible as $o(a) = n$.
if $r = 0$, then $a^r = e$ is possible. So that $r = 0$.
From (2), then we get, $m = nq$.
Conversely, Let $m = nq$. Now we will prove that $a^m = e$.
Now $a^m = a^{nq} = (a^n)^q = e^q = e \Rightarrow a^m = e$.
Remark: This theorem can also be expressed in the following ways.
Let $a \in G$ be arbitrary element.
1) If $o(a) = n$, then $a^m = e \Leftrightarrow m = nq, q \in Z$.
2) If $a \in G$ is of order n, then there exist an integer m for which $a^m = e$ if m is a multiple of n.

4. Results and Discussion

Here, we discuss the result of order of every element for addition and multiplication composition in the higher order of group. As usual we can use addition and multiplication related theorem to evaluate order of group of different orders such as order $2, 3, 4, 5, \cdots, 20$ etc., *i.e.* whose order not so high (Not Higher Order Groups). As a result we use addition and multiplication related theorem to evaluate order of group of higher order group as like as $40, 41, 42, 43, \cdots, 49$ etc. Finally, here we discuss the order of group of higher order group as 50.

4.1. ([25]-[31]) Find Order of Every Element of the Group {0, 1, 2, 3, ..., 49} the Composition Being Addition Modulo 50

Solution: Let $(G, +_{50})$ is a group. Where $G = \{0, 1, 2, 3, \cdots, 49\}$ and $+_{50}$ denotes addition modulo 50. Here $e = 0$.

In addition notation,
$na = e$ and n is a least positive integer,

$$\Rightarrow O(a) = n \quad \therefore O(0) = 1$$

for $O(e) = 1$ for identity element of every group.
To determine $O(1)$**:**

$$1(1) = 1,\ 2(1) = 1 +_{50} 1 = 2,\ 3(1) = 1 +_{50} 2(1) = 1 +_{50} 2 = 3,$$
$$4(1) = 1 +_{50} 3(1) = 1 +_{50} 3 = 4,\ 5(1) = 1 +_{50} 4(1) = 1 +_{50} 4 = 5,$$
$$6(1) = 1 +_{50} 5(1) = 1 +_{50} 5 = 6,\ 7(1) = 1 +_{50} 6(1) = 1 +_{50} 6 = 7,$$

$$8(1)=1+_{50}7(1)=1+_{50}2=8,\ 9(1)=1+_{50}8(1)=1+_{50}8=9,$$
$$10(1)=1+_{50}9(1)=1+_{50}9=10,\ 11(1)=1+_{50}10(1)=1+_{50}10=11,\cdots,$$
$$50(1)=1+_{50}49(1)=1+_{50}49=0=e$$

Thus $50(1) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(1)=50$.

To determine $O(2)$:

$$1(2)=2,\ 2(2)=2+_{50}2=4,\ 3(2)=2+_{50}2(2)=2+_{50}4=6,$$
$$4(2)=2+_{50}3(2)=2+_{50}6=8,\ 5(2)=2+_{50}4(2)=2+_{50}8=10,$$
$$6(2)=2+_{50}5(2)=2+_{50}10=12,\ 7(2)=2+_{50}6(2)=2+_{50}12=14,$$
$$8(2)=2+_{50}7(2)=2+_{50}14=16,\ 9(2)=2+_{50}8(2)=2+_{50}16=18,$$
$$10(2)=2+_{50}9(2)=2+_{50}18=20,\cdots,25(2)=2+_{50}24(2)=2+_{50}48=0=e$$

Thus $25(2) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(2)=25$.

To determine $O(3)$:

$$1(3)=3,\ 2(3)=3+_{50}3=6,\ 3(3)=3+_{50}2(3)=3+_{50}6=9,$$
$$4(3)=3+_{50}3(3)=3+_{50}9=12,\ 5(3)=3+_{50}4(3)=3+_{50}12=15,$$
$$6(3)=3+_{50}5(3)=3+_{50}15=18,\ 7(3)=3+_{50}6(3)=3+_{50}18=21,$$
$$8(3)=3+_{50}7(3)=3+_{50}21=24,\cdots,16(3)=3+_{50}15(3)=3+_{50}45=48,$$
$$17(3)=3+_{50}16(3)=3+_{50}48=1,\ 18(3)=3+_{50}17(3)=3+_{50}1=4,\cdots$$
$$26(3)=3+_{50}25(3)=3+_{50}25=28,\ 27(3)=3+_{50}26(3)=3+_{50}28=31,$$
$$28(3)=3+_{50}27(3)=3+_{50}31=34,\cdots,38(3)=3+_{50}37(3)=3+_{50}11=14,$$
$$39(3)=3+_{50}38(3)=3+_{50}14=17,\ 40(3)=3+_{50}39(3)=3+_{50}17=20,\cdots,$$
$$48(3)=3+_{50}47(3)=3+_{50}41=44,\ 49(3)=3+_{50}48(3)=3+_{50}44=47,$$
$$50(3)=3+_{50}49(3)=3+_{50}47=0=e$$

Thus $50(3) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(3)=50$.

To determine $O(4)$:

$$1(4)=4,\ 2(4)=4+_{50}4=8,\ 3(4)=4+_{50}2(4)=4+_{50}8=12,$$
$$4(4)=4+_{50}3(4)=4+_{50}12=16,\ 5(4)=4+_{50}4(4)=4+_{50}16=20,$$
$$6(4)=4+_{50}5(4)=4+_{50}20=24,\ 7(4)=4+_{50}6(4)=4+_{50}24=28,$$
$$8(4)=4+_{50}7(4)=4+_{50}28=32,\cdots,16(4)=4+_{50}15(4)=4+_{50}10=14,$$
$$17(4)=4+_{50}16(4)=4+_{50}14=18,18(4)=4+_{50}17(4)=4+_{50}18=22,\cdots,$$
$$23(4)=4+_{50}22(4)=4+_{50}38=42,\ 24(4)=4+_{50}23(4)=4+_{50}42=46,$$
$$25(4)=4+_{50}24(4)=4+_{50}46=0=e$$

Thus $25(4) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(4)=25$.

To determine $O(5)$:

$$1(5)=5,\ 2(5)=5+_{50}5=10,\ 3(5)=5+_{50}2(5)=5+_{50}10=15,$$
$$4(5)=5+_{50}3(5)=5+_{50}15=20,\ 5(5)=5+_{50}4(5)=5+_{50}20=25,$$
$$6(5)=5+_{50}5(5)=5+_{50}25=30,\ 7(5)=5+_{50}6(5)=5+_{50}30=35,$$
$$8(5)=5+_{50}7(5)=5+_{50}35=40,\ 9(5)=5+_{50}8(5)=5+_{50}40=45,$$
$$10(5)=5+_{50}9(5)=5+_{50}45=0=e$$

Thus $10(5) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(5)=10$.

To determine $O(6)$:

$$\begin{aligned}
&1(6)=6,\ 2(6)=6+_{50}6=12,\ 3(6)=6+_{50}2(6)=6+_{50}12=18,\\
&4(6)=6+_{50}3(6)=6+_{50}18=24,\ 5(6)=6+_{50}4(6)=6+_{50}24=30,\\
&6(6)=6+_{50}5(6)=6+_{50}30=36,\ 7(6)=6+_{50}6(6)=6+_{50}36=42,\\
&8(6)=6+_{50}7(6)=6+_{50}42=48,\ 9(6)=6+_{50}8(6)=6+_{50}48=4,\cdots,\\
&16(6)=6+_{50}15(6)=6+_{50}40=46,17(6)=6+_{50}16(6)=6+_{50}46=2,\\
&18(6)=6+_{50}17(6)=6+_{50}2=8,\cdots,23(6)=6+_{50}22(6)=6+_{50}32=38,\\
&24(6)=6+_{50}23(6)=6+_{50}38=44,\ 25(6)=6+_{50}24(6)=6+_{50}44=0=e
\end{aligned}$$

Thus 25(6) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(6)=25$.

To determine $O(7)$:

$$\begin{aligned}
&1(7)=7,\ 2(7)=7+_{50}7=14,\ 3(7)=7+_{50}2(7)=7+_{50}14=21,\\
&4(7)=7+_{50}3(7)=7+_{50}21=28,\ 5(7)=7+_{50}4(7)=7+_{50}28=35,\\
&6(7)=7+_{50}5(7)=7+_{50}35=42,\ 7(7)=7+_{50}6(7)=7+_{50}42=49,\\
&8(7)=7+_{50}7(7)=7+_{50}49=6,\cdots,16(7)=7+_{50}15(7)=7+_{50}5=12,\\
&17(7)=7+_{50}16(7)=7+_{50}12=19,18(7)=7+_{50}17(7)=7+_{50}19=26,\cdots,\\
&26(7)=7+_{50}25(7)=7+_{50}25=32,\ 27(7)=7+_{50}26(7)=7+_{50}32=39,\\
&28(7)=7+_{50}27(7)=7+_{50}39=46,\cdots,38(7)=7+_{50}37(7)=7+_{50}9=16,\\
&39(7)=7+_{50}38(7)=7+_{50}16=23,\ 40(7)=7+_{50}39(7)=7+_{50}23=30,\cdots,\\
&48(7)=7+_{50}47(7)=7+_{50}29=36,\ 49(7)=7+_{50}48(7)=7+_{50}36=43,\\
&50(7)=7+_{50}49(7)=7+_{50}43=0=e
\end{aligned}$$

Thus 50(7) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(7)=50$.

To determine O(8):

$$\begin{aligned}
&1(8)=8,\ 2(8)=8+_{50}8=16,\ 3(8)=8+_{50}2(8)=8+_{50}16=24,\\
&4(8)=8+_{50}3(8)=8+_{50}24=32,\ 5(8)=8+_{50}4(8)=8+_{50}32=40,\\
&6(8)=8+_{50}5(8)=8+_{50}40=48,\ 7(8)=8+_{50}6(8)=8+_{50}48=6,\cdots,\\
&16(8)=8+_{50}15(8)=8+_{50}20=28,17(8)=8+_{50}16(8)=8+_{50}28=36,\\
&18(8)=8+_{50}17(8)=8+_{50}36=44,19(8)=8+_{50}18(8)=8+_{50}44=2,\cdots,\\
&24(8)=8+_{50}23(8)=8+_{50}34=42,\ 25(8)=8+_{50}24(8)=8+_{50}42=0=e
\end{aligned}$$

Thus 25(8) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(8)=25$.

To determine $O(9)$:

$$\begin{aligned}
&1(9)=9,\ 2(9)=9+_{50}9=18,\ 3(9)=9+_{50}2(9)=9+_{50}18=27,\\
&4(9)=9+_{50}3(9)=9+_{50}27=36,\ 5(9)=9+_{50}4(9)=9+_{50}36=45,\\
&6(9)=9+_{50}5(9)=9+_{50}45=4,\cdots,16(9)=9+_{50}15(9)=9+_{50}35=44,\\
&17(9)=9+_{50}16(9)=9+_{50}44=3,18(9)=9+_{50}17(9)=9+_{50}3=12,\cdots,\\
&26(9)=9+_{50}25(9)=9+_{50}25=34,\ 27(9)=9+_{50}26(9)=9+_{50}34=43,\\
&28(9)=9+_{50}27(9)=9+_{50}43=2,\cdots,38(9)=9+_{50}37(9)=9+_{50}33=42,\\
&39(9)=9+_{50}38(9)=9+_{50}42=1,\ 40(9)=9+_{50}39(9)=9+_{50}1=10,\cdots,\\
&48(9)=9+_{50}47(9)=9+_{50}23=32,\ 49(9)=9+_{50}48(9)=9+_{50}32=41,\\
&50(9)=9+_{50}49(9)=9+_{50}41=0=e
\end{aligned}$$

Thus $50(9) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(9) = 50$.

To determine $O(10)$:

$$1(10) = 10,\ 2(10) = 10 +_{50} 10 = 20,\ 3(10) = 10 +_{50} 2(10) = 10 +_{50} 20 = 30,$$
$$4(10) = 10 +_{50} 3(10) = 10 +_{50} 30 = 40,\ 5(10) = 10 +_{50} 4(10) = 10 +_{50} 40 = 0 = e,$$

Thus $5(10) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(10) = 5$.

To determine $O(11)$:

$$1(11) = 11,\ 2(11) = 11 +_{50} 11 = 22,\ 3(11) = 11 +_{50} 2(11) = 11 +_{50} 22 = 33,$$
$$4(11) = 11 +_{50} 3(11) = 11 +_{50} 33 = 44,\ 5(11) = 11 +_{50} 4(11) = 11 +_{50} 44 = 5,$$
$$6(11) = 11 +_{50} 5(11) = 11 +_{50} 5 = 16, \cdots, 16(11) = 11 +_{50} 15(11) = 11 +_{50} 15 = 26,$$
$$17(11) = 11 +_{50} 16(11) = 11 +_{50} 26 = 37,\ 18(11) = 11 +_{50} 17(11) = 11 +_{50} 37 = 48, \cdots,$$
$$26(11) = 11 +_{50} 25(11) = 11 +_{50} 25 = 36,\ 27(11) = 11 +_{50} 26(11) = 11 +_{50} 36 = 47,$$
$$28(11) = 11 +_{50} 27(11) = 11 +_{50} 47 = 8, \cdots, 38(11) = 11 +_{50} 37(11) = 11 +_{50} 7 = 18,$$
$$39(11) = 11 +_{50} 38(11) = 11 +_{50} 18 = 29,\ 40(11) = 11 +_{50} 39(11) = 11 +_{50} 29 = 40, \cdots,$$
$$48(11) = 11 +_{50} 47(11) = 11 +_{50} 17 = 28,\ 49(11) = 11 +_{50} 48(11) = 11 +_{50} 28 = 39,$$
$$50(11) = 11 +_{50} 49(11) = 11 +_{50} 39 = 0 = e$$

Thus $50(11) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(11) = 50$.

To determine $O(12)$:

$$1(12) = 12,\ 2(12) = 12 +_{50} 12 = 24,\ 3(12) = 12 +_{50} 2(12) = 12 +_{50} 24 = 36,$$
$$4(12) = 12 +_{50} 3(12) = 12 +_{50} 36 = 48,\ 5(12) = 12 +_{50} 4(12) = 12 +_{50} 48 = 10, \cdots,$$
$$16(12) = 12 +_{50} 15(12) = 12 +_{50} 30 = 42,\ 17(12) = 12 +_{50} 16(12) = 12 +_{50} 42 = 4,$$
$$18(12) = 12 +_{50} 17(12) = 12 +_{50} 4 = 16, \cdots, 24(12) = 12 +_{50} 23(12) = 12 +_{50} 26 = 38,$$
$$25(12) = 12 +_{50} 24(12) = 12 +_{50} 38 = 0 = e$$

Thus $25(12) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(12) = 25$.

To determine $O(13)$:

$$1(13) = 13,\ 2(13) = 13 +_{50} 13 = 26,\ 3(13) = 13 +_{50} 2(13) = 13 +_{50} 26 = 39,$$
$$4(13) = 13 +_{50} 3(13) = 13 +_{50} 39 = 2,\ 5(13) = 13 +_{50} 4(13) = 13 +_{50} 2 = 15, \cdots,$$
$$16(13) = 13 +_{50} 15(13) = 13 +_{50} 45 = 8,\ 17(13) = 13 +_{50} 16(13) = 13 +_{50} 8 = 21, \cdots,$$
$$26(13) = 13 +_{50} 25(13) = 13 +_{50} 25 = 38,\ 27(13) = 13 +_{50} 26(13) = 13 +_{50} 38 = 1,$$
$$28(13) = 13 +_{50} 27(13) = 13 +_{50} 1 = 14, \cdots, 38(13) = 13 +_{50} 37(13) = 13 +_{50} 31 = 44,$$
$$39(13) = 13 +_{50} 38(13) = 13 +_{50} 44 = 7, \cdots, 48(13) = 13 +_{50} 47(13) = 13 +_{50} 11 = 24,$$
$$49(13) = 13 +_{50} 48(13) = 13 +_{50} 24 = 37,\ 50(13) = 13 +_{50} 49(13) = 13 +_{50} 37 = 0 = e$$

Thus $50(13) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(13) = 50$.

To determine $O(14)$:

$$1(14) = 14,\ 2(14) = 14 +_{50} 14 = 28,\ 3(14) = 14 +_{50} 2(14) = 14 +_{50} 28 = 42,$$
$$4(14) = 14 +_{50} 3(14) = 14 +_{50} 42 = 6, \cdots, 16(14) = 14 +_{50} 15(14) = 14 +_{50} 10 = 24,$$
$$17(14) = 14 +_{50} 16(14) = 14 +_{50} 24 = 38, \cdots, 18(14) = 14 +_{50} 17(14) = 14 +_{50} 38 = 2,$$
$$24(14) = 14 +_{50} 23(14) = 14 +_{50} 22 = 36,\ 25(14) = 14 +_{50} 24(14) = 14 +_{50} 36 = 0 = e$$

Thus $25(14) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(14) = 25$.

To determine $O(15)$:

$$1(15)=15,\ 2(15)=15+_{50}15=30,\ 3(15)=15+_{50}2(15)=15+_{50}30=45,$$
$$4(15)=15+_{50}3(15)=15+_{50}45=10,\cdots,8(15)=15+_{50}7(15)=15+_{50}5=20,$$
$$9(15)=15+_{50}8(15)=15+_{50}20=35,10(15)=15+_{50}9(15)=15+_{50}5=0=e$$

Thus 10(15) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(15)=10$.

To determine $O(16)$:

$$1(16)=16,\ 2(16)=16+_{50}16=32,\ 3(16)=16+_{50}2(16)=16+_{50}32=48,$$
$$4(16)=16+_{50}3(16)=16+_{50}48=14,\cdots,16(16)=16+_{50}15(16)=16+_{50}40=6,$$
$$17(16)=16+_{50}16(16)=16+_{50}6=22,\cdots,24(16)=16+_{50}23(16)=16+_{50}18=34,$$
$$25(16)=16+_{50}24(16)=16+_{50}34=0=e$$

Thus 25(16) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(16)=25$.

To determine $O(17)$:

$$1(17)=17,\ 2(17)=17+_{50}17=34,\ 3(17)=17+_{50}2(17)=17+_{50}34=1,$$
$$4(17)=17+_{50}3(17)=17+_{50}1=18,\cdots,16(17)=17+_{50}15(17)=17+_{50}5=22,$$
$$17(17)=17+_{50}16(17)=17+_{50}22=39,\cdots,26(17)=17+_{50}25(17)=17+_{50}25=42,$$
$$27(17)=17+_{50}26(17)=17+_{50}42=9,\cdots,38(17)=17+_{50}37(17)=17+_{50}29=46,$$
$$39(17)=17+_{50}38(17)=17+_{50}46=13,\cdots,49(17)=17+_{50}48(17)=17+_{50}16=33,$$
$$50(17)=17+_{50}49(17)=17+_{50}33=0=e$$

Thus 50(17) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(17)=50$.

To determine $O(18)$:

$$1(18)=18,\ 2(18)=18+_{50}18=36,\ 3(18)=18+_{50}2(18)=18+_{50}36=4,$$
$$4(18)=18+_{50}3(18)=18+_{50}4=22,\cdots,16(18)=18+_{50}15(18)=18+_{50}20=38,$$
$$17(18)=18+_{50}16(18)=18+_{50}38=6,\cdots,24(18)=18+_{50}23(18)=18+_{50}14=32,$$
$$25(18)=18+_{50}24(18)=18+_{50}32=0=e$$

Thus 25(18) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(18)=25$.

To determine $O(19)$:

$$1(19)=19,\ 2(19)=19+_{50}19=38,\ 3(19)=19+_{50}2(19)=19+_{50}38=7,\cdots,$$
$$16(19)=19+_{50}15(19)=19+_{50}35=4,\ 17(19)=19+_{50}16(19)=19+_{50}4=23,\cdots,$$
$$26(19)=19+_{50}25(19)=19+_{50}25=44,\ 27(19)=19+_{50}26(19)=19+_{50}44=13,\cdots,$$
$$38(19)=19+_{50}37(19)=19+_{50}3=22,\ 39(19)=19+_{50}38(19)=19+_{50}22=41,\cdots,$$
$$49(19)=19+_{50}48(19)=19+_{50}12=31,\ 50(19)=19+_{50}49(19)=19+_{50}31=0=e$$

Thus 50(19) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(19)=50$.

To determine $O(20)$:

$$1(20)=20,\ 2(20)=20+_{50}20=40,\ 3(20)=20+_{50}2(20)=20+_{50}40=10,\cdots,$$
$$8(20)=20+_{50}7(20)=20+_{50}40=10,\ 9(20)=20+_{50}8(20)=20+_{50}10=30,$$
$$10(20)=20+_{50}9(20)=20+_{50}30=0=e$$

Thus 10(20) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(20)=10$.

To determine $O(21)$:

$$1(21)=21,\ 2(21)=21+_{50}21=42,\ 3(21)=21+_{50}2(21)=21+_{50}42=13,\cdots,$$
$$16(21)=21+_{50}15(21)=21+_{50}15=36,\ 17(21)=21+_{50}16(21)=21+_{50}36=7,\cdots,$$
$$26(21)=21+_{50}25(21)=21+_{50}25=46,\ 27(21)=21+_{50}26(21)=21+_{50}46=17,\cdots,$$
$$38(21)=21+_{50}37(21)=21+_{50}27=48,\ 39(21)=21+_{50}38(21)=21+_{50}48=19,\cdots,$$
$$49(21)=21+_{50}48(21)=21+_{50}8=29,\ 50(21)=21+_{50}49(21)=21+_{50}29=0=e$$

Thus $50(21)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(21)=50$.

To determine $O(22)$:

$$1(22)=22,\ 2(22)=22+_{50}22=44,\ 3(22)=22+_{50}2(22)=22+_{50}44=16,\cdots,$$
$$16(22)=22+_{50}15(22)=22+_{50}30=2,\ 17(22)=22+_{50}16(22)=22+_{50}2=24,\cdots,$$
$$24(22)=22+_{50}23(22)=22+_{50}6=28,\ 25(22)=22+_{50}24(22)=22+_{50}28=0=e$$

Thus $25(22)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(22)=25$.

To determine $O(23)$:

$$1(23)=23,\ 2(23)=23+_{50}23=46,\ 3(23)=23+_{50}2(23)=23+_{50}46=19,\cdots,$$
$$16(23)=23+_{50}15(23)=23+_{50}45=18,\ 17(23)=23+_{50}16(23)=23+_{50}18=41,\cdots,$$
$$26(23)=23+_{50}25(23)=23+_{50}25=48,\ 27(23)=23+_{50}26(23)=23+_{50}48=21,\cdots,$$
$$38(23)=23+_{50}37(23)=23+_{50}1=24,\ 39(23)=23+_{50}38(23)=23+_{50}24=47,\cdots,$$
$$49(23)=23+_{50}48(23)=23+_{50}4=27,\ 50(23)=23+_{50}49(23)=23+_{50}27=0=e$$

Thus $50(23)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(23)=50$.

To determine $O(24)$:

$$1(24)=24,\ 2(24)=24+_{50}24=48,\ 3(24)=24+_{50}2(24)=24+_{50}48=22,\cdots,$$
$$16(24)=24+_{50}15(24)=24+_{50}10=34,\ 17(24)=24+_{50}16(24)=24+_{50}34=8,\cdots,$$
$$24(24)=24+_{50}23(24)=24+_{50}2=26,\ 25(24)=24+_{50}24(24)=24+_{50}26=0=e$$

Thus $25(24)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(24)=25$.

To determine $O(25)$:

$$1(25)=25,\ 2(25)=25+_{50}25=0=e$$

Thus $2(25)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(25)=2$.

To determine $O(26)$:

$$1(26)=26,\ 2(26)=26+_{50}26=2,\cdots,16(26)=26+_{50}15(26)=26+_{50}40=16,$$
$$17(26)=26+_{50}16(26)=26+_{50}16=42,\cdots,24(26)=26+_{50}23(26)=26+_{50}48=24,$$
$$25(26)=26+_{50}24(26)=26+_{50}24=0=e$$

Thus $25(26)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(26)=25$.

To determine $O(27)$:

$$1(27)=27,\ 2(27)=27+_{50}27=4,\cdots,16(27)=27+_{50}15(27)=27+_{50}5=32,$$
$$17(27)=27+_{50}16(27)=27+_{50}32=9,\cdots,26(27)=27+_{50}25(27)=27+_{50}25=2,$$
$$27(27)=27+_{50}26(27)=27+_{50}2=29,\cdots,38(27)=27+_{50}37(27)=27+_{50}49=26,$$
$$39(27)=27+_{50}38(27)=27+_{50}26=3,\cdots,49(27)=27+_{50}48(27)=27+_{50}46=23,$$
$$50(27)=27+_{50}49(27)=27+_{50}23=0=e$$

Thus 50 (27) = e and $n(a) \neq e$ for n < 50. $\therefore O(27) = 50$.

To determine O (28):

$$1(28) = 28,\ 2(28) = 28 +_{50} 28 = 6, \cdots, 16(28) = 28 +_{50} 15(28) = 28 +_{50} 20 = 48,$$
$$17(28) = 28 +_{50} 16(28) = 28 +_{50} 48 = 26, \cdots, 24(28) = 28 +_{50} 23(28) = 28 +_{50} 44 = 22,$$
$$25(28) = 28 +_{50} 24(28) = 28 +_{50} 22 = 0 = e$$

Thus 25(28) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(28) = 25$.

To determine O (29):

$$1(29) = 29,\ 2(29) = 29 +_{50} 29 = 8, \cdots, 16(29) = 29 +_{50} 15(29) = 29 +_{50} 35 = 14,$$
$$17(29) = 29 +_{50} 16(29) = 29 +_{50} 14 = 43, \cdots, 26(29) = 29 +_{50} 25(29) = 29 +_{50} 25 = 4,$$
$$27(29) = 29 +_{50} 26(29) = 29 +_{50} 4 = 33, \cdots, 38(29) = 29 +_{50} 37(29) = 29 +_{50} 23 = 2,$$
$$39(29) = 29 +_{50} 38(29) = 29 +_{50} 2 = 31, \cdots, 49(29) = 29 +_{50} 48(29) = 29 +_{50} 42 = 21,$$
$$50(29) = 29 +_{50} 49(29) = 29 +_{50} 21 = 0 = e$$

Thus 50(29) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(29) = 50$.

To determine O (30):

$$1(30) = 30,\ 2(30) = 30 +_{50} 30 = 10,\ 3(30) = 30 +_{50} 2(30) = 30 +_{50} 10 = 40, \cdots,$$
$$8(30) = 30 +_{50} 7(30) = 30 +_{50} 10 = 40,\ 9(30) = 30 +_{50} 8(30) = 30 +_{50} 40 = 20,$$
$$10(30) = 30 +_{50} 9(30) = 30 +_{50} 20 = 0 = e$$

Thus 10(30) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(30) = 10$.

To determine O (31):

$$1(31) = 31,\ 2(31) = 31 +_{50} 31 = 12, \cdots, 16(31) = 31 +_{50} 15(31) = 31 +_{50} 15 = 46,$$
$$17(31) = 31 +_{50} 16(31) = 31 +_{50} 46 = 27, \cdots, 26(31) = 31 +_{50} 25(31) = 31 +_{50} 25 = 6,$$
$$27(31) = 31 +_{50} 26(31) = 31 +_{50} 6 = 37, \cdots, 38(31) = 31 +_{50} 37(31) = 31 +_{50} 47 = 28,$$
$$39(31) = 31 +_{50} 38(31) = 31 +_{50} 28 = 9, \cdots, 49(31) = 31 +_{50} 48(31) = 31 +_{50} 38 = 19,$$
$$50(31) = 31 +_{50} 49(31) = 31 +_{50} 19 = 0 = e$$

Thus 50(31) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(31) = 50$.

To determine O (32):

$$1(32) = 32,\ 2(32) = 32 +_{50} 32 = 14, \cdots, 16(32) = 32 +_{50} 15(32) = 32 +_{50} 30 = 12,$$
$$17(32) = 32 +_{50} 16(32) = 32 +_{50} 12 = 44, \cdots, 24(32) = 32 +_{50} 23(32) = 32 +_{50} 36 = 18,$$
$$25(32) = 32 +_{50} 24(32) = 32 +_{50} 18 = 0 = e$$

Thus 25(32) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(32) = 25$.

To determine O (33):

$$1(33) = 33,\ 2(33) = 33 +_{50} 33 = 16, \cdots, 16(33) = 33 +_{50} 15(33) = 33 +_{50} 45 = 28,$$
$$17(33) = 33 +_{50} 16(33) = 33 +_{50} 28 = 11, \cdots, 26(33) = 33 +_{50} 25(33) = 33 +_{50} 25 = 8,$$
$$27(33) = 33 +_{50} 26(33) = 33 +_{50} 8 = 41, \cdots, 38(33) = 33 +_{50} 37(33) = 33 +_{50} 21 = 4,$$
$$39(33) = 33 +_{50} 38(33) = 33 +_{50} 4 = 37, \cdots, 49(33) = 33 +_{50} 48(33) = 33 +_{50} 34 = 17,$$
$$50(33) = 33 +_{50} 49(33) = 33 +_{50} 17 = 0 = e$$

Thus 50(33) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(33) = 50$.

To determine O (34):

$$1(34)=34,\ 2(34)=34+_{50}34=18,\cdots,16(34)=34+_{50}15(34)=34+_{50}10=44,$$
$$17(34)=34+_{50}16(34)=34+_{50}44=28,\cdots,24(34)=34+_{50}23(34)=34+_{50}32=16,$$
$$25(34)=34+_{50}24(34)=34+_{50}16=0=e$$

Thus $25(34) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(34)=25$.

To determine $O(35)$:

$$1(35)=35,\ 2(35)=35+_{50}35=20,\ 3(35)=35+_{50}2(35)=35+_{50}20=5,\cdots,$$
$$8(35)=35+_{50}7(35)=35+_{50}45=30,\ 9(35)=35+_{50}8(35)=35+_{50}30=15,$$
$$10(35)=35+_{50}9(35)=35+_{50}15=0=e$$

Thus $10(35) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(35)=10$.

To determine $O(36)$:

$$1(36)=36,\ 2(36)=36+_{50}36=22,\cdots,16(36)=36+_{50}15(36)=36+_{50}40=36,$$
$$17(36)=36+_{50}16(36)=36+_{50}36=12,\ 24(36)=36+_{50}23(36)=36+_{50}28=14,$$
$$25(36)=36+_{50}24(36)=36+_{50}14=0=e$$

Thus $25(36) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(36)=25$.

To determine $O(37)$:

$$1(37)=37,\ 2(37)=37+_{50}37=24,\cdots,16(37)=37+_{50}15(37)=37+_{50}5=42,$$
$$17(37)=37+_{50}16(37)=37+_{50}42=29,\cdots,26(37)=37+_{50}25(37)=37+_{50}25=12,$$
$$27(37)=37+_{50}26(37)=37+_{50}12=49,\cdots,38(37)=37+_{50}37(37)=37+_{50}=19,$$
$$39(37)=37+_{50}38(37)=37+_{50}19=6,\cdots,49(37)=37+_{50}48(37)=37+_{50}26=13,$$
$$50(37)=37+_{50}49(37)=37+_{50}13=0=e$$

Thus $50(37) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(37)=50$.

To determine $O(38)$:

$$1(38)=38,\ 2(38)=38+_{50}38=16,\cdots,16(38)=38+_{50}15(38)=38+_{50}20=8,$$
$$17(38)=38+_{50}16(38)=38+_{50}8=46,\ 24(38)=38+_{50}23(38)=38+_{50}24=12,$$
$$25(38)=38+_{50}24(38)=38+_{50}12=0=e$$

Thus $25(38) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(38)=25$.

To determine $O(39)$:

$$1(39)=39,\ 2(39)=39+_{50}39=28,\cdots,16(39)=39+_{50}15(39)=39+_{50}35=24,$$
$$17(39)=39+_{50}16(39)=39+_{50}24=13,\cdots,26(39)=39+_{50}25(39)=39+_{50}25=14,$$
$$27(39)=39+_{50}26(39)=39+_{50}14=13,\cdots,38(39)=39+_{50}37(39)=39+_{50}43=32,$$
$$39(39)=39+_{50}38(39)=39+_{50}32=21,\cdots,49(39)=39+_{50}48(39)=39+_{50}22=11,$$
$$50(39)=39+_{50}49(39)=39+_{50}11=0=e$$

Thus $50(39) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(39)=50$.

To determine $O(40)$:

$$1(40)=40,\ 2(40)=40+_{50}40=30,\ 3(40)=40+_{50}2(40)=40+_{50}30=20,\cdots,$$
$$8(40)=40+_{50}7(40)=40+_{50}30=20,\ 9(40)=40+_{50}8(40)=40+_{50}20=10,$$
$$10(40)=40+_{50}9(40)=40+_{50}10=0=e$$

Thus $10(40) = e$ and $n(a) \neq e$ for $n < 50$. $\therefore O(40)=10$.

To determine $O(41)$:

$$1(41)=41,\ 2(41)=41+_{50}41=32,\cdots,16(41)=41+_{50}15(41)=41+_{50}15=6,$$
$$17(41)=41+_{50}16(41)=41+_{50}6=47,\cdots,26(41)=41+_{50}25(41)=41+_{50}25=16,$$
$$27(41)=41+_{50}26(41)=41+_{50}16=7,\cdots,38(41)=41+_{50}37(41)=41+_{50}17=8,$$
$$39(41)=41+_{50}38(41)=41+_{50}8=49,\cdots,49(41)=41+_{50}48(41)=41+_{50}18=9,$$
$$50(41)=41+_{50}49(41)=41+_{50}9=0=e$$

Thus $50(41)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(41)=50$.

To determine $O(42)$:

$$1(42)=42,\ 2(42)=42+_{50}42=34,\cdots,16(42)=42+_{50}15(42)=42+_{50}30=22,$$
$$17(42)=42+_{50}16(42)=42+_{50}22=14,\cdots,24(42)=42+_{50}23(42)=42+_{50}16=8,$$
$$25(42)=42+_{50}24(42)=42+_{50}8=0=e$$

Thus $25(42)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(42)=25$.

To determine $O(43)$:

$$1(43)=43,\ 2(43)=43+_{50}43=36,\cdots,16(43)=43+_{50}15(43)=43+_{50}45=38,$$
$$17(43)=43+_{50}16(43)=43+_{50}38=31,\cdots,26(43)=43+_{50}25(43)=43+_{50}25=18,$$
$$27(43)=43+_{50}26(43)=43+_{50}18=11,\cdots,38(43)=43+_{50}37(43)=43+_{50}41=34,$$
$$39(43)=43+_{50}38(43)=43+_{50}34=27,\cdots,49(43)=43+_{50}48(43)=43+_{50}14=7,$$
$$50(43)=43+_{50}49(43)=43+_{50}7=0=e$$

Thus $50(43)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(43)=50$.

To determine $O(44)$:

$$1(44)=44,\ 2(44)=44+_{50}44=38,\cdots,16(44)=44+_{50}15(44)=44+_{50}10=4,$$
$$17(44)=44+_{50}16(44)=44+_{50}4=48,\cdots,24(44)=44+_{50}23(44)=44+_{50}12=6,$$
$$25(44)=44+_{50}24(44)=44+_{50}6=0=e$$

Thus $25(44)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(44)=25$.

To determine $O(45)$:

$$1(45)=45,\ 2(45)=45+_{50}45=40,\ 3(45)=45+_{50}2(45)=45+_{50}40=35,\cdots$$
$$8(45)=45+_{50}7(45)=45+_{50}15=10,\ 9(45)=45+_{50}8(45)=45+_{50}10=5,$$
$$10(45)=45+_{50}9(45)=45+_{50}5=0=e$$

Thus $10(45)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(45)=10$.

To determine $O(46)$:

$$1(46)=46,\ 2(46)=46+_{50}46=42,\cdots,16(46)=46+_{50}15(46)=46+_{50}40=16,$$
$$17(46)=46+_{50}16(46)=46+_{50}16=12,\cdots,24(46)=46+_{50}23(46)=46+_{50}8=4,$$
$$25(46)=46+_{50}24(46)=46+_{50}4=0=e$$

Thus $25(46)=e$ and $n(a)\neq e$ for $n<50$. $\therefore O(46)=25$.

To determine $O(47)$:

$$1(47)=47,\ 2(47)=47+_{50}47=44,\cdots,16(47)=47+_{50}15(47)=47+_{50}5=2,$$
$$17(47)=47+_{50}16(47)=47+_{50}2=49,\cdots,26(47)=47+_{50}25(47)=47+_{50}25=22,$$
$$27(47)=47+_{50}26(47)=47+_{50}22=19,\cdots,38(47)=47+_{50}37(47)=47+_{50}39=36,$$

$39(47)=47+_{50}38(47)=47+_{50}36=33,\cdots,49(47)=47+_{50}48(47)=47+_{50}6=3,$
$50(47)=47+_{50}49(47)=47+_{50}3=0=e$

Thus 50(47) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(47)=50$.

To determine $O(48)$:

$1(48)=48, 2(48)=48+_{50}48=46,\cdots,16(48)=48+_{50}15(48)=48+_{50}20=28,$
$17(48)=48+_{50}16(48)=48+_{50}28=26,\cdots,24(48)=48+_{50}23(48)=48+_{50}4=2,$
$25(48)=48+_{50}24(48)=48+_{50}2=0=e$

Thus 25(48) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(48)=25$.

To determine $O(49)$:

$1(49)=49, 2(49)=49+_{50}49=48,\cdots,16(49)=49+_{50}15(49)=49+_{50}35=34,$
$17(49)=49+_{50}16(49)=49+_{50}34=33,\cdots,26(49)=49+_{50}25(49)=49+_{50}25=24,$
$27(49)=49+_{50}26(49)=49+_{50}24=23,\cdots,38(49)=49+_{50}37(49)=49+_{50}13=12,$
$39(49)=49+_{50}38(49)=49+_{50}12=11,\cdots,49(49)=49+_{50}48(49)=49+_{50}2=1,$
$50(49)=49+_{50}49(49)=49+_{50}1=0=e$

Thus 50(49) = e and $n(a) \neq e$ for $n < 50$. $\therefore O(49)=50$.

The order of every element of the group {0, 1, 2, 3, …, 49}, the composition being addition modulo 50 in the following figure as follows:

$+_{50}$ ($na = e$)	1	2	…	5	…	10	…	25	…	50
1	1	2	…	5	…	10	…	25	…	0
2	2	4	…	10	…	20	…	0	×	×
3	3	6	…	15	…	30	…	25	…	0
4	4	8	…	20	…	40	…	0	×	×
5	5	10	…	25	…	0	×	×	×	×
6	6	12	…	30	…	10	…	0	×	×
7	6	14	…	35	…	20	…	25	…	0
8	8	16	…	40	…	30	…	0	×	×
9	9	18	…	45	…	40	…	25	…	0
10	10	20	…	0	×	×	×	×	×	×
11	11	22	…	5	…	10	…	25	…	0
12	12	24	…	10	…	20	…	0	×	×
13	13	26	…	15	…	30	…	25	…	0
14	14	28	…	20	…	40	…	0	×	×
15	15	30	…	25	…	0	×	×	×	×
16	16	32	…	30	…	10	…	0	×	×
17	17	34	…	35	…	20	…	25	…	0
18	18	36	…	40	…	30	…	0	×	×

Continued

19	19	38	…	45	…	…	…	25	…	0
20	20	40	…	0	×	×	×	×	×	×
21	21	42	…	5	…	10	…	25	…	0
22	22	44	…	10	…	20	…	0	×	×
23	23	46	…	15	…	30	…	25	…	0
24	24	48	…	20	…	40	…	0	×	×
25	25	0	×	×	×	×	×	×	×	×
26	26	2	…	30	…	10	…	0	×	×
27	27	4	…	35	…	20	…	25	…	0
28	28	6	…	40	…	30	…	0	×	×
29	29	8	…	45	…	40	…	25	…	0
30	30	10	…	0	×	×	×	×	×	×
31	31	12	…	5	…	10	…	25	…	0
32	32	14	…	10	…	20	…	0	×	×
33	33	16	…	15	…	30	…	25	…	0
34	34	18	…	20	…	40	…	0	×	×
35	35	20	…	25	…	0	×	×	×	×
36	36	22	…	30	…	10	…	0	×	×
37	37	24	…	35	…	20	…	25	…	0
38	38	26	…	40	…	30	…	0	×	×
39	39	28	…	45	…	40	….	25	….	0
40	40	30	…	0	×	×	….	×	….	×
41	41	32	…	5	…	10	….	25	….	0
42	42	34	…	10	…	20	….	0	….	×
43	43	36	…	15	…	30	….	25	….	0
44	44	38	…	20	…	40	….	0	….	×
45	45	40	…	25	…	0	….	×	….	×
46	46	42	…	30	…	10	….	0	….	×
47	47	44	…	35	…	20	….	25	….	0
48	48	46	…	40	…	30	….	0	….	×
49	49	48	…	45	…	40	….	25	….	0

4.2. ([32]-[37]) Find the Order of Every Element in the Multiplication Group $G=\{a,a^2,a^3,a^4,\cdots,a^{50}=e\}$

Solution:

The identity element of the given group is $a^{50}=e \Rightarrow o(a)=50 \therefore o(a)=50$.

We know that $o\left(a^m\right)=\frac{\lambda}{m}$, where λ = l.c.m of m and n.

To determine $o(a^2)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 2 is 50 $\therefore o\left(a^2\right)=\frac{50}{2}=25 \Rightarrow o\left(a^2\right)=25$

To determine $o(a^3)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 3 is 150 $\therefore o\left(a^3\right)=\frac{150}{3}=50 \Rightarrow o\left(a^3\right)=50$

To determine $o(a^4)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 4 is 100 $\therefore o\left(a^4\right)=\frac{100}{4}=25 \Rightarrow o\left(a^4\right)=25$

To determine $o(a^5)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 5 is 50 $\therefore o\left(a^5\right)=\frac{50}{5}=10 \Rightarrow o\left(a^5\right)=10$

To determine $o(a^6)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 6 is 150 $\therefore o\left(a^6\right)=\frac{150}{6}=25 \Rightarrow o\left(a^6\right)=25$

To determine $o(a^7)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 7 is 350 $\therefore o\left(a^7\right)=\frac{350}{7}=50 \Rightarrow o\left(a^7\right)=50$

To determine $o(a^8)$

Here, $o(a^{50}) = e$, l.c.m of 50 and 8 is 200 $\therefore o\left(a^8\right)=\frac{200}{8}=50 \Rightarrow o\left(a^8\right)=25$

To determine $o(a^9)$

Here, o(a^50)=e, l.c.m of 50 and 9 is 450 $\therefore o\left(a^9\right)=\frac{450}{9}=50 \Rightarrow o\left(a^9\right)=50$

To determine $o(a^{10})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 10 is 50 $\therefore o\left(a^{10}\right)=\frac{50}{10}=5 \Rightarrow o\left(a^{10}\right)=5$

To determine $o(a^{11})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 11 is 550 $\therefore o\left(a^{11}\right)=\frac{550}{11}=50 \Rightarrow o\left(a^{11}\right)=50$

To determine $o(a^{12})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 12 is 300 $\therefore o\left(a^{12}\right)=\frac{300}{12}=25 \Rightarrow o\left(a^{12}\right)=25$

To determine $o(a^{13})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 13 is 650 $\therefore o\left(a^{13}\right)=\frac{650}{13}=50 \Rightarrow o\left(a^{13}\right)=50$

To determine $o(a^{14})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 14 is 350 $\therefore o\left(a^{14}\right)=\frac{350}{14}=25 \Rightarrow o\left(a^{14}\right)=25$

To determine $o(a^{15})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 15 is 150 $\therefore o\left(a^{15}\right)=\frac{150}{15}=10 \Rightarrow o\left(a^{15}\right)=10$

To determine $o(a^{16})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 16 is 400 $\therefore o\left(a^{16}\right) = \dfrac{400}{16} = 25 \Rightarrow o\left(a^{16}\right) = 25$

To determine $o(a^{17})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 17 is 850 $\therefore o\left(a^{17}\right) = \dfrac{850}{17} = 50 \Rightarrow o\left(a^{17}\right) = 50$

To determine $o(a^{19})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 18 is 450 $\therefore o\left(a^{18}\right) = \dfrac{450}{18} = 25 \Rightarrow o\left(a^{18}\right) = 25$

To determine $o(a^{19})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 19 is 950 $\therefore o\left(a^{19}\right) = \dfrac{950}{19} = 50 \Rightarrow o\left(a^{19}\right) = 50$

To determine $o(a^{20})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 20 is 20 $\therefore o\left(a^{20}\right) = \dfrac{500}{20} = 25 \Rightarrow o\left(a^{20}\right) = 25$

To determine $o(a^{21})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 21 is 1050 $\therefore o\left(a^{21}\right) = \dfrac{1050}{21} = 25 \Rightarrow o\left(a^{21}\right) = 50$

To determine $o(a^{22})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 22 is 550 $\therefore o\left(a^{22}\right) = \dfrac{150}{22} = 25 \Rightarrow o\left(a^{22}\right) = 25$

To determine $o(a^{23})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 23 is 1150 $\therefore o\left(a^{23}\right) = \dfrac{1150}{23} = 50 \Rightarrow o\left(a^{23}\right) = 50$

To determine $o(a^{24})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 24 is 600 $\therefore o\left(a^{24}\right) = \dfrac{600}{24} = 25 \Rightarrow o\left(a^{24}\right) = 25$

To determine $o(a^{25})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 25 is 50 $\therefore o\left(a^{25}\right) = \dfrac{50}{25} = 2 \Rightarrow o\left(a^{25}\right) = 2$

To determine $o(a^{26})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 26 is 650 $\therefore o\left(a^{26}\right) = \dfrac{650}{26} = 25 \Rightarrow o\left(a^{26}\right) = 25$

To determine $o(a^{27})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 27 is 1350 $\therefore o\left(a^{27}\right) = \dfrac{1350}{27} = 50 \Rightarrow o\left(a^{27}\right) = 50$

To determine $o(a^{28})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 28 is 700 $\therefore o\left(a^{28}\right) = \dfrac{700}{28} = 25 \Rightarrow o\left(a^{28}\right) = 25$

To determine $o(a^{29})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 29 is 1450 $\therefore o\left(a^{29}\right) = \dfrac{1450}{29} = 50 \Rightarrow o\left(a^{29}\right) = 50$

To determine $o(a^{30})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 30 is 150 $\therefore o\left(a^{30}\right) = \dfrac{150}{30} = 5 \Rightarrow o\left(a^{25}\right) = 5$

To determine $o(a^{31})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 31 is 1550 $\therefore o\left(a^{31}\right)=\frac{1550}{31}=50 \Rightarrow o\left(a^{31}\right)=50$

To determine $o(a^{32})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 32 is 800 $\therefore o\left(a^{32}\right)=\frac{800}{32}=25 \Rightarrow o\left(a^{32}\right)=25$

To determine $o(a^{33})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 33 is 1650 $\therefore o\left(a^{33}\right)=\frac{1650}{33}=50 \Rightarrow o\left(a^{33}\right)=50$

To determine $o(a^{34})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 34 is 850 $\therefore o\left(a^{34}\right)=\frac{850}{34}=25 \Rightarrow o\left(a^{34}\right)=25$

To determine $o(a^{35})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 35 is 350 $\therefore o\left(a^{35}\right)=\frac{1550}{35}=10 \Rightarrow o\left(a^{35}\right)=10$

To determine $o(a^{36})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 36 is 900 $\therefore o\left(a^{36}\right)=\frac{900}{36}=25 \Rightarrow o\left(a^{36}\right)=25$

To determine $o(a^{37})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 31 is 1850 $\therefore o\left(a^{37}\right)=\frac{1850}{37}=50 \Rightarrow o\left(a^{37}\right)=50$

To determine $o(a^{38})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 38 is 950 $\therefore o\left(a^{38}\right)=\frac{950}{38}=25 \Rightarrow o\left(a^{38}\right)=25$

To determine $o(a^{39})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 39 is 1950 $\therefore o\left(a^{39}\right)=\frac{1950}{39}=50 \Rightarrow o\left(a^{39}\right)=50$

To determine $o(a^{40})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 40 is 200 $\therefore o\left(a^{40}\right)=\frac{200}{40}=5 \Rightarrow o\left(a^{40}\right)=5$

To determine $o(a^{41})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 41 is 2050 $\therefore o\left(a^{41}\right)=\frac{2050}{41}=50 \Rightarrow o\left(a^{41}\right)=50$

To determine $o(a^{42})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 42 is 1050 $\therefore o\left(a^{42}\right)=\frac{1050}{42}=25 \Rightarrow o\left(a^{42}\right)=5$

To determine $o(a^{43})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 43 is 2150 $\therefore o\left(a^{43}\right)=\frac{2150}{43}=50 \Rightarrow o\left(a^{43}\right)=50$

To determine $o(a^{44})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 44 is 1100 $\therefore o\left(a^{44}\right)=\frac{1100}{44}=25 \Rightarrow o\left(a^{44}\right)=25$

To determine $o(a^{45})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 45 is 450 $\therefore o\left(a^{45}\right)=\frac{450}{45}=10 \Rightarrow o\left(a^{45}\right)=10$

To determine $o(a^{46})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 46 is 1150 $\therefore o\left(a^{46}\right)=\frac{1150}{46}=25 \Rightarrow o\left(a^{46}\right)=25$

To determine $o(a^{47})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 47 is 2350 $\therefore o\left(a^{47}\right)=\frac{2350}{47}=50 \Rightarrow o\left(a^{47}\right)=50$

To determine $o(a^{48})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 48 is 1200 $\therefore o\left(a^{48}\right)=\frac{1200}{48}=25 \Rightarrow o\left(a^{48}\right)=25$

To determine $o(a^{49})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 49 is 2450 $\therefore o\left(a^{49}\right)=\frac{2450}{49}=50 \Rightarrow o\left(a^{49}\right)=50$

To determine $o(a^{50})$

Here, $o(a^{50}) = e$, l.c.m of 50 and 50 is 50 $\therefore o\left(a^{50}\right)=\frac{50}{50}=1 \Rightarrow o\left(a^{50}\right)=1$

Remarkable:

1) The addition modulo m, denoted by $+_m$ on the set Z as follows:

$$a +_m b = r\,,\quad 0 \le r < m$$

where r is the least non-negative remainder when ordinary of (a + b) is divided by m;

2) In order to find out the order of an element $a^m \in G$ in which $a^n = e$ = identity element, first find out least common multiple (*i.e.* (l.c.m)) = λ) of m and n. Then $o\left(a^m\right)=\frac{\lambda}{m}$.

5. Conclusion

We hope that this work will be useful for group theory related to order of element of a group. Our result is the order of every element of a group in different types of the higher order group. This result has found extensive use in statistics, information theory and geometrics etc. Then all expected results in this paper will help us to understand better solution of complicated order.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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