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On Standard Concepts Using *ii*-Open Sets

Beyda S. Abdullah, Amir A. Mohammed

Department of Mathematics, College of Education for pure Sciences, University of Mosul, Mosul, Iraq
Email: baedaa419@gmail.com

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Abstract

Following Caldas in [1] we introduce and study topological properties of *ii*-derived, *ii*-border, *ii*-frontier, and *ii*-exterior of a set using the concept of *ii*-open sets. Moreover, we prove some further properties of the well-known notions of *ii*-closure and *ii*-interior. We also study a new decomposition of *ii*-continuous functions. Finally, we introduce and study some of the separation axioms specifically T_{0ii}, T_{1ii}.

Subject Areas

Mathematical Analysis

Keywords

α-Open Set, *ii*-Open Set, Separation Axioms

1. Introduction

The notion of α-open set was introduced by Njastad in [2]. Caldas in [1] introduced and studied topological properties of α-derived, α-border, α-frontier, α-exterior of a set by using the concept of α-open sets. In this paper, we introduce and study the same above concepts by using *ii*-open sets. A subset A of X is called *ii*-open set [3] if there exists an open set G in the topology τ of X, such that: $G \neq \phi, X$, $A \subseteq CL(A \cap G)$ and $Int(A) = G$, the complement of an *ii-open* set is an *ii*-closed set. We denote the family of *ii*-open sets in (X, τ) by τ^{ii}. It is shown in [4] that each of $\tau \subset \tau^{ii}$ and τ^{ii} is a topology on X. This property allows us to prove similar properties of α-open set. Also, we define *ii*-continuous functions and we study the relation between this type of function and continuous, semi-continuous, α-continuous and *i*-continuous functions. Finally, we introduce a new type of separation axioms namely T_{0ii}, T_{1ii}. We prove similar properties and characterizations of T_0 and T_1.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (simply *X* and *Y*) always mean topological spaces. For a subset *A* of a space *X*, *Cl* (*A*) and *Int* (*A*) denote the closure of *A* and the interior of *A* respectively. We recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset *A* of a space *X* is called

1) Semi-open set [5] if $A \subseteq CL(Int(A))$.

2) *α*-open set [2] if $A \subseteq Int(CL(Int(A)))$.

3) *i*-open set [3] if there exist an open set *G* in the topology τ of *X*, such that

i) $G \neq \phi, X$

ii) $A \subseteq CL(A \cap G)$

The complement of an *i*-open set is an *i*-closed set.

4) *ii*-open set [4] if there exist an open set *G* in the topology τ of *X*, such that

i) $G \neq \phi, X$

ii) $A \subseteq CL(A \cap G)$

iii) $Int(A) = G$

The complement of an *ii-open* set is an *ii*-closed set.

5) *int*-open set [4] if there exist an open set *G* in the topology τ of *X* and $G \neq \phi, X$ such that $Int(A) = G$. The complement of *int*-open set is *int*-closed set.

6) *αo* (*X*), *So* (*X*), *io* (*X*), *iio* (*X*), *into* (*X*) are family of *α*-open, semi-open, *i*-open, *ii-open*, *int*-open sets respectively.

7) τ^{i}, τ^{ii} denote the family of all *i*-open sets and *ii*-open sets respectively.

Definition 2.2. [3] A topological space *X* is called

1) T_{0i} if *a*, *b* are to distinct points in *X*, there exist an *i*-open set *U* such that either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$.

2) T_{1i} if $a, b \in X$ and $a \neq b$, there exist *i*-open sets *U*, *V* containing *a*, *b* respectively, such that $b \notin U$ and $a \notin V$.

Definition 2.3. A function $f:(X,\tau) \to (Y,\sigma)$ is called

1) Continuous [6], if $f^{-1}(G)$ is open in (X,τ) for every open set *G* of (Y,σ).

2) *α*-continuous [6], if $f^{-1}(G)$ is *α*-open in (X,τ) for every open set *G* of (Y,σ).

3) Semi-Continuous [5], if $f^{-1}(G)$ is semi-open in (X,τ) for every open set *G* of (Y,σ).

4) *i*-Continuous [3], if $f^{-1}(G)$ is *i*-open in (X,τ) for every open set *G* of (Y,σ).

3. Applications of *ii*-Open Sets

Definition 3.1. Let *A* be a subset of a topological space (X,τ). A derived set of *A* denoted by *D*(*A*) is defined as follows:

$D(A) = \{x \in X : (G \cap A) \setminus \{x\} \neq \phi, \forall x \in G\}$. A point $x \in X$ is said to be *ii*-limit

point of A if it satisfies the following assertion:
$(\forall G\in\tau^{ii})(x\in G\Rightarrow(G\cap A)\setminus\{x\}\neq\phi)$. The set of all *ii*-limit points of A is called the *ii*-derived set of A and is denoted by $D_{ii}(A)$. Note that $x\in X$ is not *ii*-limit point of A if and only if there exist an *ii-open* set G in X such that ($x\in G$ and $(G\cap A)\setminus\{x\}=\phi$).

Theorem 3.2. For subsets A, B of a space X, the following statements hold:

1) $D_{ii}(A)\subset D(A)$
2) If $A\subseteq B$, then $D_{ii}(A)\subseteq D_{ii}(B)$
3) $D_{ii}(A)\cup D_{ii}(B)\subset D_{ii}(A\cup B)$ and $D_{ii}(A\cap B)\subset D_{ii}(A)\cap D_{ii}(B)$
4) $D_{ii}(D_{ii}(A))\setminus A\subset D_{ii}(A)$
5) $D_{ii}(A\cup D_{ii}(A))\subset A\cup D_{ii}(A)$

Proof. 1) Since every open set is *ii*-open [4], it follows that $D_{ii}(A)\subset D(A)$.

2) Let $x\in D_{ii}(A)$. Then G is *ii-open* set containing x such that

$$(A\cap G)\setminus\{x\}\neq\phi \tag{3.1}$$

Since $A\subseteq B$ we get $(A\cap G)\subseteq(B\cap G)$, it implies that $(A\cap G)\setminus\{X\}\subseteq(B\cap G)\setminus\{X\}\neq\phi$, from (3.1) we get $(B\cap G)\setminus\{X\}\neq\phi$.

Hence, $x\in D_{ii}(B)$. Therefore $D_{ii}(A)\subseteq D_{ii}(B)$.

3) Since $A\subseteq A\cup B$ and $B\subseteq A\cup B$, from (2) we get $D_{ii}(A)\subseteq D_{ii}(A\cup B)$, $D_{ii}(B)\subseteq D_{ii}(A\cup B)$.

This implies to $D_{ii}(A)\cup D_{ii}(B)\subset D_{ii}(A\cup B)$.

We shall prove that $D_{ii}(A\cap B)\subset D_{ii}(A)\cap D_{ii}(B)$. Since $A\cap B\subseteq A$, $A\cap B\subseteq B$, from (2) we get $D_{ii}(A\cap B)\subset D_{ii}(A)$ and $D_{ii}(A\cap B)\subset D_{ii}(B)$. Therefore $D_{ii}(A\cap B)\subset D_{ii}(A)\cap D_{ii}(B)$.

4) If $x\in D_{ii}(D_{ii}(A))\setminus A$ and G is an *ii*-open set containing x, then $G\cap(D_{ii}(A)\setminus\{X\})\neq\phi$. Let $y\in G\cap(D_{ii}(A)\setminus\{x\})$. Then, since $y\in D_{ii}(A)$ and $y\in G$, $G\cap(A\setminus\{Y\})\neq\phi$. Let $z\in G\cap(A\setminus\{Y\})$. Then, $z\neq x$ for $z\in A$ and $x\notin A$. Hence, $G\cap(A\setminus\{X\})\neq\phi$. Therefore, $x\in D_{ii}(A)$.

5) Let $x\in D_{ii}(A\cup D_{ii}(A))$. If $x\in A$, the result is obvious. So, let, $x\in D_{ii}(A\cup D_{ii}(A))\setminus A$ then for *ii-open* set G containing x, $(G\cap(A\cup D_{ii}(A)\setminus\{x\}))\neq\phi$. Thus, $G\cap(A\setminus\{X\})\neq\phi$ or $G\cap(D_{ii}(A)\setminus\{x\})\neq\phi$.

Now, it follows similarly from (4) that $G\cap(A\setminus\{X\})\neq\phi$. Hence, $x\in D_{ii}(A)$.

Therefore, in any case, $D_{ii}(A\cup D_{ii}(A))\subset A\cup D_{ii}(A)$.

In general, the converse of (1) may not true and the equality does not hold in (3) of theorem 3.2.

Example 3.3. Let $X=\{a,b,c\}$ and $\tau=\{\phi,X,\{b\}\}$. Thus, $iio(x)=\{\phi,X,\{b\},\{a,b\},\{b,c\}\}$. Take the following:

1) $A=\{c\}$. Then, $D(A)=\{a,b\}$ and $D_{ii}(A)=\phi$. Hence, $D(A)\not\subset D_{ii}(A)$;
2) $C=\{a,b\}$ and $E=\{c\}$. Then $D_{ii}(c)=\{a,c\}$ and $D_{ii}(E)=\phi$. Hence $D_{ii}(C\cup E)\neq D_{ii}(C)\cup D_{ii}(E)$.

Theorem 3.4. For any subset A of a space X, $CL_{ii}(A)=A\cup D_{ii}(A)$.

Proof. Since $D_{ii}(A)\subset CL_{ii}(A)$, $A\cup D_{ii}(A)\subset CL_{ii}(A)$. On the other hand, Let $x\in CL_{ii}(A)$. If $x\in A$, then the proof is complete. If $x\notin A$, each *ii-open* set

G containing x intersects A at a point distinct from x; so $x \in D_{ii}(A)$. Thus, $CL_{ii}(A) \subset A \cup D_{ii}(A)$, which completes the proof.

Definition 3.5. A point $x \in X$ is said to be *ii*-interior point of *A* if there exist an *ii-open* set G containing x such that $G \subset A$. The set of all *ii*-interior points of *A* is said to be *ii*-interior of *A* and denoted by $Int_{ii}(A)$.

Theorem 3.6. For subset *A*, *B* of a space *X*, the following statements are true:

1) $Int_{ii}(A)$ is the union of all ii-open subset of *A*
2) *A* is ii-open if and only if $A = Int_{ii}(A)$
3) $Int_{ii}(Int_{ii}(A)) = Int_{ii}(A)$
4) $Int_{ii}(A) = A \setminus D_{ii}(X \setminus A)$
5) $X \setminus Int_{ii}(A) = CL_{ii}(X \setminus A)$
6) $X \setminus CL_{ii}(A) = Int_{ii}(X \setminus A)$
7) If $A \subseteq B$, then $Int_{ii}(A) \subseteq Int_{ii}(B)$
8) $Int_{ii}(A) \cup Int_{ii}(B) \subseteq Int_{ii}(A \cup B)$
9) $Int_{ii}(A) \cap Int_{ii}(B) \supseteq Int_{ii}(A \cap B)$

Proof. 1) Let $\{G_{ii} \setminus ii \in \wedge\}$ be a collection of all *ii*-open subsets of *A*. If $x \in Int_{ii}(A)$, then there exist $j \in \wedge$ such that $x \in G_j \subseteq A$. Hence $x \in \bigcup_{ii \in \wedge} G_{ii}$, and so $Int_{ii}(A) \subseteq \bigcup_{ii \in \wedge} G_{ii}$. On the other hand, if $y \in \bigcup_{ii \in \wedge} G_{ii}$, then $y \in G_k \subseteq A$ for some $k \in \wedge$. Thus $y \in Int_{ii}(A)$, and $\bigcup_{ii \in \wedge} G_{ii} \subseteq Int_{ii}(A)$. Accordingly, $\bigcup_{ii \in \wedge} G_{ii} \subseteq Int_{ii}(A)$.

2) Straightforward.

3) It follows from (1) and (2).

4) If $x \in A \setminus D_{ii}(X \setminus A)$, then $x \notin D_{ii}(X \setminus A)$ and so there exist an *ii-open* set G containing x such that $G \cap (X \setminus A) = \phi$. Thus, $x \in G \subset A$ and hence $x \in Int_{ii}(A)$. This shows that $A \setminus D_{ii}(X \setminus A) \subset Int_{ii}(A)$. Now let $x \in Int_{ii}(A)$. Since $Int_{ii}(A) \in \tau^{ii}$ and $Int_{ii}(A) \cap (X \setminus A) = \phi$. We have $x \notin D_{ii}(X \setminus A)$. Therefore, $Int_{ii}(A) = A \setminus D_{ii}(X \setminus A)$.

5) Using (4) and Theorem (3.4), we have

$$X \setminus Int_{ii}(A) = X \setminus (A \setminus D_{ii}(X \setminus A)) = (X \setminus A) \cup D_{ii}(X \setminus A) = CL_{ii}(X \setminus A).$$

6) Using (4) and Theorem (3.4), we get.

$$Int_{ii}(X \setminus A) = (X \setminus A) \setminus D_{ii}(A) = X \setminus (A \cup D_{ii}(A)) = X \setminus CL_{ii}(A)$$

7) Since $A \subseteq B$ and $Int_{ii}(A) \subseteq A$, $Int_{ii}(B) \subseteq B$, we get $Int_{ii}(A) \subseteq Int_{ii}(B)$.

8) Since $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$, from (7) we get $Int_{ii}(A) \subseteq Int_{ii}(A \cup B)$, $Int_{ii}(B) \subseteq Int_{ii}(A \cup B)$. Therefore $Int_{ii}(A) \cup Int_{ii}(B) \subseteq Int_{ii}(A \cup B)$.

9) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (7) we get $Int_{ii}(A \cap B) \subseteq Int_{ii}(A)$, $Int_{ii}(A \cap B) \subseteq Int_{ii}(B)$. Therefore $Int_{ii}(A \cap B) \subseteq Int_{ii}(A) \cap (B)$.

Definition 3.7. $b_{ii}(A) = A \setminus Int_{ii}(A)$ is said to be the *ii*-border of *A*.

Theorem 3.8. For a subset *A* of a space *X*, the following statements hold:

1) $b_{ii}(A) \subset b(A)$ where $b(A)$ denotes the border of *A*
2) $Int_{ii}(A) \cup b_{ii}(A) = A$

3) $Int_{ii}(A)\cap b_{ii}(A)=\phi$
4) $b_{ii}(A)=\phi$ if and only if A is *ii*-open set
5) $b_{ii}(Int_{ii}(A))=\phi$
6) $Int_{ii}(b_{ii}(A))=\phi$
7) $b_{ii}(b_{ii}(A))=b_{ii}(A)$
8) $b_{ii}(A)=A\cap CL_{ii}(X\setminus A)$
9) $b_{ii}(A)=A\cap D_{ii}(X\setminus A)$

Proof.

1) Since $Int(A)\subset Int_{ii}(A)$, we have $b_{ii}(A)=A\setminus Int_{ii}(A)\subseteq A\setminus Int(A)=b(A)$.

2) and (3). Straightforward.

4) Since $Int_{ii}(A)\subseteq A$, it follows from Theorem 3.6 (2). That A is *ii*-open $\Leftrightarrow$ $A=Int_{ii}(A)\Leftrightarrow b_{ii}(A)=A\setminus Int_{ii}(A)=\phi$.

5) Since $Int_{ii}(A)$ is *ii*-open, it follows from (4) that $b_{ii}(Int_{ii}(A))=\phi$.

6) If $x\in Int_{ii}(b_{ii}(A))$, then $x\in b_{ii}(A)$. On the other hand, since $b_{ii}(A)\subset A$, $x\in Int_{ii}(b_{ii}(A))\subset Int_{ii}(A)$. Hence, $x\in Int_{ii}(A)\cap b_{ii}(A)$.

Which contradicts (3). Thus $Int_{ii}(b_{ii}(A))=\phi$.

7) Using (6), we get $b_{ii}(b_{ii}(A))=b_{ii}(A)\setminus Int_{ii}(b_{ii}(A))=b_{ii}(A)$.

8) Using Theorem 3.6 (6), we have $b_{ii}(A)=A\setminus Int_{ii}(A)=A\setminus(X\setminus CL_{ii}(X\setminus A))=A\cap CL_{ii}(X\setminus A)$

9) Applying (8) and the Theorem (3.4), we have $b_{ii}(A)=A\cap CL_{ii}(X\setminus A)=A\cap((X\setminus A)\cup D_{ii}(X\setminus A))=A\cap D_{ii}(X\setminus A)$.

Example 3.9. Consider the topological space (X,τ) given in Example (3.3). If $A=\{a,b\}$, then $b_{ii}(A)=\phi$ and $b(A)=\{a\}$. Hence, $b(A)\not\subset b_{ii}(A)$, that is, in general, the converse Theorem 3.9 (1) may not be true.

Definition 3.10. $Fr_{ii}(A)=CL_{ii}(A)\setminus Int_{ii}(A)$ is said to be the *ii*-frontier of *A*.

Theorem 3.11. For a subset *A* of a space *X*, the following statements hold:

1) $Fr_{ii}(A)\subset Fr(A)$ where $Fr(A)$ denotes the frontier of A
2) $CL_{ii}(A)=Int_{ii}(A)\cup Fr_{ii}(A)$
3) $Int_{ii}(A)\cap Fr_{ii}(A)=\phi$
4) $b_{ii}(A)\subset Fr_{ii}(A)$
5) $Fr_{ii}(A)=b_{ii}(A)\cup D_{ii}(A)$
6) $Fr_{ii}(A)=D_{ii}(A)$ if and only if A is *ii*-open set
7) $Fr_{ii}(A)=CL_{ii}(A)\cap CL_{ii}(X\setminus A)$
8) $Fr_{ii}(A)=Fr_{ii}(X\setminus A)$
9) $Fr_{ii}(A)$ is *ii*-closed
10) $Fr_{ii}(Fr_{ii}(A))\subset Fr_{ii}(A)$
11) $Fr_{ii}(Int_{ii}(A))\subset Fr_{ii}(A)$
12) $Fr_{ii}(CL_{ii}(A))\subset Fr_{ii}(A)$
13) $Int_{ii}(A)=A\setminus Fr_{ii}(A)$

Proof.

1) Since $CL_{ii}(A)\subseteq CL(A)$ and $Int(A)\subseteq Int_{ii}(A)$, it follows that $Fr_{ii}(A)=CL_{ii}(A)\setminus Int_{ii}(A)\subseteq CL(A)\setminus Int_{ii}(A)\subseteq CL(A)\setminus Int(A)\subseteq Fr(A)$.

2) $Int_{ii}(A)\cup Fr_{ii}(A)=Int_{ii}(A)\cup(CL_{ii}(A)\setminus Int_{ii}(A))=CL_{ii}(A)$.

3) $Int_{ii}(A)\cap Fr_{ii}(A)=Int_{ii}(A)\cap(CL_{ii}(A)\setminus Int_{ii}(A))=\phi$.

4) Since $A\subseteq CL_{ii}(A)$, we have $b_{ii}(A)=A\setminus Int_{ii}(A)\subseteq CL_{ii}(A)\setminus Int_{ii}(A)=Fr_{ii}(A)$.

5) Since $Int_{ii}(A)\cup Fr_{ii}(A)=Int_{ii}(A)\cup b_{ii}(A)\cup D_{ii}(A)$, $Fr_{ii}(A)=b_{ii}(A)\cup D_{ii}(A)$.

6) Assume that A is *ii*-open. Then $Fr_{ii}(A)=b_{ii}(A)\cup D_{ii}(A)\setminus Int_{ii}(A)=\phi\cup(D_{ii}(A)\setminus A)=D_{ii}(A)\setminus A=b_{ii}(X\setminus A)$, by using (5), Theorem 3.6 (2), Theorem 3.8 (4) and Theorem 3.8 (9).

Conversely, suppose that $Fr_{ii}(A)=b_{ii}(X\setminus A)$. Then $\phi=Fr_{ii}(A)\setminus b_{ii}(X\setminus A)=(CL_{ii}(A)\setminus Int_{ii}(A))\setminus(X\setminus A)\setminus Int_{ii}(X\setminus A)=A\setminus Int_{ii}(A)$. by using (4) and (5) of Theorem 3.6, and so $A\subseteq Int_{ii}(A)$. Since $Int_{ii}(A)\subseteq A$ in general, it follows that $Int_{ii}(A)=A$ so from Theorem 3.6 (2) that A is *ii*-open set.

7) $Fr_{ii}(A)=CL_{ii}(A)\setminus Int_{ii}(A)=CL_{ii}(A)\cap(CL_{ii}(X\setminus A))$.

8) It follows from (7).

9) $$CL_{ii}(Fr_{ii}(A))=CL_{ii}(CL_{ii}(A))\cap(CL_{ii}(X\setminus A))$$ $$\subset CL_{ii}(CL_{ii}(A))\cap CL_{ii}(CL_{ii}(X\setminus A))=Fr_{ii}(A).$$ Hence, $Fr_{ii}(A)$ is *ii*-closed.

10) $$Fr_{ii}(Fr_{ii}(A))$$ $$=CL_{ii}(Fr_{ii}(A))\cap CL_{ii}(X\setminus Fr_{ii}(A))\subset CL_{ii}(Fr_{ii}(A))=Fr_{ii}(A).$$

11) Using Theorem 3.6 (3), we get $Fr_{ii}(Int_{ii}(A))=CL_{ii}(Int_{ii}(A))\setminus Int_{ii}(Int_{ii}(A))\subseteq CL_{ii}(A)\setminus Int_{ii}(A)=Fr_{ii}(A)$.

12) $$Fr_{ii}(CL_{ii}(A))=CL_{ii}(CL_{ii}(A))\setminus Int_{ii}(CL_{ii}(A))=CL_{ii}(A)\setminus Int_{ii}(CL_{ii}(A))$$ $$=CL_{ii}(A)\setminus Int_{ii}(A)=Fr_{ii}(A).$$

13) $A\setminus Fr_{ii}(A)=(A\setminus CL_{ii}(A))\setminus Int_{ii}(A)=Int_{ii}(A)$.

The converses of (1) and (4) of Theorem 3.11 are not true in general, as shown by Example

Example 3.12. Consider the topological space (X,τ) given in Example 3.3. If $A=\{c\}$, then $Fr(A)=\{a,c\}\not\subset\{c\}=Fr_{ii}(A)$, and if $B=\{a,b\}$, then $Fr_{ii}(B)=\{c\}\not\subset b_{ii}(B)$.

Definition 3.13. $Ext_{ii}(A)=Int_{ii}(X\setminus A)$ is said to be an *ii*-exterior of *A*.

Theorem 3.14. For a subset A of a space *X*, the following statements hold:

1) $Ext(A)\subset Ext_{ii}(A)$ where $Ext(A)$ denotes the exterior of A

2) $Ext_{ii}(A)$ *is ii-open*

3) $Ext_{ii}(A)=Int_{ii}(X\setminus A)=X\setminus CL_{ii}(A)$

4) $Ext_{ii}(Ext_{ii}(A))=Int_{ii}(CL_{ii}(A))$

5) If $A\subseteq B$, then $Ext_{ii}(A)\supset Ext_{ii}(B)$

6) $Ext_{ii}(A\cup B)\subset Ext_{ii}(A)\cup Ext_{ii}(B)$

7) $Ext_{ii}(A\cap B)\supset Ext_{ii}(A)\cap Ext_{ii}(B)$

8) $Ext_{ii}(X)=\phi$

9) $Ext_{ii}(\phi)=X$

10) $Ext_{ii}(A)=Ext_{ii}(X\setminus Ext_{ii}(A))$

11) $Int_{ii}(A) \subset Ext_{ii}(Ext_{ii}(A))$

12) $X = Ext_{ii}(A) \cup Ext_{ii}(A) \cup Fr_{ii}(A)$

Proof. 1) It follows from Theorem 3.6 (1).

2) It is straightforward by Theorem 3.6 (6).

3) $$\begin{aligned} & Ext_{ii}(Ext_{ii}(A)) = Ext_{ii}(X \setminus CL_{ii}(A)) \\ & = Int_{ii}(X \setminus X \setminus CL_{ii}(A)) = Int_{ii}(CL_{ii}(A)) \end{aligned}.$$

4) Assume that $A \subset B$. Then

$Ext_{ii}(B) = Ext_{ii}(X \setminus B) \subseteq Ext_{ii}(X \setminus A) = Ext_{ii}(A)$, by using Theorem 3.6 (7).

5) Applying Theorem 3.6 (8), we get

$$\begin{aligned} & Ext_{ii}(A \cup B) = Int_{ii}(X \setminus (A \cup B)) = Int_{ii}((X \setminus A) \cup (X \setminus B)) \\ & \subseteq Int_{ii}(X \setminus A) \cup Int_{ii}(X \setminus B) = Ext_{ii}(A) \cup Ext_{ii}(B). \end{aligned}$$

6) Applying Theorem 3.6 (9), we obtain

$$\begin{aligned} & Ext_{ii}(A \cap B) = Int_{ii}(X \setminus (A \cap B)) = Int_{ii}((X \setminus A) \cap (X \setminus B)) \\ & \supset Int_{ii}(X \setminus A) \cap Int_{ii}(X \setminus B) = Ext_{ii}(A) \cap Ext_{ii}(B). \end{aligned}.$$

7) Straightforward.

8) Straightforward.

9) $$\begin{aligned} & Ext_{ii}(X \setminus Ext_{ii}(A)) = Ext_{ii}(X \setminus Int_{ii}(X \setminus A)) = Int_{ii}(X \setminus (X \setminus Int_{ii}(X \setminus A))) \\ & = Int_{ii}(Int_{ii}(X \setminus A)) = Int_{ii}(X \setminus A) = Ext_{ii}(A). \end{aligned}$$

10) $$\begin{aligned} & Int_{ii}(A) \subset Int_{ii}(CL_{ii}(A)) = Int_{ii}(X \setminus Int_{ii}(X \setminus A)) \\ & = Int_{ii}(X \setminus Ext_{ii}(A)) = Ext_{ii}(Ext_{ii}(A)). \end{aligned}$$

Example 3.15. Let $X = \{a,b,c,d\}$ and $\tau = \{\phi, X, \{c,d\}\}$. Thus, $iio(x) = \{\phi, X, \{c,d\}, \{b,c,d\}, \{a,c,d\}\}$. If $A = \{a\}$ and $B = \{b\}$. Then $Ext_{ii}(A) \not\subset Ext(A)$. $Ext_{ii}(A \cap B) \neq Ext_{ii}(A) \cap Ext_{ii}(B)$ and $Ext_{ii}(A \cup B) \neq Ext_{ii}(A) \cup Ext_{ii}(B)$.

4. A New Decomposition of *ii*-Continuity

We begin by the following definition:

Definition 4.1. A function $f:(X,\tau) \to (Y,\sigma)$ is called *ii*-continuous if $f^{-1}(G)$ is *ii-open* set in (X,τ) for any open set *G* of (Y,σ).

Theorem 4.2. Let $f:(X,\tau) \to (Y,\sigma)$ be a function then:

1) Every continuous function is an *ii*-continuous,

2) Every *ii*-continuous function is an *i*-continuous,

3) Every *α*-continuous function is an *ii*-continuous.

Proof. 1) Let *G* be open set in (Y,σ). Since *f* is continuous, it follows that $f^{-1}(G)$ is open set in (X,τ). But every open set is *ii-open* set [4]. Hence $f^{-1}(G)$ is *ii-open* set in (X,τ). Thus *f* is *ii*-continuous.

2) Let *G* be open set in (Y,σ). Since *f* is an *ii*-continuous, it follows that $f^{-1}(G)$ is an *ii-open* set in (X,τ). But every *ii-open* set is *i*-open set [4]. Hence $f^{-1}(G)$ is *i*-open set in (X,τ). Thus *f* is *i*-continuous.

3) Let *G* be open set in (Y, σ). Since *f* is *α*-continuous, it follows that

$f^{-1}(G)$ is α-open set in (X,τ). But every α-open set is *ii-open* set [4]. Hence $f^{-1}(G)$ is *ii-open* set in (X,τ). Thus *f* is an *ii*-continuous.

The converse need not be true by the following example.

Example 4.3. Let

$$X=\{a,b,c,d\},\ \ \tau=\{\phi,X,\{a\},\{b\},\{a,b\}\}$$

and

$$Y=\{a,b,c,d\},\ \ \sigma=\{\phi,Y,\{a\},\{b\},\{a,b\},\{a,d\},\{a,b,d\}\}$$

and

$$iio(x)=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,d\},\{a,b,c\},\{a,b,d\}\},$$

$$io(x)=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},$$
$$\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\},$$

$$\alpha o(x)=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,b,c\},\{a,b,d\}\}.$$

Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be the identity function then $f^{-1}(\{a\})=\{a\}$, $f^{-1}(\{b\})=\{b\}$, $f^{-1}(\{c\})=\{c\}$, $f^{-1}(\{d\})=\{d\}$. Then *f* is *ii*-continuous, but *f* is not α-continuous, since for the open set $\{a,d\}$ in (Y,σ), $f^{-1}(\{a,d\})=\{a,d\}$ is not α-open in (X,τ) *and f* is not continuous, since for the open set $\{a,d\}$ in (Y,σ), $f^{-1}(\{a,d\})=\{a,d\}$ is not open in (X,τ). Now when $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by $f^{-1}(\{a\})=\{b\}$, $f^{-1}(\{b\})=\{a\}$, $f^{-1}(\{c\})=\{c\}$, $f^{-1}(\{d\})=\{d\}$ we get *f* is *i*-continuous, but *f* is not *ii*-continuous, since for the open set $\{a,d\}$ in (Y,σ), $f^{-1}(\{a,d\})=\{b,d\}$ is not *ii*-open in (X,τ).

Theorem 4.4. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function then every semi-continuous function is an *ii*-continuous.

Proof. Let *G* be open set in (Y,σ). Since *f* is semi-continuous, it follows that $f^{-1}(G)$ is semi-open set in (X,τ). But every semi-open set is *ii-open* set [4]. Hence $f^{-1}(G)$ is *ii-open* set in (X,τ). Thus *f* is an *ii*-continuous.

Definition 4.5. A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called *int*-continuous if $f^{-1}(G)$ is *int*-open set in (X,τ) for any open set *G* in (Y,σ).

Theorem 4.6. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function then:

1) Every continuous function is *int*-continuous,

2) Every *ii*-continuous function is *int*-continuous,

3) Every *α-continuous* function is *int*-continuous.

Proof. 1) Let *G* be open set in (Y,σ). Since *f* is continuous, it follows that $f^{-1}(G)$ is open set in (X,τ). But every open set is *int*-open set [4]. Hence $f^{-1}(G)$ is *int*-open set in (X,τ). Thus *f* is *int*-continuous.

2) Let *G* be open set in (Y,σ). Since *f* is *ii*-continuous, it follows that $f^{-1}(G)$ is an *ii-open* set in (X,τ). But every *ii-open* set is *int*-open set [4]. Hence $f^{-1}(G)$ is *int*-open set in (X,τ). Thus *f* is *int*-continuous.

3) Let *G* be open set in (Y,σ). Since *f* is α-continuous, it follows that $f^{-1}(G)$ is α-open set in (X,τ). But every α-open set is *int*-open set [4]. Hence $f^{-1}(G)$ is *int*-open set in (X,τ). Thus *f* is *int*-continuous.

The converse need not be true by the following example.

Example 4.7. Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\},\{b,c\}\}$ and $Y=\{a,b,c\}$, $\sigma=\{\phi,Y,\{a\},\{a,c\}\}$ and $into(x)=\{\phi,X,\{a\},\{a,b\},\{b,c\},\{a,c\}\}$, $iio(x)=\alpha o(x)=\{\phi,X,\{a\},\{b,c\}\}$. *Let* $f:(X,\tau)\to(Y,\sigma)$ be the identity function then $f^{-1}(\{a\})=\{a\}$, $f^{-1}(\{b\})=\{b\}$, $f^{-1}(\{c\})=\{c\}$. *Then* f is *int*-continuous, but f is not *ii*-continuous, since for the open set $\{a,c\}$ in (Y,σ), $f^{-1}(\{a,c\})=\{a,c\}$ is not *ii-open* in (X,τ) and f is not continuous, since for the open set $\{a,c\}$ in (Y,σ), $f^{-1}(\{a,c\})=\{a,c\}$ is not open in (X,τ) and f is not α-continuous, since for the open set $\{a,c\}$ in (Y,σ), $f^{-1}(\{a,c\})=\{a,c\}$ is not α-open.

Definition 4.8. A subset A of X is called weakly *ii*-open set if A is *ii*-open set and $A\subseteq CL(Int(A)\cap A)$.

Theorem 4.9. A subset A of a space X is α-open set if and only if A is weakly *ii*-open.

Proof. Let A be α-open set. Since $A\subseteq Int(CL(Int(A)))$ and $A\subseteq CL(A)$. Therefore $A\subseteq CL(Int(A))\cap CL(A)$, this implies that $A\subseteq CL(Int(A)\cap A)$. Now, put $G=Int(A)$ where $G\neq\phi,X$, then A is *ii*-open set. Therefore, A is weakly *ii*-open set.

Conversely, Let A be weakly *ii*-open set, then there exist an open set $G\neq\phi,X$, such that $G=Int(A)$ satisfying $A\subseteq CL(Int(A)\cap A)$ and A is *ii*-open set. Since $A\subseteq CL(Int(A)\cap A)$, this implies that $A\subseteq CL(Int(A))$ and $Int(A)\subseteq Int(CL(Int(A)))$. Since A is *ii*-open set, using (2) from Theorem (3.6), we get $A=Int(A)$. Therefore $A\subseteq Int(CL(Int(A)))$. Thus A is α-open set.

As a summary the following Figure 1 shows the relations among semi-continuous, *ii*-continuous, *i*-continuous, *int*-continuous, α-continuous and continuous.

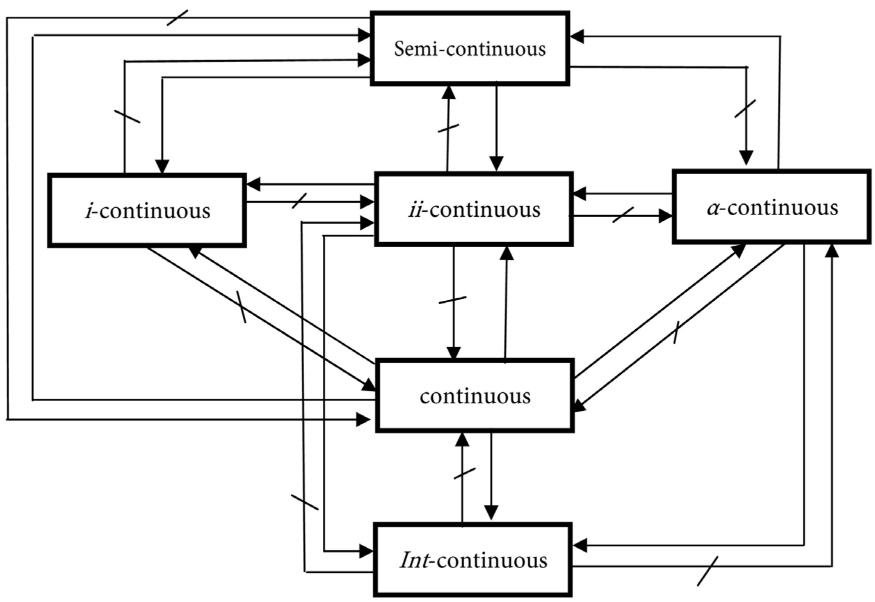

Figure 1. Relations among semi-continuous, *ii*-continuous, *i*-continuous, *int*-continuous, α-continuous and continuous.

Corollary 4.10. A function $f:(X,\tau)\to(Y,\sigma)$ is *α-continuous* if and only if it is weakly *ii*-continuous.

Proof. Clear from Theorem 4.9.

5. *ii*-Separating Axioms

In this section we define T_{0ii} and T_{1ii} spaces for *ii*-open sets and we determine them by giving many examples. Specially, we define T_1, $T_{1\alpha}$ and T_{1i} spaces to compare them with T_{1ii} space.

Definition 5.1. A topological space *X* is called

1) T_{0ii} if *a*, *b* are to distinct points in *X*, there exists *ii-open* set *U* such that either $a\in U$ *and* $b\notin U$, and $b\in U$ *and* $a\notin U$.

2) T_{1ii} if $a,b\in X$ and $a\neq b$, there exist *ii*-open sets *U*, *V* containing *a*, *b* respectively, such that $b\notin U$ and $a\notin V$.

Example 5.2. Let $X=\{a,b\}$, $\tau^{ii}=\tau=\{\phi,X,\{a\},\{b\}\}$ (X,τ) and (X,τ^{ii}) are topological spaces.

1) $a,b\in X$ ($a\neq b$) there exists $\{a\}\in\tau^{ii}$ such that $a\in\{a\}$, $b\notin\{a\}$. Therefore (X,τ) is T_{0ii}.

2) $a,b\in X$ ($a\neq b$) there exists $\{a\},\{b\}\in\tau^{ii}$ such that $a\in\{a\}$, $b\in\{b\}$. Therefore (X,τ) is T_{1ii}.

Theorem 5.3.

1) Every T_0-space is T_{0ii}-space,

2) Every T_1-space is T_{0ii}-space,

3) Every T_1-space is T_{1ii}-space,

4) Every T_{1ii}-space is T_{0ii}-space.

Proof. (1), (2), (3) and (4) follow using the fact that every open set is *ii*-open [4].

The converse needs not to be true by the following example.

Example 5.4. Let

$$X=\{a,b,c\},\ \tau=\{\phi,X,\{a\}\} \text{ and } \tau^{ii}=\{\phi,X,\{a\},\{a,b\},\{a,c\}\}.$$

(X,τ) and (X,τ^{ii}) are topological spaces.

(X,τ) is not T_0-space because, $b,c\in X$ ($b\neq c$) there is no open set G such that $b\in G$, $c\notin G$.

(X,τ) is T_{0ii}-space because, $a,b\in X$ ($a\neq b$) there exists $\{a\}\in\tau^{ii}$ such that $a\in\{a\}$, $b\notin\{a\}$.

$a,c\in X$ ($a\neq c$) there exists $\{a\}\in\tau^{ii}$ such that $a\in\{a\}$, $c\notin\{a\}$.

$b,c\in X$ ($b\neq c$) there exists $\{a,b\}\in\tau^{ii}$ such that $b\in\{a,b\}$, $c\notin\{a,b\}$.

(X,τ) is not T_1-space because, $a,b\in X$ ($a\neq b$) there exists $X\in\tau$ such that $a\in X$, $b\in X$.

(X,τ) is not T_{1ii}-space because, $b,a\in X$ ($a\neq b$) there exists $\{a,b\}\in\tau^{ii}$ such that $a\in\{a,b\}$, $b\in\{a,b\}$.

Theorem 5.5. Every $T_{1\alpha}$-*space* is T_{1ii}-*space*.

Proof. Let *X* be $T_{1\alpha}$-space. Let *a*, *b* be two distinct points in *X*. Since *X* is $T_{1\alpha}$-space there exist two *α*-open sets *U*, *V* in *X* such that $a\in U$, $b\notin U$, $a\notin V$,

$b \in V$. Since every α-open set is *ii*-open set [4], *U*, *V* is an *ii*-open set in *X*. Hence *X* is T_{1ii}-space.

Theorem 5.6. Every T_{1ii}-*space* is T_{1i}-space.

Proof. Let *X* be a T_{1ii}-space. Let *a*, *b* be two distinct points in *X*. Since *X* is T_{1ii}-space there exist two *ii*-open sets *U*, *V* in *X* such that $a \in U$, $b \notin U$, $a \notin V$, $b \in V$. Since every *ii*-open set is *i*-open set [4], *U*, *V* is an *i*-open set in *X*. Hence *X* is T_{1i}-space.

The converse needed not to be true by the following example.

Example 5.7. Let $X = \{a,b,c\}$, $\tau = \{\phi, X, \{a\}, \{b,c\}\}$ and.

$\tau^{i} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{b,c\}\}$. $\tau^{ii} = \{\phi, X, \{a\}, \{b,c\}\}$.

(X,τ) and (X,τ^{ii}) are topological spaces.

(X,τ) is T_{1i}-space because, $a,b \in X$ ($a \neq b$) there exists $\{a\},\{b\} \in \tau^{i}$ such that $a \in \{a\}$, $b \notin \{a\}$ and $b \in \{b\}$, $a \notin \{b\}$.

$a,c \in X$ ($a \neq c$) there exists $\{a\},\{c\} \in \tau^{i}$ such that $a \in \{a\}$, $c \notin \{a\}$ and $c \in \{c\}$, $a \notin \{c\}$.

$b,c \in X$ ($b \neq c$) there exists $\{c\},\{b\} \in \tau^{i}$ such that $c \in \{c\}$, $b \notin \{c\}$ and $b \in \{b\}$, $c \notin \{b\}$.

(X,τ) is not T_{1ii}-space because, $b,c \in X$ ($c \neq b$) there exists $\{b\},\{c\} \in \tau^{ii}$ such that $c \in \{b,c\}$, $b \in \{b,c\}$.

Theorem 5.8. A space *X* is T_{0ii} if and only if $CL_{ii}(\{x\}) \neq CL_{ii}(\{y\})$ for every pair of distinct points *x*, *y* of *X*.

Proof. Let *X* be a T_{0ii}-space. Let $x, y \in X$ such that $x \neq y$, then there exists an *ii*-open set *U* containing one of the points but not the other, then $x \in U$ and $y \notin U$. Then $X \setminus U$ is *ii*-closed set containing *y* but not *x*. But $CL_{ii}(\{y\})$ is the smallest *ii*-closed set containing *y*. Therefore $CL_{ii}(\{y\}) \subset X \setminus U$ and hence $x \notin CL_{ii}(\{y\})$. Thus $CL_{ii}(\{x\}) \neq CL_{ii}(\{y\})$.

Conversely, Suppose for any $x, y \in X$ with $x \neq y$, $CL_{ii}(\{x\}) \neq CL_{ii}(\{y\})$. Let $z \in X$ such that $z \in CL_{ii}(\{x\})$ but $z \notin CL_{ii}(\{y\})$. If $x \in CL_{ii}(\{y\})$ then $CL_{ii}(\{x\}) \subset CL_{ii}(\{y\})$ and hence $z \in CL_{ii}(\{y\})$. This is contradiction. Therefore $x \notin CL_{ii}(\{y\})$. That is $x \in X \setminus CL_{ii}(\{y\})$. Therefore $X \setminus CL_{ii}(\{y\})$ is *ii*-open set containing *x* but not *y*. Hence *X* is an T_{0ii}-space.

Theorem 5.9. A space (X,τ) is T_{1ii}-space if and only if the singletons are *ii*-closed sets.

Proof. Let *X* be T_{1ii}-space and let $x \in X$, to prove that $\{x\}$ is *ii*-closed set. We will prove $X \setminus \{x\}$ is *ii*-open set in *X*. Let $y \in X \setminus \{x\}$, implies $x \neq y$ and since *X* is T_{1ii}-space then their exist two *ii*-open sets *U*, *V* such that $x \notin U$, $y \in V \subset X \setminus \{x\}$. Since $y \in V \subset X \setminus \{x\}$, then $X \setminus \{x\}$ is *ii*-open set. Hence $\{x\}$ is *ii*-closed set.

Conversely, Let $x \neq y \in X$ then $\{x\},\{y\}$ are *ii*-closed sets. That is $X \setminus \{x\}$ is *ii*-open set clearly, $x \notin X \setminus \{x\}$ and $y \in X \setminus \{x\}$. Similarly $X \setminus \{y\}$ is *ii*-open set, $y \notin X \setminus \{y\}$ and $x \in X \setminus \{y\}$. Hence *X* is an T_{1ii}-space.

As a consequence the following **Figure 2** shows the relations among T_0, T_{0ii}, T_1, T_{1ii} and $T_{1\alpha}$.

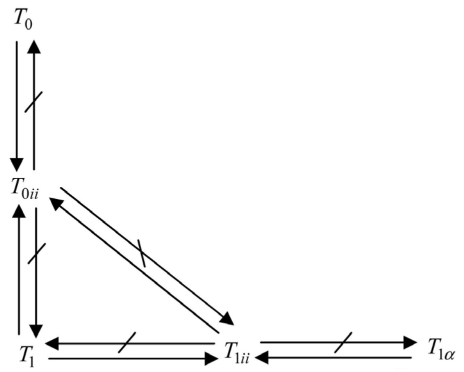

Figure 2. Relations among T_0, T_{0ii}, T_1, T_{1ii} and $T_{1\alpha}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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