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Unicity of Meromorphic Solutions of Some Nonlinear Difference Equations

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Abstract

This paper is to study the unicity of transcendental meromorphic solutions to some nonlinear difference equations. Let $m \in \{\pm 2, \pm 1, 0\}$ be a nonzero rational function. Consider the uniqueness of transcendental meromorphic solutions to some nonlinear difference equations of the form $w(z+1)w(z-1) = R(z)w^m(z)$. For two finite order transcendental meromorphic solutions of the equation above, it shows that they are almost equal to each other except for a nonconstant factor, if they have the same zeros and poles counting multiplicities, when $m \in \{2, \pm 1, 0\}$. Two relative results are proved, and examples to show sharpness of our results are provided.

Keywords

Unicity, Meromorphic Solution, Difference Equation

1. Introduction

It is well known that a given nonconstant monic polynomial is determined by its zeros. But it is not true for transcendental entire or meromorphic functions. Take e^z and e^{-z} for example, they are essentially different even have the same zeros, 1-value points and poles. This indicates that it is complex and interesting to determine a transcendental meromorphic function uniquely. Nevanlinna then proves his famous Nevanlinna's 5 CM (4 IM) Theorem (see e.g. [1] [2]):

Theorem A: Let $w(z)$ and $u(z)$ be two nonconstant meromorphic functions. If $w(z)$ and $u(z)$ share 5 values IM (4 values CM, respectively) in the extended complex plane, then $w(z) \equiv u(z) \left(w(z) = T(u(z)) \right)$, where T is a Möbius transformation, respectively).

Here and in the following, for two nonconstant meromorphic functions $w(z)$

and $u(z)$, and a complex constant a, we say $w(z)$ and $u(z)$ share a IM (CM), if $w(z)-a$ and $u(z)-a$ have the same zeros ignoring multiplicities (counting multiplicities); and we say $w(z)$ and $u(z)$ share ∞ IM(CM), if they have the same poles ignoring multiplicities (counting multiplicities).

Our aim is to study the unicity of meromorphic solutions to the nonlinear difference equation of the form

$$w(z+1)w(z-1)=R(z)w^m(z), \tag{1.1}$$

where $R(z)$ is a nonzero rational function and $m\in\{\pm2,\pm1,0\}$ The Equation (1.1) comes from the family of Painlevé III equations which are given by Ronkainen in [3] when he classifies the difference equation

$$w(z+1)w(z-1)=R(z,w),$$

where $R(z, w)$ is irreducible and rational in w and meromorphic in z. This is a natural idea which comes from the topic on the growth, value distribution and unicity on the meromorphic solutions to difference equations (see e.g. [4] [5] [6] [7] [8]). The first result is as follows.

Theorem 1.1. Let $w(z)$ and $u(z)$ be two finite order transcendental meromorphic solutions to the Equation (1.1), where $m\in\{2,\pm1,0\}$. If $w(z)$ and $u(z)$ share 0, ∞ CM, then $w(z)\equiv\lambda u(z)$, where λ is a constant such that $\lambda^{2-m}=1$.

The following examples show that all cases in Theorem 1.1. can happen, and the "CM" cannot be relaxed to "IM".

Example 1. In the following examples, $w_j(z)$ and $u_j(z)$ share 0, ∞ CM, while $w_j(z)$ and $v_j(z)$ share 0, ∞ IM $(j=1,2,3,4)$:

1) $u_1(z)=\tan\left(\frac{\pi z}{2}\right), w_1(z)=iu_1(z)$ and $v_1(z)=u_1^2(z)$ satisfy the difference equation

$$w(z+1)w(z-1)=w^{-2}(z).$$

here $m=-2,\lambda=i$ such that $\lambda^{2-(-2)}=1$.

2) $u_2(z)=\tan^2\left(\frac{\pi z}{3}\right)\tan^2\left[\frac{(2z-1)\pi}{6}\right], w_2(z)=\mathrm{e}^{i\frac{2\pi}{5}}u_2(z)$ and $v_2(z)=u_2^2(z)$ satisfy the difference equation

$$w(z+1)w(z-1)=w^{-1}(z).$$

here $m=-1,\lambda=\mathrm{e}^{\frac{2\pi i}{3}}$ such that $\lambda^{2-(-1)}=1$.

3) $u_3(z)=\tan\left(\frac{\pi z}{4}\right), w_3(z)=-u_3(z)$ and $v_3(z)=iu_3^2(z)$ satisfy the difference equation

$$w(z+1)w(z-1)=-1.$$

here $m=0,\lambda=-1$ such that $\lambda^{2-0}=1$.

4) $u_4(z)=\tan\left(\frac{\pi z}{6}\right)\tan\left[\frac{\pi(z-1)}{6}\right], w_4(z)=u_4(z)$ and $v_4(z)=u_4^3(z)$ satisfy

the difference equation

$$w(z+1)w(z-1)=-w(z).$$

here $m=1,\lambda=1$ such that $\lambda^{2-1}=1$.

Theorem 1.2. Let *w*(*z*) and *u*(*z*) be two finite order transcendental meromorphic solutions to the Equation (1.1), where $m\in\{2,\pm1,0\}$. If *w*(*z*) and *u*(*z*) share 0, ∞ CM, then

$$w(z)\equiv e^{a_2z^2+a_1z+a_0}u(z), \tag{1.2}$$

where a_0,a_1,a_2 are constants such that $e^{2a_2}=1$. What is more, $w(z)\equiv u(z)$ if $w(z)-u(z)$ has a zero z_1 of multiplicity ≥ 3 such that $w(z_1)=u(z_1)=c\neq 0$.

The following example shows that all conclusions in Theorem 1.2 can happen, and the "CM" cannot be relaxed to "IM".

Example 2. Let $u(z)=\tan(\pi z), v(z)=u^2(z)$ and $w_1(z)=e^{\pi i z^2}u(z)$, $w_2(z)=e^z u(z)$, $w_3(z)=u(z)$. Then $w_j(z)$ and $u(z)$ share 0, ∞ CM, while $w_j(z)$ and $v(z)$ share 0, ∞ IM (*j* = 1, 2, 3), and they solve the equation

$$w(z+1)w(z-1)=w^2(z).$$

Theorem 1.3. Let *w*(*z*) and *u*(*z*) be two finite order transcendental meromorphic solutions to the Equation (1.1), where $m\in\{\pm1,0\}$. If *w*(*z*) and *u*(*z*) share 1, ∞ CM, then

$$w(z)-1\equiv e^{a_1z+a_0}\left(u(z)-1\right), \tag{1.3}$$

where a_0,a_1 are constants such that:

1) $a_1=\frac{k_1}{2}\pi i$, when $m=0$; 2) $a_1=\frac{2k_2}{3}\pi i$, when $m=-1$; (3) $a_1=\frac{k_3}{3}\pi i$, when

$m=1$, where k_1,k_2,k_3 are some integers. What is more, $w(z)\equiv u(z)$ if one of the following additional condition holds:

a) $w(z)-u(z)$ has a zero z_1 of multiplicity ≥ 2 such that $w(z_1)=u(z_1)=0$;

b) there exist two constants z_2,z_3 such that $w(z_j)=u(z_j)\neq 1\ (j=2,3)$ and $z_2-z_3\notin\mathbb{Q}$.

Remark 1. We have tried hard but failed to provide some similar results as Theorem 1.3 for the cases $m=\pm2$ so far.

2. Proof of Theorem 1.1

Since *w*(*z*) and *u*(*z*) are finite order transcendental meromorphic functions and share 0, ∞ CM, we see that

$$\frac{w(z)}{u(z)}=e^{p(z)},$$

where $p(z)$ is a polynomial such that it is of degree

$$\deg p(z)=p\leq\max\{\rho(w),\rho(u)\}.$$

Next, we discuss case by case.

Case 1: $m=-2$. From (1.1) and (2.1) we get

$$u(z+1)u(z-1)u^2(z)\mathrm{e}^{p(z+1)+p(z-1)+2p(z)}$$
$$=w(z+1)w(z-1)w^2(z)=R(z)=u(z+1)u(z-1)u^2(z),$$

which gives

$$\left(\mathrm{e}^{p(z+1)+p(z-1)+2p(z)}-1\right)u(z+1)u(z-1)u^2(z)\equiv 0.$$

Thus, we have

$$\mathrm{e}^{p(z+1)+p(z-1)+2p(z)}\equiv 1. \tag{2.2}$$

Since

$$\deg\left(p(z+1)+p(z-1)+2p(z)\right)=\deg p(z)=p,$$

from (2.2), it is easy to find that $p=0$. Therefore, there exists some constant p_0, such that $p(z)\equiv p_0$ and

$$\mathrm{e}^{4p_0}=\mathrm{e}^{p(z+1)+p(z-1)+2p(z)}\equiv 1.$$

That is, for $\lambda=\mathrm{e}^{p_0}$, we have $w(z)\equiv\lambda u(z)$ and $\lambda^4=1$.

Case 2: $m=-1$. Now, we obtain from (1.1) and (2.1) that

$$u(z+1)u(z-1)u(z)\mathrm{e}^{p(z+1)+p(z-1)+2p(z)}$$
$$=w(z+1)w(z-1)w(z)=R(z)=u(z+1)u(z-1)u(z).$$

With this equation and similar reasoning as in Case 1, we can deduce that $w(z)\equiv\lambda u(z)$ holds for some λ such that $\lambda^3=1$.

Case 3: $m=0$. From (1.1) and (2.1), we have

$$u(z+1)u(z-1)\mathrm{e}^{p(z+1)+p(z-1)}=w(z+1)w(z-1)=R(z)=u(z+1)u(z-1).$$

Similarly, we can prove that $w(z)\equiv\lambda u(z)$ holds for some λ such that $\lambda^2=1$.

Case 4: $m=1$. Now (1.1) is of the form

$$w(z+1)w(z-1)=R(z)w(z). \tag{2.3}$$

Thus,

$$w(z+2)w(z)=R(z+1)w(z+1).$$

It follows from these two equations above and (2.1) that

$$u(z+2)u(z-1)\mathrm{e}^{p(z+2)+p(z-1)}$$
$$=w(z+2)w(z-1)=R(z+1)R(z)=u(z+2)u(z-1),$$

with which we can show that $w(z)\equiv\lambda u(z)$ holds for some λ such that $\lambda^2=1$. However, if $w(z)\equiv -u(z)$, we find that

$$\left(-w(z+1)\right)\left(-w(z-1)\right)=u(z+1)u(z-1)=R(z)u(z)=-R(z)w(z). \tag{2.4}$$

Combining (2.3) and (2.4), we get $R(z)w(z)\equiv 0$, which is impossible. Thus, $\lambda=1$.

3. Proof of Theorem 1.2

Notice that (2.1) still holds for this case. We can get from (1.1) and (2.1) that

$$\frac{u(z+1)u(z-1)e^{p(z+1)+p(z-1)}}{u^2(z)e^{2p(z)}}=\frac{w(z+1)w(z-1)}{w^2(z)}=R(z)=\frac{u(z+1)u(z-1)}{u^2(z)}.$$

Thus, we have

$$e^{p(z+1)+p(z-1)+2p(z)}\equiv 1. \tag{3.1}$$

If $p\le 1$, then our conclusion holds for $a_2=0$. If $p\ge 2$, set

$$p(z)=a_pz^p+a_{p-1}z^{p-1}+\cdots+a_1z+a_0, \tag{3.2}$$

where $a_p\ne 0,a_{p-1},\cdots,a_1,a_0$ are constants.

From (3.2), we see that

$$p(z+1)+p(z-1)-2p(z)=p(p-1)a_pz^{p-2}+q(z), \tag{3.3}$$

where $q(z)$ is a polynomial such that $q(z)\equiv 0$ when $p=2$, or $\deg q(z)<p-2$ when $p\ge 3$.

Suppose that $p\ge 3$, we obtain from (3.1) and (3.3) that

$$1\equiv e^{p(z+1)+p(z-1)+2p(z)}=e^{p(p-1)a_pz^{p-2}+q(z)},$$

which is impossible. Thus, $p=2$, then from (3.1) and (3.3), we get $e^{2a_2}=1$ immediately. To sum up, we prove that (1.2) holds.

Next, we use $p(z)=a_2z^2+a_1z+a_0$ and prove our additional conclusion. From (1.2), we see that $e^{p(z_1)}=1$.

Differentiating both sides of (1.2), we can deduce that

$$p'(z)e^{p(z)}u(z)=w'(z)-e^{p(z)}u'(z)$$

and

$$p''(z)e^{p(z)}u(z)=w''(z)e^{p(z)}u(z)=(p'(z))^2e^{p(z)}u(z)-2p'(z)e^{p(z)}u'(z).$$

By our assumption, (1.2}), (3.4) and the fact that $e^{p(z_1)}=1$, we have

$$\begin{aligned}p'(z_1)=p'(z_1)u(z_1)&=p'(z_1)e^{p(z_1)}u(z_1)\\&=w'(z_1)-e^{p(z_1)}u'(z_1)\\&=w'(z_1)-u'(z_1)=0.\end{aligned}$$

Therefore, similarly, it follows from (3.5) that

$$\begin{aligned}p''(z_1)&=p''(z_1)e^{p(z_1)}u(z_1)\\&=w''(z_1)-e^{p(z_1)}u''(z_1)-(p'(z_1))^2e^{p(z_1)}u(z_1)-2p'(z_1)e^{p(z_1)}u'(z_1)\\&=w''(z_1)-u''(z_1)=0.\end{aligned}$$

As a result, we obtain

$$2a_2=p''(z_1)=0,2a_2z_1+a_1=p'(z_1)=0,e^{2a_2z_1^2+a_0z_1+a_0}=e^{p(z_1)}=1,$$

that is, $a_2=a_1=0,e^{a_0}=1$. Hence, $w(z)\equiv e^{a_2z^2+a_1z+a_0}u(z)=u(z)$.

4. Proof of Theorem 1.3

Here, we need the lemma below, where the case that $R(z)$ is a nonzero constant has been proved by Zhang and Yang [7] and the case that $R(z)$ is a nonconstant rational function by Lan and Chen [8].

Lemma 4.1. [7] [8] Let $w(z)$ be a finite order transcendental meromorphic solution to

the Equation (1.1), where $m \in \{-2, \pm 1, 0\}$ and a be a constant. Then

$$\lambda(w-a) = \lambda(1/w) = \rho(w) \geq 1.$$

Proof of Theorem 1.3. Since $w(z)$ and $u(z)$ are finite order transcendental meromorphic functions and share 1, ∞ CM, we see that

$$\frac{w(z)-1}{u(z)-1} = \mathrm{e}^{p(z)}, \tag{4.1}$$

where $p(z)$ is a polynomial such that

$$p(z) = a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0, \tag{4.2}$$

where $a_p \neq 0, \cdots, a_0$ are constants and $p = \deg p(z) \leq \max\{\rho(w), \rho(u)\}$.

Case 1: $m = 0$. From (1.1) and (4.1), we obtain

$$\frac{u(z+4)}{u(z)} = \frac{R(z+3)}{R(z+1)} := R_1(z) \tag{4.3}$$

and

$$\frac{\mathrm{e}^{p(z+4)}(u(z+4)-1)+1}{\mathrm{e}^{p(z)}(u(z)-1)+1} = \frac{w(z+4)}{w(z)} = \frac{R(z+3)}{R(z+1)} = R_1(z), \tag{4.4}$$

where $R_1(z)$ is a rational function. Combining (4.1}), (4.3) and (4.4), we have

$$\left(\mathrm{e}^{p(z+4)} - \mathrm{e}^{p(z)}\right) R_1(z)(u(z)-1) = (1-R_1(z))\left(\mathrm{e}^{p(z+4)}-1\right). \tag{4.5}$$

Now, if $\mathrm{e}^{p(z+4)} \not\equiv \mathrm{e}^{p(z)}$, then $p \geq 1$ and it follows from (4.5) that

$$u(z) = \frac{1-R_1(z)}{R_1(z)} \frac{1-\mathrm{e}^{-p(z+4)}}{1-\mathrm{e}^{p(z)-p(z+4)}} + 1. \tag{4.6}$$

Notice that $\deg(p(z) - p(z+4)) \leq p-1$. From (4.6), we can find that

$$\lambda(u-1) = p > p-1 \geq \rho\left(1-\mathrm{e}^{p(z)-p(z+4)}\right) \geq \lambda\left(\frac{1}{u}\right).$$

This is a contradiction to the conclusion of Lemma 4.1. Thus, $\mathrm{e}^{p(z+4)} \equiv \mathrm{e}^{p(z)}$. From (4.2) there exists some integer k_1 such that

$$2k_1\pi i = p(z+4) - p(z) = 4pa_p z^{p-1} + \cdots,$$

which yields obviously that $p = 1$. Therefore, we see that

$$a_p = a_1 = \frac{k_1}{2}\pi i \text{ and hence } p(z) = \frac{k_1}{2}\pi i z + a_0 \text{ for some constant } a_0.$$

Case 2: $m = -1$. Now (1.1) is of the form

$$u(z+1)u(z-1)u(z) = R(z),$$

which gives

$$\frac{u(z+3)}{u(z)}=\frac{R(z+2)}{R(z+1)}:=R_2(z).$$

With this equation and a similar arguing as in Case 1, we can prove that $p(z)=\frac{2k_2}{3}\pi iz+a_0$ for some integer k_2 and some constant a_0.

Case 3: $m=1$. Now (1.1) is of the form

$$u(z+1)u(z-1)=R(z)u(z),$$

which gives

$$u(z+3)u(z)=R(z+2)R(z+1).$$

And hence we have

$$\frac{u(z+6)}{u(z)}=\frac{R(z+5)R(z+4)}{R(z+2)R(z+1)}:=R_3(z).$$

It follows this equation that $p(z)=\frac{k_3}{3}\pi iz+a_0$ for some integer k_3 and some constant a_0, and (1.3) holds.

Now, if $w(z)-u(z)$ has a zero z_1 of multiplicity ≥ 2 such that $w(z_1)=0$, then from (4.1), we see that $e^{p(z_1)}=1$.

Rewrite (4.1) as the form

$$w(z)-1=e^{p(z)}(u(z)-1).$$

Differentiating both sides of the equation above, we have

$$p'(z)e^{p(z)}(1-u(z))=e^{p(z)}u'(z)-w'(z).$$

Since z_1 is a zero of $w(z)-u(z)$ with multiplicity ≥ 2 such that $w(z_1)=u(z_1)=0$, from the fact that $e^{p(z_1)}=1$ and (4.7), we find that

$$p'(z_1)=p'(z_1)e^{p(z_1)}(1-u(z_1))=e^{p(z_1)}u'(z_1)-w'(z_1)=0.$$

Thus, $a_1=p'(z_1)=0$, and hence $e^{p(z)}\equiv e^{p(z_1)}=1$. This implies that $w(z)\equiv u(z)$.

Finally, we discuss the Case 2). Since $w(z_j)=u(z_j)\neq 1$ and $z_2-z_3\notin\mathbb{Q}$, then from (4.1), we can deduce that $e^{p(z_2)}=1=e^{p(z_3)}$. Therefore, there exists an integer k_0 such that

$$a_1(z_2-z_3)=p(z_2)-p(z_3)=2k_0\pi i.$$

If $a_1\neq 0$, from the equation above, considering each form of a_1 for $m=-1,0,1$, we can find that z_2-z_3 must be a nonzero rational number. This contradicts our assumption that $z_2-z_3\notin\mathbb{Q}$. Thus $a_1=0$, and hence $e^{p(z)}\equiv 1$. This gives $w(z)\equiv u(z)$ again.

5. Conclusion

It is shown that the finite order transcendental meromorphic solution of the

Equation (1.1) is mainly determined by its zeros (or 1-value points) and poles. Examples are provided to show sharpness of our results.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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