Scientific Research Publishing

Advances in Pure Mathematics, 2019, 9, 480-522
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Estimates $L_1 - L_\infty$ for the Reduced Radial Equation of Schrödinger

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How to cite this paper: Blancarte, H. (2019) The Estimates $L_1 - L_\infty$ for the Reduced Radial Equation of Schrödinger. *Advances in Pure Mathematics*, **9**, 480-522.
https://doi.org/10.4236/apm.2019.95023

Received: February 16, 2019
Accepted: May 26, 2019
Published: May 29, 2019

Abstract

Estimates of the type $L_1 - L_\infty$ for the *Schrödinger Equation on the Line and on Half-Line* with a regular potential $V(x)$, express the dispersive nature of the Schrödinger Equation and are the essential elements in the study of the problems of initial values, the asymptotic times for large solutions and Scattering Theory for the Schrödinger equation and non-linear in general; for other equations of Non-linear Evolution. In general, the estimates $Lp - Lp'$ express the dispersive nature of this equation. And its study plays an important role in problems of non-linear initial values; likewise, in the study of problems nonlinear initial values; see [1] [2] [3]. On the other hand, following a series of problems proposed by V. Marchenko [4], that we will name Marchenko's formulation, and relate it to a generalized version of *Theorem* 1 given in [1], the main theorem (*Theorem* 1) of this article provides a transformation operator $\mathbb{W}$ that transforms the *Reduced Radial Schrödinger Equation* (*RRSE*) (whose main characteristic is the addition a singular term of quadratic order to a regular potential $V(x)$) in the *Schrödinger Equation on Half-Line* (*RSEHL*) under $\mathbb{W}$. That is to say; $\mathbb{W}$ eliminates the singular term of quadratic order of potential $V(x)$ in the asymptotic development towards zero and adds to the potential $V(x)$ a bounded term and a term exponentially decrease fast enough in the asymptotic development towards infinity, which continues guaranteeing the uniqueness of the potential $V(x)$ in the condition of the infinity boundary. Then the $L_1 - L_\infty$ estimates for the (*RRSE*) are preserved under the transformation operator $\mathbb{W}$, as in the case of (*RSEHL*) where they were established in [3]. Finally, as an open question, the possibility of extending the $L_1 - L_\infty$ estimates for the case (*RSEHL*), where added to the potential $V(x)$ an *analytical perturbation* is mentioned.

Keywords

The Schrödinger Equation on the Half-Line, Reduced Radial Equation of Schrödinger, Conditions Sufficient to Establish the Uniqueness of the Potential and Boundary Conditions Are Named the Generalized Theorem 1, The Marchenko's Formulation, Reduction of Estimates $L_1 - L_\infty$ for the Reduced Radial Equation of Schrôdinger to Equation on Half-Line

1. Introduction

The inverse problem addressed in [1], *Theorem* 1, pages 481-483, where the uniqueness of the potential $q_2(x)$ and the condition of value boundary:

$$y'(0) - h_2 y(0) = 0, \tag{1}$$

were established, for the second problem of boundary value which will be named simply *Problem* 2 (similarly the first problem of boundary value will be named *Problem* 1: with hypotheses: $0 < x < a$, the boundary condition:

$$y'(0) - h_1 y(0) = 0, \tag{2}$$

and the potential $q_1(x)$ is given in the *Problem* 1.

Similarly now assuming that distribution spectral function R extends when $a = \infty$, see [4], Chapter 2, Section 3, page 153, *Problem* 6 A and the boundary condition

$$y(0) = 0, \tag{3}$$

in the *Problem* 1, that is

$$-y''(x) + q_1(x) y(x) = s^2 y(x), 0 < x < \infty, s \in \{\lambda, \mu : \lambda, \mu \in \mathbb{R}\} \tag{4}$$

$$y(0) = 0.$$

Then the uniqueness of the potential $q_2(x)$ is established for the *Problem* 2

$$-y''(x) + q_2(x) y(x) = s^2 y(x), 0 < x < \infty, s \in \{\lambda, \mu : \lambda, \mu \in \mathbb{R}\} \tag{5}$$

$$y(0) = 0.$$

Conditions Sufficient to establish the uniqueness of the potential and boundary conditions in *Theorem* 1, established above will be named *Generalized Theorem* 1.

Now if we name *Problem* 1 for $i \in \{1, 2\}$ to the boundary value problem:

$$-y''(x) + q_i(x) y(x) = s^2 y(x), 0 < x < \infty, \tag{6}$$

$$y_i(0) = 0.$$

and the solutions $\omega_i(s, x)$, respective for $s \in \{\lambda, \mu : \lambda, \mu \in \mathbb{R}\}$. We obtain de reformulation of results in terms of the Transformation Operators.

The final conclusion of [1], formula (42), Thm 1, page 483, Section 4,

$$\left.\begin{aligned} &\omega_2(\lambda,x)=\omega_1(\lambda,x)-\frac{c\omega_1(\mu,x)}{1+c\int_0^x\omega_1(\mu,t)^2\,\mathrm{d}t}\int_0^x\omega_1(\mu,t)\omega_1(\lambda,t)\,\mathrm{d}t,\\ &h_2=\omega_2'(\lambda,0)=h_1-c,\\ &q_2=\frac{1}{2}q_1-c\left\{\frac{\omega_1(\mu,x)}{1+c\int_0^x\omega_1(\mu,t)^2\,\mathrm{d}t}\right\}'. \end{aligned}\right\} \tag{7}$$

and *Generalized Theorem* 1 can rewrite in a formula for a solution $\omega_2(\lambda,x)$ of the *Problem* 2, for $s=\lambda$ of the following manner

$$\omega_2(\lambda,x)=\omega_1(\lambda,x)+\frac{c\omega_2(\mu,x)W(\omega_1(\mu,x),\omega_1(\lambda,x))}{\mu^2-\lambda^2}, \tag{8}$$

where $\omega_1(\mu,x),\omega_1(\lambda,x)$ are solutions of the *Problem* 1, for $s\in\{\lambda,\mu\}$ respectively. The Wronskian

$$W(\omega_1(\mu,x),\omega_1(\lambda,x)):=\omega_1'(\mu,x)\omega_1(\lambda,x)-\omega_1(\mu,x)\omega_1'(\lambda,x)\neq 0,':=\frac{\mathrm{d}}{\mathrm{d}x}.$$

Now in terms of the transformation operators. Let us define

$$\mathbb{I}f:=f,\text{ and }\mathbb{W}f:=\frac{c\omega_2(\mu,x)W(\omega_1(\mu,x),f)}{\mu^2-\lambda^2},\mu\neq\lambda. \tag{9}$$

If $\omega_2(\mu,x)$ is the solution of the *Problem* 2, for $s=\mu$. For $\omega_2(\mu,x)$ solution know of the *Problem* 2, $\omega_1(\mu,x)$ denote a fixed solution of *Problem* 1, in both problems: fixed $s:=\mu$ then the equation above suggests the following transformation operator

$$\omega_2(\lambda,x)=(\mathbb{I}+\mathbb{W})\omega_1(\lambda,x), \tag{10}$$

with the property that a solution $\omega_1(\lambda,x)$ for the *Problem* 1 is transformed in a solution $\omega_2(\lambda,x)$ for the *Problem* 2, for $s:=\lambda$.

In a complementary way to our approach and following a series of problems proposed by [4], Chapter 2, Section 3, *Problem*s: 4, 5, 6. Pages 149-153 that *the formulation of Marchenko* will be named. One get the *Theorem* 1, which is formulated in terms of the transformation operators as follows: if we assumed that $y_1(x,\mu)$ denote a fixed solution de *Problem* 1 for $s:=\mu$, $\varphi_1(\lambda,x)$ denota an arbitrary solution of *Problem* 1 and $y_1(x,\mu)\neq 0$ on $(0,\infty)$, then defined us the operator

$$\mathbb{W}:=\frac{W(y_1,\cdot)}{y_1(\mu^2-\lambda^2)}\Rightarrow\mathbb{W}\varphi_1(\lambda,x)=\frac{W(y_1(x,\mu),\varphi_1(\lambda,x))}{y_1(x,\mu)(\mu^2-\lambda^2)}:=\varphi_2(\lambda,x) \tag{11}$$

where $\varphi_2(\lambda,x)$ is the solution of the equation:

$$-\varphi_2''(\lambda,x)+q_2(x)\varphi_2(\lambda,x)=\lambda^2\varphi_2(\lambda,x),0<x<\infty, \tag{12}$$
$$\varphi_2(\lambda,0)=0.$$

and

$$q_2(x) := -q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}. \tag{13}$$

Two *Examples* for *Theorem* 1 are proposed, in the context of the previous *Marchenko's formulation*. *Example* 1 for the *Problem* 1 corresponds to *Schrödinger Equation on the Half-Line* with a regular potential $q(x) := V(x)$, that is,

$$\int_0^\infty x|q(x)|\mathrm{d}x < \infty, \tag{14}$$

which will be denoted by the abbreviation (*RSEHL*) to which we have added the singular term $\frac{l(l+1)}{x^2}$, obtain us the named *Reduced Radial Schrödinger Equation* (*RRSE*)

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi_l(k,r) + \left[q(r) + \frac{l(l+1)}{r^2}\right]\psi_l(k,r) = k^2\psi_l(k,r), r > 0, \tag{15}$$

$$\psi_l(k,0) = 0.$$

for the partial wave of angular momentum *l* and wave number *k*. See [5], Chapter 1, Sections 1.1-1.3, 1.5, pages 1-10, 13-16. And it will correspond to the *Example* 2 for the *Problem* 2 the (*RSEHL*)

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), 0 < x < \infty, \tag{16}$$

$$\phi(0) = 0.$$

The two proposed examples for the above for *Theorem* 1 are constructed by using *the regular solution* $\varphi_l(s,x)$ and *the Jost solution* $f_l(s,x)$ for (RRSQ) and its properties, the pair of solutions $y_1(\mu,x)$ and $\varphi_1(\lambda,x)$ for the parameters $s=\mu$ and $s=\lambda$ will be built and the asymptotic developments and estimates are made in neighborhoods of zero and infinity respectively. The Wronskian $W(y_1(\mu,x),\varphi_1(\lambda,x))$ and the solution $\varphi_2(\lambda,x) := \mathbb{W}\varphi_1(\lambda,x)$ will be expressed in terms of *the function of Jost* $F_l(k), k=\mu,\lambda$ and its properties. See [5], Chapter 1, Sections 1.4-1.5, pages 11-16. Subsequently, the asymptotic developments and estimates are made in neighborhoods of zero and infinity for the terms

$$\frac{y_1''}{y_1}, \tag{17}$$

$$\frac{\varphi_1'}{\varphi_2}, \tag{18}$$

and

$$\frac{y_1'}{y_1^2}, \tag{19}$$

of potential $q_2(x)$. When performing the previous calculation using the sum of the respective asymptotic developments, the elimination of the singular term of the potencial $q_1(x)$ is obtained in the case of the neighborhood of zero and bounded terms and exponentially decrease fast enought are added for the potential $q_1(x)$ in a neighborhood of infinity.

The elimination of the singular term $\frac{l(l+1)}{x^2}$ when $x \to 0$ and the addition of an exponential decay term and a bounded term when $x \to \infty$ of the potential $q_1(x)$ in the *Example* 1, when applying the transformation operator $\mathbb{W}$ transforms the potential $q_1(x)$ into the regular potential $q_2(x) := q(x) = \mathbb{W}q_1$ in the *Problem* 2, as a consequence of the potential obtained

$$q_2(x) := -q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}$$

in *Theorem* 1. Where it is used strongly the fundamental properties of preserving the initial data $y_1(0) = 0$ in *Problem* 1, as well as, the preservation of spectral and scattering properties in both *Problems* 1 *and* 2, by the operator $\mathbb{W}$. See [1], Section 6, pages 486-487.

The previous fact, constitutes a fundamental part of the previous *Marchenko's formulation* and gives the possibility of extending the $L^1 - L^\infty$ Estimates for the (*RSEHL*) established in [6] was proved with the potential $q(x) := V(x)$ to be real and regular (*RSEHL*) now for the *Reduced Radial Schrödinger Equation* (*RRSE*). That is, the *RRSE* is transformed into the equation *RSEHL* through the transformation operator $\mathbb{W}$. And the following generalization of *Theorem* 2.1 of [6] is obtained.

***Theorem* 2.** (The $L^1 - L^\infty$ estimate). Suppose that V is regular and

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) + \frac{l(l+1)}{x^2}, 0 < x < \infty, \tag{20}$$

$$\phi(0) = 0.$$

Then

$$\left\| \mathrm{e}^{-itH} P_c \right\|_{B(L^1, L^\infty)} \le C \frac{1}{\sqrt{t}}, \tag{21}$$

of moreover, $V \in L^1$, then

$$\left\| \mathrm{e}^{-itH} P_c \right\|_{B(W_{1,1}, W_{1,\infty})} \le C \frac{1}{\sqrt{t}}. \tag{22}$$

The article is organized as follows. In Section 2: Existence of the transformation operator $\mathbb{W}$, *Theorem* 1 is demonstrated in which for the solution given by $\mathbb{W}\varphi_1(\lambda, x) := \varphi_2(\lambda, x)$ one get a formula for the potential

$$q_2(x) = q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}.$$

In Section 3: the two examples: Reducced Radial Schrödinger Equation (*RRSE*) and Schrödinger Equation on the Half-Line (*RSEHL*), the Example for the potential $q_2(x)$ stated in *Theorem* 1 are constructed calculating the terms: $\frac{y_1''}{y_1}, \frac{\varphi_1'}{\varphi_2}, \frac{y_1'}{y_1^2}$ using the regular solution $\varphi_l(s, x)$, the solution of Jost $f_l(s, x)$, the function of Jost $F_l(k)$ and its properties, for (*RRSE*) in the (*Problem* 1). In section 4 the demonstration of *Theorem* 2 is given. In Section 5, we establish the Conclusions and Open *Problems*. An appendix has been added in which, the asymptotic developments and estimates are made in neighborhoods of zero and infinity of the addends that form the potential $q_2(x)$ are obtained.

Warning: In order to respect the notations of the texts [3] [4] and [5] throughout the paper we will use the letters

$$k := s, r := x, V(x) := q(x)$$

the potential regular, indistinctly.

2. Existence of the Transformation Operator $\mathbb{W}$

Theorem **1.** *Let's consider two Sturm-Liouville equations*

$$-y_j'' + q_j(x) y_j = \lambda^2 y_j, j = 1,2, x \in (a,b), b \geq \infty, \tag{23}$$

$q_j(x)$ continuous only in the interior points of (a,b). Consider in particular, the following pair of boundary value problems of Sturm-Liouville on the Half Line

$$-y'' + q_1(x) y = s^2 y, y(0) = 0, \text{ where } s \in \{\lambda, \mu\}, x > 0, \tag{24}$$

and

$$-y'' + q_2(x) y = s^2 y, y(0) = 0 \text{ where } s \in \{\lambda, \mu\}, x > 0, \tag{25}$$

Let $q_1(x), q_2(x)$ are continuous on $(0,\infty)$. If $y_1(x,\mu) := y_1$ is a fixed solution of the first Equation (2) for $s = \mu$ and let $\varphi_1(\lambda, x) := \varphi_1$ an arbitrary solution of (2) for $s = \lambda$, then

$$-y_1'' + q_1(x) y_1 = \mu^2 y_1, y_1(0,\mu) = 0 \text{ and } -\varphi_1'' + q_1(x)\varphi_1 = \lambda^2 \varphi_1, \varphi_1(0,\lambda) = 0.$$

Suppose

$$y_1(x,\mu) \neq 0, \forall x > 0, \tag{26}$$

If

$$\varphi_2(\lambda, x) := \frac{W(y_1(x,\mu), \varphi_1(\lambda, x))}{y_1(x)(\mu^2 - \lambda^2)}, \mu \neq \lambda, \tag{27}$$

is solution of (3) for $s = \lambda$ where the Wronskian

$$W(y_1(x,\mu), \varphi_1(\lambda,x)) := y_1(x,\mu)\varphi_1'(\lambda,x) - y_1'(x,\mu)\varphi_1(\lambda,x) \neq 0, \tag{28}$$

$' := \frac{\mathrm{d}}{\mathrm{d}x}$. Then $\varphi_2(\lambda, x)$ satisfies the equation

$$-\varphi_2'' + q_2(x)\varphi_2 = \lambda^2 \varphi_2, \varphi_2(0,\lambda) = 0, \tag{29}$$

where

$$q_2(x) = \mu^2 - 2q_1 + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2} \tag{30}$$

Remark 2. *In order to facilitate the reading we will carry out the abuse of the*

$\varphi_1(\lambda,x) := \varphi_1 := \varphi_1(x)$ *and* $y_1(x,\mu) := y_1 := y_1(x)$ *notation.*

Proof. Since $\left\{\frac{y_1'}{y_1}\right\}' = \frac{y_1 y_1'' - [y_1']^2}{y_1^2} = \frac{y_1''}{y_1} - \frac{[y_1']^2}{y_1^2} = \frac{y_1''}{y_1} - \frac{[y_1']^2}{y_1^2} = \frac{y_1''}{y_1} + y_1'\left(\frac{1}{y_1}\right)'$, then

$$\left\{\frac{y_1'}{y_1}\right\}' = \frac{y_1''}{y_1} + y_1'\left(\frac{1}{y_1}\right)', \tag{31}$$

and $-y_1'' + q_1(x) y_1 = \mu^2 y_1$ then

$$\frac{y_1''}{y_1} = q_1(x) - \mu^2. \tag{32}$$

Also

$$\begin{aligned} q_1(x) - 2\left\{\frac{y_1'}{y_1}\right\}' &= q_1(x) - 2\frac{y_1''}{y_1} - 2y_1'\left(\frac{1}{y_1}\right)' \\ &= q_1(x) - 2\left[q_1(x) - \mu^2\right] - 2y_1'\left(\frac{1}{y_1}\right)' \\ &= -q_1(x) + 2\mu^2 - 2y_1'\left(\frac{1}{y_1}\right)', \end{aligned}$$

that is,

$$q_1(x) - 2\left\{\frac{y_1'}{y_1}\right\}' = -q_1(x) + 2\mu^2 - 2y_1'\left(\frac{1}{y_1}\right)'. \tag{33}$$

Since $\varphi_2(\lambda,x) := \frac{W(y_1(x), \varphi_1(\lambda,x))}{y_1(x)(\mu^2 - \lambda^2)}$ then let's calculate

$$\varphi_2'(\lambda,x) := \left(\frac{W(y_1(x), \varphi_1(\lambda,x))}{y_1(x)(\mu^2 - \lambda^2)}\right)' = \frac{1}{(\mu^2 - \lambda^2)}\left(\frac{W(y_1(x), \varphi_1(x))}{y_1(x)}\right)'$$

$$= \frac{1}{(\mu^2 - \lambda^2)} \frac{y_1(x) W'(y_1(x), \varphi_1(x)) - W(y_1(x), \varphi_1(x)) y_1'(x)}{y_1^2(x)}$$

$$= \frac{1}{(\mu^2 - \lambda^2)} \frac{y_1(x) W'(y_1(x), \varphi_1(x)) - W(y_1(x), \varphi_1(x)) y_1'(x)}{y_1^2(x)}$$

$$= \frac{1}{(\mu^2 - \lambda^2)} \frac{W'(y_1(x), \varphi_1(x))}{y_1(x)} - \frac{1}{(\mu^2 - \lambda^2)} \frac{W(y_1(x), \varphi_1(x))}{y_1(x)} \times \frac{y_1'(x)}{y_1(x)}$$

$$= \frac{(\mu^2 - \lambda^2)}{(\mu^2 - \lambda^2)} \frac{\{\varphi_2(\lambda,x) y_1\}'}{y_1(x)} - \varphi_2(\lambda,x)\frac{y_1'(x)}{y_1(x)} = \frac{\{\varphi_2(\lambda,x) y_1\}'}{y_1(x)} - \varphi_2(\lambda,x)\frac{y_1'(x)}{y_1(x)}$$

that is

$$\varphi_2'(\lambda,x) = \frac{\{\varphi_2(\lambda,x) y_1\}'}{y_1(x)} - \frac{\varphi_2(\lambda,x) y_1'(x)}{y_1(x)}, \tag{34}$$

Using (34) let's calculate

$$\varphi_2''(\lambda,x)=\left[\frac{\{\varphi_2(\lambda,x)y_1\}'}{y_1(x)}\right]'-\left[\frac{\varphi_2(\lambda,x)y_1'(x)}{y_1(x)}\right]'$$

$$=\frac{y_1\{\varphi_2(\lambda,x)y_1\}''-\{\varphi_2(\lambda,x)y_1\}'y_1'}{y_1^2(x)}$$

$$-\frac{y_1(x)\left[\varphi_2(\lambda,x)y_1'(x)\right]'-\varphi_2(\lambda,x)y_1'(x)y_1'(x)}{y_1^2(x)}$$

$$=\frac{y_1\{\varphi_2(\lambda,x)y_1'+\varphi_2'(\lambda,x)y_1\}'-\{\varphi_2(\lambda,x)y_1'+\varphi_2'(\lambda,x)y_1\}y_1'}{y_1^2(x)}$$

$$-\frac{y_1(x)\left[\varphi_2(\lambda,x)y_1''(x)+\varphi_2'(\lambda,x)y_1'(x)\right]-\varphi_2(\lambda,x)y_1'(x)y_1'(x)}{y_1^2(x)}$$

$$=\frac{y_1\left\{\left[\varphi_2(\lambda,x)y_1'\right]'+\left[\varphi_2'(\lambda,x)y_1\right]'\right\}-\varphi_2(\lambda,x)[y_1']^2-\varphi_2'(\lambda,x)y_1y_1'}{y_1^2(x)}$$

$$-\frac{y_1(x)\varphi_2(\lambda,x)y_1''(x)+y_1(x)\varphi_2'(\lambda,x)y_1'(x)-\varphi_2(\lambda,x)\left[y_1'(x)\right]^2}{y_1^2(x)}$$

$$=\frac{y_1\{\varphi_2(\lambda,x)y_1''+\varphi_2'(\lambda,x)y_1'+\varphi_2'(\lambda,x)y_1'+\varphi_2''(\lambda,x)y_1}{y_1^2(x)}$$

$$+\frac{\varphi_2''(\lambda,x)y_1+\varphi_2'(\lambda,x)y_1'}{y_1^2(x)}\frac{-\varphi_2(\lambda,x)[y_1']^2-\varphi_2'(\lambda,x)y_1y_1'}{y_1^2(x)}$$

$$-\frac{y_1(x)\varphi_2(\lambda,x)y_1''(x)+y_1(x)\varphi_2'(\lambda,x)y_1'(x)-\varphi_2(\lambda,x)\left[y_1'(x)\right]^2}{y_1^2(x)}$$

$$=\frac{\varphi_2(\lambda,x)y_1y_1''+3\varphi_2'(\lambda,x)y_1y_1'+2\varphi_2''(\lambda,x)y_1^2}{y_1^2(x)}$$

$$-\frac{\varphi_2(\lambda,x)[y_1']^2+\varphi_2'(\lambda,x)y_1y_1'}{y_1^2(x)}$$

$$-\frac{y_1(x)\varphi_2(\lambda,x)y_1''(x)+y_1(x)\varphi_2'(\lambda,x)y_1'(x)-\varphi_2(\lambda,x)\left[y_1'(x)\right]^2}{y_1^2(x)}$$

$$=\frac{\varphi_2(\lambda,x)y_1y_1''+3\varphi_2'(\lambda,x)y_1y_1'+2\varphi_2''(\lambda,x)y_1^2-\varphi_2(\lambda,x)[y_1']^2-\varphi_2'(\lambda,x)y_1y_1'}{y_1^2(x)}$$

$$-\frac{y_1(x)\varphi_2(\lambda,x)y_1''(x)+y_1(x)\varphi_2'(\lambda,x)y_1'(x)-\varphi_2(\lambda,x)\left[y_1'(x)\right]^2}{y_1^2(x)}$$

$$=\frac{2\varphi_2(\lambda,x)y_1y_1''+\varphi_2'(\lambda,x)y_1y_1'+2\varphi_2''(\lambda,x)y_1^2}{y_1^2(x)}$$

$$=\frac{2\varphi_2(\lambda,x)y_1''}{y_1(x)}+\frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)}+2\varphi_2''(\lambda,x)$$

That is

$$\varphi_2''(\lambda,x)=\frac{2\varphi_2(\lambda,x)y_1''}{y_1(x)}+\frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)}+2\varphi_2''(\lambda,x) \tag{35}$$

then

$$-\varphi_2''(\lambda,x)=\frac{2\varphi_2(\lambda,x)y_1''}{y_1(x)}+\frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)}. \tag{36}$$

Now using (31), (32), (33), Equation (36) can be written as

$$\begin{aligned}
-\varphi_2''(\lambda,x)&=\frac{\varphi_2(\lambda,x)y_1''+\varphi_2(\lambda,x)y_1''+\varphi_2'(\lambda,x)y_1'}{y_1(x)}\\
&=\frac{\varphi_2(\lambda,x)y_1''+\{\varphi_2(\lambda,x)y_1'\}'}{y_1(x)}\\
&=\frac{\varphi_2(\lambda,x)y_1''}{y_1(x)}+\frac{\{\varphi_2(\lambda,x)y_1'\}'}{y_1(x)}\\
&=\varphi_2(\lambda,x)\{q_1(x)-\mu^2\}+\frac{\varphi_2(\lambda,x)y_1''+\varphi_2'(\lambda,x)y_1'}{y_1(x)}\\
&=\varphi_2(\lambda,x)\{q_1(x)-\mu^2\}+\frac{y_1''}{y_1(x)}\varphi_2(\lambda,x)+\varphi_2(\lambda,x)\{q_1(x)-\mu^2\}\\
&\quad+\{q_1(x)-\mu^2\}\varphi_2(\lambda,x)+\frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)}\\
&=2\varphi_2(\lambda,x)\{q_1(x)-\mu^2\}+\frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)}
\end{aligned}$$

Therefore

$$-\varphi_2''(\lambda,x)=2\varphi_2(\lambda,x)\{q_1(x)-\mu^2\}+\frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)}. \tag{37}$$

One can expressed $\varphi_2'(\lambda,x)$ in terms of Wronskian

$$\varphi_2'(\lambda,x)=\frac{1}{\mu^2-\lambda^2}\frac{W'(y_1,\varphi_1)}{y_1}-\frac{\varphi_2}{y_1}, \tag{38}$$

so

$$\varphi_2'(\lambda,x)=\left(\frac{W'(y_1,\varphi_1)}{W(y_1,\varphi_1)}-\frac{1}{y_1}\right)\varphi_2(\lambda,x), \tag{39}$$

then (26) and (28) imply that

$$\varphi_2(\lambda,x)\neq 0, \forall x\in(0,\infty), \tag{40}$$

and

$$\frac{\varphi_2'(\lambda,x)}{\varphi_2(\lambda,x)}=\frac{W'(y_1,\varphi_1)}{W(y_1,\varphi_1)}-\frac{1}{y_1}. \tag{41}$$

From Equation (27)

$$
\begin{aligned}
W'(y_1,\varphi_1) = y_1''\varphi_1 - y_1\varphi_1'' &= y_1\left(\left[\frac{y_1''}{y_1}\right]\varphi_1 - \varphi_1''\right) \\
&= y_1\left(\left[q_1(x)-\mu^2\right]\varphi_1 - \varphi_1''\right) \\
&= y_1\left(\left[q_1(x)-\mu^2\right]\varphi_1 + \left[\lambda^2 - q_1(x)\right]\varphi_1\right) \\
&= y_1\left(-\mu^2+\lambda^2\right)\varphi_1.
\end{aligned}
$$

Then

$$
W'(y_1,\varphi_1) = -y_1\left(\mu^2-\lambda^2\right)\varphi_1. \tag{42}
$$

And from (41) it is obtained

$$
\begin{aligned}
\varphi_2'(\lambda,x) &= \left(\frac{W'(y_1,\varphi_1)}{W(y_1,\varphi_1)} - \frac{1}{y_1}\right)\varphi_2(\lambda,x) \\
&= \left(-\frac{y_1\left(\mu^2-\lambda^2\right)\varphi_1}{W(y_1,\varphi_1)} - \frac{1}{y_1}\right)\varphi_2(\lambda,x) \\
&= \left(-\frac{\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)} - \frac{1}{y_1}\right)\varphi_2(\lambda,x) \\
&= -\varphi_1(\lambda,x) - \frac{\varphi_2(\lambda,x)}{y_1},
\end{aligned}
$$

that is

$$
\varphi_2'(\lambda,x) = -\varphi_1(\lambda,x) - \frac{\varphi_2(\lambda,x)}{y_1}. \tag{43}
$$

Now, replacing (43) in (37) we get

$$
\begin{aligned}
-\varphi_2''(\lambda,x) &= 2\varphi_2(\lambda,x)\left\{q_1(x)-\mu^2\right\} + \frac{\varphi_2'(\lambda,x)y_1'}{y_1(x)} \\
&= 2\varphi_2(\lambda,x)\left\{q_1(x)-\mu^2\right\} + \frac{y_1'}{y_1(x)}\left[-\varphi_1(\lambda,x) - \frac{\varphi_2(\lambda,x)}{y_1}\right] \\
&= 2\varphi_2(\lambda,x)\left\{q_1(x)-\mu^2\right\} - \frac{y_1'\varphi_1(\lambda,x)}{y_1(x)} - \frac{\varphi_2(\lambda,x)y_1'}{y_1^2} \\
&= \varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right] - \frac{y_1'}{y_1^2}\right\} - \frac{y_1'\varphi_1(\lambda,x)}{y_1(x)}.
\end{aligned}
$$

Therefore

$$
-\varphi_2''(\lambda,x) = \varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right] - \frac{y_1'}{y_1^2}\right\} - \frac{y_1'\varphi_1(\lambda,x)}{y_1(x)}. \tag{44}
$$

Using the representation for a given solution $\varphi_2(\lambda,x)$ in (28)

$$
\varphi_2(\lambda,x) := \frac{W\left(y_1(x),\varphi_1(x)\right)}{y_1(x)\left(\mu^2-\lambda^2\right)},
$$

we get the equation

$$
\frac{y_1'\varphi_1(\lambda,x)}{y_1(x)} = \frac{\varphi_2(\lambda,x)\left(\mu^2-\lambda^2\right)y_1'\varphi_1(\lambda,x)}{W\left(y_1(x),\varphi_1(x)\right)}, \tag{45}
$$

then the right member of Equation (44) can be written as

$$\varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right]-\frac{y_1'}{y_1^2}\right\}-\frac{y_1'\varphi_1(\lambda,x)}{y_1(x)}$$
$$=\varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right]-\frac{y_1'}{y_1^2}\right\}-\frac{y_1'\varphi_1(\lambda,x)\left(\mu^2-\lambda^2\right)\varphi_2(\lambda,x)}{W\left(y_1(x),\varphi_1(x)\right)}$$
$$=\varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right]-\frac{y_1'}{y_1^2}-\frac{y_1'\varphi_1(\lambda,x)\left(\mu^2-\lambda^2\right)}{W\left(y_1(x),\varphi_1(x)\right)}\right\}.$$

That is

$$-\varphi_2''(\lambda,x)=\varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right]-\frac{y_1'}{y_1^2}-\frac{y_1'\varphi_1(\lambda,x)\left(\mu^2-\lambda^2\right)}{W\left(y_1(x),\varphi_1(x)\right)}\right\}. \tag{46}$$

Let us observe that

$$\varphi_2(\lambda,x)=\frac{W\left(y_1,\varphi_1\right)}{\left(\mu^2-\lambda^2\right)y_1(x)}\neq 0,\forall x\in(0,\infty),$$

from (26) and (28). Then

$$\frac{1}{\varphi_2(\lambda,x)}=\frac{\left(\mu^2-\lambda^2\right)y_1(x)}{W\left(y_1,\varphi_1\right)}\Leftrightarrow\frac{\left(\mu^2-\lambda^2\right)y_1'(x)\varphi_1(\lambda,x)}{W\left(y_1,\varphi_1\right)}, \tag{47}$$

since $\varphi_2(\lambda,x)$ is solution for the Equation (25) for $s=\lambda$. Rewriting (24) one get

$$q_2(x)-\lambda^2-\frac{\varphi_2''}{\varphi_2}=0 \text{ and } q_1(x)-\mu^2=\frac{y_1''}{y_1} \tag{48}$$

Then we can rewrite the right term of the Equation (46) as

$$\varphi_2(\lambda,x)\left\{2\left[q_1(x)-\mu^2\right]-\frac{y'}{y_1^2}-\frac{y_1'\varphi_1(\lambda,x)\left(\mu^2-\lambda^2\right)}{W\left(y_1(x),\varphi_1(x)\right)}\right\}$$
$$=\varphi_2(\lambda,x)\left\{2\left[q_2(x)-\lambda^2-\frac{\varphi_2''}{\varphi_2}+\frac{y_1''}{y_1}\right]-\frac{y_1'}{y_1^2}-\frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right\},$$

associating the terms and using the equations given in (24) and (25) one get

$$\varphi_2(\lambda,x)\left\{2\left[q_2(x)-\lambda^2-\frac{\varphi_2''}{\varphi_2}+\frac{y_1''}{y_1}\right]-\frac{y_1'}{y_1^2}-\frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right\}$$
$$=\varphi_2(\lambda,x)\left\{2q_2(x)-2\lambda^2+2\left(\frac{y_1''}{y_1}-\frac{\varphi_2''}{\varphi_2}\right)-\frac{y_1'}{y_1^2}-\frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right\}$$
$$=\varphi_2(\lambda,x)\left\{\lambda^2-\left(-2q_2(x)+3\lambda^2-2\left[\frac{y_1''}{y_1}-\frac{\varphi_2''}{\varphi_2}\right]+\frac{y_1'}{y_1^2}+\frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right)\right\}$$
$$=\varphi_2(\lambda,x)\left\{\lambda^2-\left(\lambda^2-2\left[q_2(x)-\lambda^2\right]-2\left[q_1-q_2-\left(\mu^2-\lambda^2\right)\right]\right.\right.$$
$$\left.\left.+\frac{y_1'}{y_1^2}+\frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(\lambda^2 - 2q_2(x) + 2\lambda^2 - 2q_1 + 2q_2 + 2\left(\mu^2-\lambda^2\right)\right.\right.$$

$$\left.\left. + \frac{y_1'}{y_1^2} + \frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(\lambda^2 - 2\left[q_1-\mu^2\right] + \frac{y_1'}{y_1^2} + \frac{y_1'\varphi_1(\lambda,x)}{\varphi_2(\lambda,x)y_1(x)}\right)\right\}\varphi_2(\lambda,x)$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 - \left[\mu^2-\lambda^2\right] + \frac{y_1'}{y_1^2} + \frac{y_1'}{y_1}\frac{\varphi_1 y_1\left(\mu^2-\lambda^2\right)}{W(y_1,\varphi_1)}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 - \left[\mu^2-\lambda^2\right] + \frac{y_1'}{y_1^2} + \frac{y_1'\varphi_1\left(\mu^2-\lambda^2\right)}{W(y_1,\varphi_1)}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \left[\mu^2-\lambda^2\right]\left\{-1 + \frac{y_1'\varphi_1}{W(y_1,\varphi_1)}\right\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \left[\mu^2-\lambda^2\right]\left\{-1 + \frac{y_1'\varphi_1}{y_1'\varphi_1 - y_1\varphi_1'}\right\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \left[\mu^2-\lambda^2\right]\left\{\frac{-y_1'\varphi + y_1\varphi_1' + y_1'\varphi_1}{y_1'\varphi_1 - y_1\varphi_1'}\right\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \left[\mu^2-\lambda^2\right]\left\{\frac{y_1\varphi_1'}{y_1'\varphi_1 - y_1\varphi_1'}\right\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \left[\mu^2-\lambda^2\right]\left\{\frac{y_1\varphi_1'}{W(y_1,\varphi_1)}\right\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \frac{\left[\mu^2-\lambda^2\right]y_1}{W(y_1,\varphi_1)}\{\varphi_1'\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 + \mu^2 + \frac{1}{\varphi_2}\{\varphi_1'\} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-2q_1 - \frac{y_1''}{y_1} + q_1 + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}\right)\right\}$$

$$= \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}\right)\right\}$$

Then we yield the equation sought

$$-\varphi_2''(\lambda,x) = \varphi_2(\lambda,x)\left\{\lambda^2 - \left(-q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}\right)\right\}, \tag{49}$$

that is

$$-\varphi_2''(\lambda,x) + \left(-q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}\right)\varphi_2(\lambda,x) = \lambda^2\varphi_2(\lambda,x), \tag{50}$$

if

$$q_2(x) := -q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}, \tag{51}$$

one yield to the initial Equation (29)

$$-\varphi_2''(\lambda, x) + q_2(x)\varphi_2(\lambda, x) = \lambda^2 \varphi_2(\lambda, x).$$

and the potential (30).

Since

$$-\frac{y_1''}{y_1} + q_1 = \mu^2 \Leftrightarrow -\frac{y_1''}{y_1} - q_1 = \mu^2 - 2q_1,$$

then the potential $q_2(x)$ can also be represented as

$$q_2(x) = \mu^2 - 2q_1 + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}. \tag{52}$$

it is uniquely determined.

Finally, according to (27) we define the corresponding transformation operator as

$$\mathbb{W} := \frac{W(y_1(x,\mu), \cdot)}{y_1(x)(\mu^2 - \lambda^2)} \Rightarrow \mathbb{W}\varphi_1(x) := \frac{W(y_1(x,\mu), \varphi_1(x))}{y_1(x)(\mu^2 - \lambda^2)} = \varphi_2(x) \tag{53}$$

3. The Two Examples: Reducced Radial Schrödinger Equation and Schrödinger Equation on the Half-Line

The named *Problem* 1 of *Theorem* 1 of the preceding section is equivalent to the following system of boundary value problems on the Half Line

$$\begin{cases} -y_1''(x,\mu) + \left\{ q(x) + \dfrac{l(l+1)}{x^2} \right\} y_1(x,\mu) = \mu^2 y_1(x,\mu), 0 < x < \infty, \\ y_1(0,\mu) = 0. \\ -\varphi_1''(x,\lambda) + \left\{ q(x) + \dfrac{l(l+1)}{x^2} \right\} \varphi_1(x,\lambda) = \lambda^2 \varphi_1(x,\lambda), 0 < x < \infty, \\ \varphi_1(0,\lambda) = 0. \end{cases} \tag{54}$$

The previous system corresponds to the reducced radial Schrödinger equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi_l(k,r) + \left[q(r) + \frac{l(l+1)}{r^2} \right] \psi_l(k,r) = k^2 \psi_l(k,r). \tag{55}$$

for the partial wave of angular momentum *l* and wave number *k*.

This equation and the solution has the following properties:

1) The potential $q(r)$ is regular, that is

$$\int_0^\infty r|q(r)|\mathrm{d}r < \infty, \tag{56}$$

2) The solution ψ_l exists and it is the only one which vanishes at the origin.

3) For physical reasons know in scattering theory, the physical wave function ψ_l, also must vanish at the origin.

4) The asymptotic behavior is of ψ_l

$$\psi_l \approx \exp(i\delta_l)\sin\left(kr - \frac{\pi}{2}\left(\lambda - \frac{1}{2}\right) + \delta_l\right), r \to \infty, \tag{57}$$

where δ_l, is the phase shift, is a real quantity which depends of energy k and the momentum l, that is, $\delta_l = \delta_l(k,l)$.

5) The momentum angular fixed $\lambda = l_0 + \frac{1}{2}$ and all (positive) values of $k^2 := \lambda^2$.

6) The Jost solution becomes:

$$f_l(k,r) \underset{\substack{r\to\infty \\ k\neq 0}}{=} i^l e^{ikr} + e^{-\operatorname{Im} kr} o(1) \tag{58}$$

then

$$f_l'(k,r) \underset{\substack{r\to\infty \\ k\neq 0}}{=} i^{l+1} k e^{ikr} + e^{-\operatorname{Im} kr} o(1) \tag{59}$$

7) In the case $\operatorname{Im} k \geq 0$, the Jost solution is given by

$$f_l(k,r) \underset{\substack{r\to\infty \\ k\neq 0}}{=} i^l e^{ikr} \text{ and } f_l'(k,r) \underset{\substack{r\to\infty \\ k\neq 0}}{=} i^{l+1} k e^{ikr}. \tag{60}$$

8) The regular solution $\varphi_l(k,r)$, near $r = 0$, is given by the boundary condition

$$\varphi_l(k,r) \underset{r\to 0}{=} \frac{r^{l+1}}{(2l+1)!!} \tag{61}$$

then

$$\varphi_l'(k,r) \underset{\substack{r\to 0 \\ \lambda\neq 0}}{=} \frac{(l+1)r^l}{(2l+1)!!}. \tag{62}$$

See [Ch-S] [5], Chapter 1, Sections: 1.1-1.5, 1.7 pages 1-16, 19. If

$$r := x, k := s, \tag{63}$$

the solution $\psi_l(k,r)$ for

$$-\psi_l''(k,x) + q_1(x)\psi_l(k,x) = \lambda^2 \psi_l(k,x), x > 0, \tag{64}$$
$$\psi_l(k,0) = 0.$$

where

$$q_1(x) := q(x) + \frac{l(l+1)}{x^2}, \tag{65}$$

is proposed as

$$\psi_l(\lambda,x) := \begin{cases} \varphi_l(\lambda,x), & \text{if } 0 < x < 1 \\ f_l(\lambda,x), & \text{if } x > 1. \end{cases} \tag{66}$$

or

$$\psi_l(\lambda,x) := \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1 \\ i^l e^{i\lambda x} + e^{-\operatorname{Im} \lambda x}, & \text{if } x > 1. \end{cases} \tag{67}$$

Since

$$\frac{\psi_l''(\lambda,x)}{\psi_l(\lambda,x)} = q(x) + \frac{l(l+1)}{x^2} - \lambda^2, \text{ for } x>0, \tag{68}$$

it follows that

$$\frac{\psi_l''(\lambda,x)}{\psi_l(\lambda,x)} = \begin{cases} q(x) + \dfrac{l(l+1)}{x^2} - \lambda^2, & 0<x<1 \\ q(x) + \dfrac{l(l+1)}{x^2} - \lambda^2, & x>1. \end{cases} \tag{69}$$

Next let $y_j(x,\mu)$ denote a fixed solution of the *j*-th equation for $k=\mu$, and let $\phi_1(k,x)$ denote an arbitrary solution of the first equation. Then for $\operatorname{Im} k \geq 0$

$$\begin{aligned} \phi_1(k,x) = y_1(k,x) &:= \begin{cases} \varphi_l(k,x), & \text{if } 0<x<1, \\ f_l(k,x), & \text{if } x>1. \end{cases} \\ &= \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^l e^{ikx} + e^{-\operatorname{Im} kx} o(1), & \text{if } x>1. \end{cases} \end{aligned} \tag{70}$$

and

$$\begin{aligned} \phi_1'(k,x) = y_1'(k,x) &:= \begin{cases} \varphi_l'(k,x), & \text{if } 0<x<1, \\ f_l'(k,x), & \text{if } x>1. \end{cases} \\ &= \begin{cases} \dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^{l+1} k e^{ikr} - \operatorname{Im} k e^{-\operatorname{Im} kx} o(1), & \text{if } x>1. \end{cases} \end{aligned} \tag{71}$$

where $\phi_1(k,x), y_1(k,x)$ are solutions for the equation $j=1$ for $k=\mu$ and $k=\lambda$ respectively.

Then

$$\phi_1(\lambda,x) = \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^l e^{i\lambda x}, & \text{if } x>1. \end{cases} \tag{72}$$

and

$$\phi_1(\mu,x) = \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^l e^{i\mu x}, & \text{if } x>1. \end{cases} \tag{73}$$

then

$$\phi_1'(\lambda,x) = \begin{cases} \dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^{l+1} \lambda e^{i\lambda x}, & \text{if } x \geq 1. \end{cases} \tag{74}$$

$$\phi_1'(\mu,x)=\begin{cases}\dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}, & \text{if } x\geq 1.\end{cases} \tag{75}$$

for

$$y_1(x,\mu)=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\mu x}, & \text{if } x\geq 1.\end{cases} \tag{76}$$

and

$$y_1(x,\lambda)=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\lambda x}, & \text{if } x\geq 1.\end{cases} \tag{77}$$

then

$$y_1'(x,\mu)=\begin{cases}\dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}, & \text{if } x> 1.\end{cases} \tag{78}$$

and

$$y_1'(x,\lambda)=\begin{cases}\dfrac{(l+1)x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\lambda e^{i\lambda x}, & \text{if } x> 1.\end{cases} \tag{79}$$

And

$$\phi_1(x,\lambda)=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\lambda x}, & \text{if } x\geq 1.\end{cases}, \tag{80}$$

$$\frac{1}{\phi_1(\lambda,x)}=\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x\geq 1.\end{cases}, \tag{81}$$

Now, from Equation (74),

$$\begin{aligned}\phi_1''(\lambda,x)&=\begin{cases}\dfrac{(l+1)l}{(2l+1)!!}x^{l-1}, & \text{if } 0<x<1,\\ i^{l+1}\lambda i\lambda e^{i\lambda x}, & \text{if } x\geq 1.\end{cases}\\ &=\begin{cases}\dfrac{(l+1)l}{(2l+1)!!}x^{l-1}, & \text{if } 0<x<1,\\ i^l i^2\lambda^2 e^{i\lambda x}, & \text{if } x\geq 1.\end{cases}\\ &=\begin{cases}\dfrac{(l+1)l}{(2l+1)!!}x^{l-1}, & \text{if } 0<x<1,\\ -i^l\lambda^2 e^{i\lambda x}, & \text{if } x\geq 1.\end{cases}\end{aligned}$$

That is,

$$\phi_1''(\lambda,x)=\begin{cases}\dfrac{(l+1)l}{(2l+1)!!}x^{l-1}, & \text{if } 0<x<1,\\ -i^l\lambda^2\mathrm{e}^{i\lambda x}, & \text{if } x\geq 1.\end{cases} \tag{82}$$

Then

$$\frac{\phi_1''(\lambda,x)}{\phi_1(\lambda,x)}=\begin{cases}\dfrac{(l+1)l}{(2l+1)!!}x^{l-1}, & \text{if } 0<x<1,\\ -i^l\lambda^2\mathrm{e}^{i\lambda x}, & \text{if } x\geq 1.\end{cases}\times\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{1}{i^l\mathrm{e}^{i\lambda x}}, & \text{if } x\geq 1.\end{cases}$$

$$=\begin{cases}\dfrac{(l+1)l}{(2l+1)!!}x^{l-1}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ -i^l\lambda^2\mathrm{e}^{i\lambda x}\left(\dfrac{1}{i^l\mathrm{e}^{i\lambda x}}\right), & \text{if } x\geq 1.\end{cases}$$

$$=\begin{cases}\dfrac{l(l+1)}{x^2}, & \text{if } 0<x<1,\\ -\lambda^2, & \text{if } x\geq 1.\end{cases}$$

That is

$$\frac{\phi_1''(\lambda,x)}{\phi_1(\lambda,x)}=\begin{cases}\dfrac{l(l+1)}{x^2}, & \text{if } 0<x<1,\\ -\lambda^2, & \text{if } x\geq 1.\end{cases} \tag{83}$$

Now since

$$q_2:=-q_1-\frac{y_1''}{y_1}+\frac{\varphi_1'}{\varphi_2}+\frac{y_1'}{y_1^2},$$

the corresponding summations $\dfrac{y_1''}{y_1},\dfrac{\varphi_1'}{\varphi_2},\dfrac{y_1'}{y_1^2}$ of the potential $q_2(x)$ (calculated in **Appendix**), Are given by

$$\frac{y_1''}{y_1}=\begin{cases}\dfrac{l(l+1)}{x^2}, & \text{if } 0<x<1,\\ -\mu^2, & \text{if } x\geq 1.\end{cases} \tag{84}$$

and

$$\frac{y_1'}{y_1^2}=\begin{cases}\dfrac{(2l+1)!!(l+1)}{x^2}, & \text{if } 0<x<1,\\ \dfrac{\mu}{i^{l-1}\mathrm{e}^{i\mu x}}, & \text{if } x\geq 1.\end{cases} \tag{85}$$

Next we defined the solution

$$\phi_2(\lambda,x):=\frac{W\big(y_1(x,\mu),\phi_1(\lambda,x)\big)}{y_1(x,\mu)\big(\mu^2-\lambda^2\big)}.$$

In terms of the Jost function

$$F_l(k)=(-k)^l\,W\big(f_l(k,x),\varphi_l(k,x)\big), \tag{86}$$

for

$$k \in \{\lambda, \mu\}, \tag{87}$$

Let's start with the Wroskin

$$W\left(y_1(x,\mu),\phi_1(\lambda,x)\right) = y_1(x,\mu)\phi_1'(\lambda,x) - y_1'(x,\mu)\phi_1(\lambda,x) \tag{88}$$

then

$$(-\lambda)^{-l} F_l(\lambda) = W\left(y_1(x,\lambda),\phi_1(\lambda,x)\right) = y_1(x,\lambda)\phi_1'(\lambda,x) - y_1'(x,\lambda)\phi_1(\lambda,x)$$

and

$$(-\mu)^{-l} F_l(\mu) = W\left(y_1(x,\mu),\phi_1(\mu,x)\right) = y_1(x,\mu)\phi_1'(\mu,x) - y_1'(x,\mu)\phi_1(\mu,x),$$

that is

$$\begin{cases} y_1(x,\lambda)\phi_1'(\lambda,x) - y_1'(x,\lambda)\phi_1(\lambda,x) = (-\lambda)^{-l} F_l(\lambda), \\ y_1(x,\mu)\phi_1'(\mu,x) - y_1'(x,\mu)\phi_1(\mu,x) = (-\mu)^{-l} F_l(\mu). \end{cases} \tag{89}$$

And

$$y_1(x,\lambda)\phi_1'(\lambda,x) - y_1'(x,\lambda)\phi_1(\lambda,x) = (-\lambda)^{-l} F_l(\lambda), \tag{90}$$

implies

$$\phi_1'(\lambda,x) = \frac{(-\lambda)^{-l} F_l(\lambda) + y_1'(x,\lambda)\phi_1(\lambda,x)}{y_1(x,\lambda)}, \tag{91}$$

and

$$y_1(x,\mu)\phi_1'(\lambda,x) = \frac{y_1(x,\mu)}{y_1(x,\lambda)}\left[(-\lambda)^{-l} F_l(\lambda) + y_1'(x,\lambda)\phi_1(\lambda,x)\right]. \tag{92}$$

Similary

$$W\left(y_1(x,\mu),\phi_1(\mu,x)\right) := y_1(x,\mu)\phi_1'(\mu,x) - y_1'(x,\mu)\phi_1(\mu,x) = (-\mu)^{-l} F_l(\mu), \tag{93}$$

that is

$$y_1'(x,\mu) = \frac{y_1(x,\mu)\phi_1'(\mu,x) - (-\mu)^{-l} F_l(\mu)}{\phi_1(\mu,x)}. \tag{94}$$

so

$$\phi_1(\lambda,x) y_1'(x,\mu) = \frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\left[y_1(x,\mu),\phi_1'(\mu,x) - (-\mu)^{-l} F_l(\mu)\right] \tag{95}$$

The

$$\begin{aligned} &W\left(y_1(x,\mu),\phi_1(\lambda,x)\right) \\ &= y_1(x,\mu)\phi_1'(\lambda,x) - y_1'(x,\mu)\phi_1(\lambda,x) \\ &= \frac{y_1(x,\mu)}{y_1(x,\lambda)}\left[(-\lambda)^{-l} F_l(\lambda) + y_1'(x,\lambda)\phi_1(\lambda,x)\right] \\ &\quad - \frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\left[y_1(x,\mu)\phi_1'(\mu,x) - (-\mu)^{-l} F_l(\mu)\right] \\ &= (-\lambda)^{-l} F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)} + y_1'(x,\lambda)\phi_1(\lambda,x)\frac{y_1(x,\mu)}{y_1(x,\lambda)} \end{aligned}$$

$$
\begin{aligned}
&-y_1(x,\mu)\phi_1'(\mu,x)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+\frac{y_1'(x,\lambda)\phi_1(\lambda,x)y_1(x,\mu)}{y_1(x,\lambda)}\\
&\quad-\frac{y_1(x,\mu)\phi_1'(\mu,x)\phi_1(\lambda,x)}{\phi_1(\mu,x)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)\\
&\quad-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\
&\quad+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)
\end{aligned}
$$

That is

$$
\begin{aligned}
&W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\
&\quad+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x).
\end{aligned}
\tag{96}
$$

The calculation of $W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)$ is shown in **Appendix**. The result obtained is:

$$
\begin{aligned}
&W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)\\
&=\begin{cases}(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu), & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\mathrm{e}^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)\mathrm{e}^{i(\lambda-\mu)x}+i^{l+1}\mathrm{e}^{i(\lambda+\mu)x}(i\lambda-\mu), & \text{if } x>1.\end{cases}
\end{aligned}
\tag{97}
$$

Now let's write the solution $\phi_2(\lambda,x)$ obtained in **Appendix**:

$$
\phi_2(\lambda,x)=\frac{1}{\left(\mu^2-\lambda^2\right)}\begin{cases}\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right]\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x} & \\ +\left(\mu^2-\lambda^2\right)\left(i^{2l+1}\mu\mathrm{e}^{ix\mu}-i^{l+1}\lambda\mathrm{e}^{(2\lambda-\mu)ix}\right), & \text{if } x>1,\mu\neq\lambda\end{cases}
$$

That is,

$$
\begin{aligned}
&\frac{1}{\phi_2(\lambda,x)}\\
&=\begin{cases}\dfrac{\left(\mu^2-\lambda^2\right)x^{l+1}}{\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1,\\ \dfrac{\left(\mu^2-\lambda^2\right)}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+\left(\mu^2-\lambda^2\right)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+\left(\lambda^2-\mu^2\right)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}} & \text{if } x>1,\mu\neq\lambda.\end{cases}
\end{aligned}
\tag{98}
$$

Since

$$\varphi_1' = \begin{cases} \dfrac{(l+1)lx^l}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^{l+1}\lambda e^{i\lambda x}, & \text{if } x \ge 1. \end{cases}$$

it is obtained (see **Appendix**)

$$\frac{\varphi_1'}{\varphi_2} = \begin{cases} \dfrac{(\mu^2-\lambda^2)l(l+1)x^{2l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)i^{l+1}\lambda e^{i\lambda x}}{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu e^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda e^{i(2\lambda-\mu)x}} & \text{if } x>1, \mu\ne\lambda \end{cases} \tag{99}$$

According to **Appendix** one gets the next asymptotic relation

$$\frac{\varphi_1'}{\varphi_2} = \begin{cases} Cx^{2l+1}, & \text{if } 0<x<1, \\ O(1), & \text{if } x>1, \forall l \text{ even}. \end{cases} \tag{100}$$

Since

$$\frac{y_1'}{y_1^2} = \begin{cases} \dfrac{(2l+1)!!(l+1)}{x^{l+2}}, & \text{if } 0<x<1, \\ \dfrac{\mu}{i^{l-1}}e^{-i\mu x}, & \text{if } x>1. \end{cases}$$

Then

$$\begin{aligned} &\frac{\varphi_1'}{\varphi_2}+\frac{y_1'}{y_1^2} \\ &= \begin{cases} Cx^{2l+1}+\dfrac{(2l+1)!!(l+1)}{x^{l+2}}, & \text{if } 0<x<1, \\ O(1)+\dfrac{\mu}{i^{l-1}}e^{-i\mu x}, & \text{if } x>1, \forall l \text{ even}. \end{cases} \\ &= \begin{cases} \dfrac{(2l+1)!!(l+1)}{x^{l+2}}+Cx^{2l+1}, & \text{if } 0<x<1, \\ \dfrac{\mu}{i^{l-1}}e^{-i\mu x}+O(1), & \text{if } x>1, \forall l \text{ even}. \end{cases} \end{aligned}$$

Therefore

$$\frac{\varphi_1'}{\varphi_2}+\frac{y_1'}{y_1^2} = \begin{cases} \dfrac{(2l+1)!!(l+1)}{x^{l+2}}+Cx^{2l+1}, & \text{if } 0<x<1, \\ \dfrac{\mu}{i^{l-1}}e^{-i\mu x}+O(1), & \text{if } x>1, \forall l \text{ even}. \end{cases} \tag{101}$$

Now

$$q_1(x) = q(x)+\frac{l(l+1)}{x^2} := \begin{cases} q(x)+\dfrac{l(l+1)}{x^2}, & 0<x<1, \\ q(x), & x>1. \end{cases} \tag{102}$$

then

$$\mu^2-2q_1=\begin{cases}\mu^2-2q(x)-\dfrac{2l(l+1)}{x^2}, & 0<x<1,\\ \mu^2-2q(x), & x>1.\end{cases} \tag{103}$$

and

$$q_2=\mu^2-2q_1+\frac{\varphi_1'}{\varphi_2}+\frac{y_1'}{y_1^2},$$

then

$$\begin{aligned}&q_2(x)\\&=\begin{cases}\mu^2-2q(x)-\dfrac{2l(l+1)}{x^2}, & \text{if } 0<x<1,\\ \mu^2-2q(x), & \text{if } x>1.\end{cases}+\begin{cases}\dfrac{(2l+1)!!(l+1)}{x^{l+2}}+Cx^{2l+1}, & \text{if } 0<x<1,\\ \dfrac{\mu}{i^{l-1}}\mathrm{e}^{-i\mu x}+O(1), & \text{if } x>1,\forall l \text{ even.}\end{cases}\\&=\begin{cases}\mu^2-2q(x)-\dfrac{2l(l+1)}{x^2}+\dfrac{(2l+1)!!(l+1)}{x^{l+2}}+Cx^{2l+1}, & \text{if } 0<x<1,\\ \mu^2-2q(x)+\dfrac{\mu}{i^{l-1}}\mathrm{e}^{-i\mu x}+O(1), & \text{if } x>1,\forall l \text{ even.}\end{cases}\end{aligned}$$

Therefore

$$q_2(x)=\begin{cases}\mu^2-2q(x)-\dfrac{2l(l+1)}{x^2}+\dfrac{(2l+1)!!(l+1)}{x^{l+2}}+Cx^{2l+1}, & \text{if } 0<x<1,\\ \mu^2-2q(x)+\dfrac{\mu}{i^{l-1}}\mathrm{e}^{-i\mu x}+O(1), & \text{if } x>1,\forall l \text{ even.}\end{cases} \tag{104}$$

For the case

$$0<x<1, \tag{105}$$

Let's consider the term

$$-\frac{2l(l+1)}{x^2}+\frac{(2l+1)!!(l+1)}{x^{l+2}}.$$

If we define

$$a:=2l(2l+1)!!,b:=(2l+1)!!(l+1)\Rightarrow a>b>1 \text{ and } 0<x\le\sqrt[l]{\frac{l+1}{2l}}<1 \tag{106}$$

then

$$\begin{aligned}-\frac{2l(l+1)}{x^2}+\frac{(2l+1)!!(l+1)}{x^{l+2}}&:=-\frac{a}{x^2}+\frac{b}{x^{l+2}},\quad x\le\sqrt[l]{\frac{l+1}{2l}}\\ \Rightarrow x^l\le\frac{l+1}{2l}=\frac{b}{a}&\Leftrightarrow ax^l<b\Rightarrow ax^{l+2}<bx^2\\ \Rightarrow\frac{1}{ax^{l+2}}>\frac{1}{bx^2}&\Leftrightarrow\frac{b}{x^{l+2}}>\frac{a}{x^2}\Leftrightarrow\frac{b}{x^{l+2}}-\frac{a}{x^2}>0.\end{aligned} \tag{107}$$

Let the infimum

$$\alpha:=\inf\left\{\frac{b}{x^{l+2}}-\frac{a}{x^2}:0<x<\sqrt[l]{\frac{l+1}{2l}}\right\} \tag{108}$$

then

$$\alpha \geq 0, \tag{109}$$

If

$$\alpha > 0, \tag{110}$$

let $\varepsilon > 0, \exists x \in \left(0, \sqrt[l]{\frac{l+1}{2l}}\right)$ such that $\alpha \leq \frac{b}{x^{l+2}} - \frac{a}{x^2} < \alpha + \varepsilon, \forall \varepsilon > 0$. Then

$$\frac{b}{x^{l+2}} - \frac{a}{x^2} = 0\,, \tag{111}$$

!which is impossible!

Hence

$$\alpha = 0. \tag{112}$$

If

$$0 < x \leq \sqrt[l]{\frac{l+1}{2l}}, \tag{113}$$

it is obtained that

$$-\frac{2l(l+1)}{x^2} + \frac{(2l+1)!!(l+1)}{x^{l+2}} \underset{x\to 0}{=} 0, \tag{114}$$

since

$$C\left(\frac{l(l+1)}{(2l+1)!!}\right)^2 x^{2l+1} \underset{x\to 0}{=} 0. \tag{115}$$

for

$$0 < x \leq \sqrt[l]{\frac{l+1}{2l}} < 1, \tag{116}$$

so

$$\begin{aligned} q_2(x) &= \mu^2 - 2q(x) - \frac{2l(l+1)}{x^2} + \frac{(2l+1)!!(l+1)}{x^{l+2}} + C\left(\frac{l(l+1)}{(2l+1)!!}\right)^2 x^{2l+1} \\ &\underset{x\to 0}{=} \mu^2 - 2q(x). \end{aligned} \tag{117}$$

That is

$$q_2(x) \underset{x\to 0}{=} \mu^2 - 2q(x), \text{ for } 0 < x \leq \sqrt[l]{\frac{l+1}{2l}} < 1. \tag{118}$$

Therefore

$$q_2(x) = \begin{cases} \mu^2 - 2q(x), \text{ for } 0 < x \leq \sqrt[l]{\frac{l+1}{2l}} < 1 \\ \mu^2 - 2q(x) + \frac{\mu}{i^{l-1}} e^{-i\mu x} + O(1), \text{ if } x > 1, l \text{ even.} \end{cases} \tag{119}$$

So our final conclusion is, *that we have obtained the uniqueness of this potential which is regular when* $x \to 0$ *and, bounded with exponentially decreases fast enough when* $x \to \infty$.

4. The Estimates $L_1 - L_\infty$ for the Reduced Radial Equation of Schrödinger

***Theorem* 2.** (The $L^1 - L^\infty$ estimates). Suppose that V is regular and

$$H = -\frac{d^2}{dx^2} + V(x) + \frac{l(l+1)}{x^2}, 0 < x < \infty, \tag{120}$$

$$\phi(0) = 0.$$

Then

$$\left\| e^{-itH} P_c \right\|_{B(L^1, L^\infty)} \le C \frac{1}{\sqrt{t}}, \tag{121}$$

of moreover, $V \in L^1$, then

$$\left\| e^{-itH} P_c \right\|_{B(W_{1,1}, W_{1,\infty})} \le C \frac{1}{\sqrt{t}}. \tag{122}$$

Proof. For *Theorem* 1, the operator $\mathbb{W}$ transforms the potential $V(x) + \frac{l(l+1)}{x^2}$ in univocally determinated potential

$$q_2(x) := \begin{cases} \mu^2 - 2V(x), & \text{for } 0 < x \le \sqrt[l]{\frac{l+1}{2l}} < 1 \\ \mu^2 - 2V(x) + \frac{\mu}{i^{l-1}} e^{-i\mu x} + O(1), & \text{if } x > 1, l \text{ even.} \end{cases}$$

And if $\varphi_1(x)$ is solution of (120) then

$$\varphi_2(x) = \mathbb{W}\varphi_1(x), \tag{123}$$

is solution of

$$-\varphi_2''(x) + q_2(x)\varphi_2(x) = \mu^2 \varphi_2(x), \tag{124}$$

For

$$0 < x < 1 \tag{125}$$

we have $q_2(x) = \mu^2 - 2V(x) \Leftrightarrow q_2(x) + V(x) = \mu^2 - V(x)$

$$\Leftrightarrow \left[q_2(x) + V(x) \right] \varphi_2(x) = \left[\mu^2 - V(x) \right] \varphi_2(x) = -\varphi_2''(x).$$

That is

$$-\varphi_2''(x) + V(x)\varphi_2(x) = \mu^2 \varphi_2(x), \tag{126}$$

which is *Schrödinger Equation on the Half-Line* (*RSEHL*) *for* $0 < x < 1$.

For the case

$$x > 1, \tag{127}$$

The sum of the terms added to the potential

$$\frac{\mu}{i^{l-1}} e^{-i\mu x} + O(1), \tag{128}$$

form a term bound with exponentially decreases fast enough which allow that

$$\int_b^\infty x \left| q_2(x) \right| dx < \infty, \quad \text{for } b > 0. \tag{129}$$

See [5], Chapter 1, Secc 1.1, formula (1.1.3), page 2 which guarantees that the equation (*RSEHL*) is also fulfilled in this case. Therefore (121) and (122) are satisfied. □

5. Conclusions and Open Problems

As a comment by *J. Bourgain* in [7], page 27, the importance of the $L^1 - L^\infty$ estimates is mentioned, in the context of search of global solutions for the equations of non-linear Schrödinger. In particular, the obtained ones in [8] perform the form

$$\left\| e^{it(\Delta - V)}\phi \right\|_\infty \le C|t|^{-d/2}\|\phi\|_1, \text{ for } d \ge 3. \tag{130}$$

Also, in [7], page 27, it is mentioned that "*it would be most interesting to prove the analogue* (19) *in low dimension* $d = 1,2$. *This is certainly a project of independent importance*". In [3] and [6] these estimations were demonstrated for the case $d = 1$ on the line and half-line respectively. See [1], page 487. Now we have established the $L^1 - L^\infty$ estimates for the case Reduced Radial Schrödinger Equation (*RRSE*).

Finally as an open problem the possibility of extending the $L^1 - L^\infty$ estimates for the (*RSEHL*) but *now adding an analytical perturbation to the potential* $q(x)$; as mentioned in [2], 13 *Open Problems*, *page* 55, where is considered the differential operator

$$Lu = -u'' + q(x)u, 0 < x < \infty. \tag{131}$$

Since according to the Analytic Perturbation Theory the potential $q(x)$ can be written

$$q(x) = q_1(x) + q_2(x) + q_3(x), \tag{132}$$

where all the $q_i(x)$ are real valued, $q_1(x) \ge 0$ is locally integrable, $q_2(x)$ is uniformly locally integrable and $q_3(x)$ is majorized by $1/4x^2$. Then a closed form $b[u] = \int_0^\infty \left(|u'|^2 + q(x)|u|^2\right)dx$ can be constructed and the associated selfadjoint operator (self-adjoint extension) H of L in $L^2(0,\infty)$ with an appropriate domain (including the boundary condition $u(0) = 0$). See [9], Chapter 7, Section 8, pages 408-409.

Dedication

Dedicated to Professor Vladimir Alexandrovich Marchenko to celebrate his ninety-seventh birthday: born July 7, 1922. Sponsor: Fellow Sistema Nacional de Investigadores.

Acknowledgements

The author is grateful to the anonymous referees for their valuable comments. Also, to Miguel Navarro Saad and Gustavo Gómez Ros for their valuable help in technical editing and drafting the manuscript respectively. Especially to Jasmyn Chen, Editorial Assistant of APM for her generous willingness and professional-

ism.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Next, the calculations of the addends

$$\frac{y_1''}{y_1}, \frac{\varphi_1'}{\varphi_2}, \frac{y_1'}{y_1^2} \tag{133}$$

that form the potential

$$q_2(x) = q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}$$

will be shown.

It starts with

$$y_1(x,\mu) := \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l e^{i\mu x}, & \text{if } x \geq 1. \end{cases}$$

then

$$y_1'(x,\mu) := \begin{cases} \dfrac{l+1}{(2l+1)!!} x^l, & \text{if } 0 < x < 1, \\ i^{l+1} \mu e^{i\mu x}, & \text{if } x \geq 1. \end{cases}$$

and since

$$\phi_1(\lambda, x) = \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l e^{i\lambda x}, & \text{if } x \geq 1. \end{cases}$$

$$\phi_1'(x,\lambda) := \begin{cases} \dfrac{l+1}{(2l+1)!!} x^l, & \text{if } 0 < x < 1, \\ i^{l+1} \lambda e^{i\lambda x}, & \text{if } x \geq 1. \end{cases}$$

Since

$$y_1(x,\mu) := \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l e^{i\mu x}, & \text{if } x \geq 1. \end{cases}$$

then

$$\frac{1}{y_1(x,\mu)} = \begin{cases} \dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0 < x < 1, \\ \dfrac{1}{i^l e^{i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

$$\frac{1}{y_1(x,\mu)^2} = \begin{cases} \dfrac{(2l+1)!!^2}{x^{2(l+1)}}, & \text{if } 0 < x < 1, \\ \dfrac{1}{i^{2l} e^{2i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

and

$$y_1'(x,\mu) := \begin{cases} \dfrac{l+1}{(2l+1)!!} x^l, & \text{if } 0 < x < 1, \\ i^{l+1} \mu e^{i\mu x}, & \text{if } x \geq 1. \end{cases}$$

and

$$y_1''(x,\mu) := \begin{cases} \dfrac{l(l+1)}{(2l+1)!!} x^{l-1}, & \text{if } 0 < x < 1, \\ i^{l+2} \mu^2 e^{i\mu x}, & \text{if } x \geq 1. \end{cases}$$

Therefore

$$\frac{y_1''}{y_1} = \begin{cases} \dfrac{l(l+1)}{(2l+1)!!} x^{l-1}, & \text{if } 0 < x < 1, \\ i^{l+2} \mu^2 e^{i\mu x}, & \text{if } x \geq 1. \end{cases} \times \begin{cases} \dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0 < x < 1, \\ \dfrac{1}{i^l e^{i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

$$= \begin{cases} \dfrac{l(l+1)}{(2l+1)!!} x^{l-1} \times \dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0 < x < 1, \\ i^{l+2} \mu^2 e^{i\mu x} \dfrac{1}{i^l e^{i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

$$= \begin{cases} \dfrac{l(l+1)}{x^2}, & \text{if } 0 < x < 1, \\ -\mu^2, & \text{if } x \geq 1. \end{cases}$$

That is

$$\frac{y_1''}{y_1} = \begin{cases} \dfrac{l(l+1)}{x^2}, & \text{if } 0 < x < 1, \\ -\mu^2, & \text{if } x \geq 1. \end{cases} \tag{134}$$

Now

$$\frac{y_1'}{y_1^2} = y_1'(x,\mu) := \begin{cases} \dfrac{l+1}{(2l+1)!!} x^l, & \text{if } 0 < x < 1, \\ i^{l+1} \mu e^{i\mu x}, & \text{if } x \geq 1. \end{cases} \times \begin{cases} \dfrac{(2l+1)!!^2}{x^{2(l+1)}}, & \text{if } 0 < x < 1, \\ \dfrac{1}{i^{2l} e^{2i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

$$= \begin{cases} \dfrac{l+1}{(2l+1)!!} x^l \times \dfrac{(2l+1)!!^2}{x^{2(l+1)}}, & \text{if } 0 < x < 1, \\ i^{l+1} \mu e^{i\mu x} \dfrac{1}{i^{2l} e^{2i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

$$= \begin{cases} \dfrac{(2l+1)!!(l+1)}{x^2}, & \text{if } 0 < x < 1, \\ \dfrac{\mu}{i^{l-1} e^{i\mu x}}, & \text{if } x \geq 1. \end{cases}$$

That is

$$\frac{y_1'}{y_1^2} = \begin{cases} \dfrac{(2l+1)!!(l+1)}{x^2}, & \text{if } 0 < x < 1, \\ \dfrac{\mu}{i^{l-1} e^{i\mu x}}, & \text{if } x \geq 1. \end{cases} \tag{135}$$

Now since

$$\begin{aligned}&W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)\\&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\&\quad+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x).\end{aligned}$$

let us calculate each of the $W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)$ terms. Since

$$\frac{y_1(x,\mu)}{y_1(x,\lambda)}=\begin{cases}1, & \text{if } 0<x<1,\\ e^{i(\mu-\lambda)x}, & \text{if } x>1.\end{cases}$$

$$\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}=\begin{cases}1, & \text{if } 0<x<1,\\ e^{i(\lambda-\mu)x}, & \text{if } x>1.\end{cases}$$

$$y_1(x,\mu):=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\mu x}, & \text{if } x>1.\end{cases}$$

$$y_1'(x,\mu):=\begin{cases}\dfrac{l+1}{(2l+1)!!}x^l, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}, & \text{if } x>1.\end{cases}$$

$$\phi_1(\lambda,x)=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\lambda x}, & \text{if } x>1.\end{cases}$$

$$\phi_1'(\lambda,x)=\begin{cases}\dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\lambda e^{i\lambda x}, & \text{if } x>1.\end{cases}$$

first let's calculate the terms

$$\begin{aligned}&\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)\\&=\begin{cases}\dfrac{l+1}{(2l+1)!!}x^l, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}, & \text{if } x>1.\end{cases}\times\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\lambda x}, & \text{if } x>1.\end{cases}\times\begin{cases}1, & \text{if } 0<x<1,\\ e^{i(\mu-\lambda)x}, & \text{if } x>1.\end{cases}\\&=\begin{cases}\dfrac{(l+1)}{(2l+1)!!^2}x^{2l+1}, & \text{if } 0<x<1,\\ i^{2l+1}\lambda e^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}\end{aligned}$$

That is

$$\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)=\begin{cases}\dfrac{(l+1)}{(2l+1)!!^2}x^{2l+1}, & \text{if } 0<x<1,\\ i^{2l+1}\lambda e^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}$$

Next

$$\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)=\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}y_1(x,\mu)\phi_1'(\mu,x)$$
$$=\begin{cases}1, & \text{if } 0<x<1,\\ \mathrm{e}^{i(\lambda-\mu)x}, & \text{if } x>1.\end{cases}\times\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l\mathrm{e}^{i\mu x}, & \text{if } x>1.\end{cases}\times\begin{cases}\dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\lambda\mathrm{e}^{i\lambda x}, & \text{if } x>1.\end{cases}$$

That is

$$\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)=\begin{cases}\dfrac{l+1}{(2l+1)!!^2}x^{2l+1}, & \text{if } 0<x<1,\\ i^{l+1}\mu\mathrm{e}^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}$$

Then

$$\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)$$
$$=\begin{cases}\dfrac{(l+1)}{(2l+1)!!^2}x^{2l+1}, & \text{if } 0<x<1,\\ i^{2l+1}\lambda\mathrm{e}^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}-\begin{cases}\dfrac{l+1}{(2l+1)!!^2}x^{2l+1}, & \text{if } 0<x<1,\\ i^{l+1}\mu\mathrm{e}^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}$$
$$=\begin{cases}0, & \text{if } 0<x<1,\\ \left(i^l\lambda-\mu\right)i^{l+1}\mathrm{e}^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}$$

Therefore

$$\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)$$
$$=\begin{cases}0, & \text{if } 0<x<1,\\ \left(i^l\lambda-\mu\right)i^{l+1}\mathrm{e}^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}$$

Now let's calculate the term:

$$(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}$$
$$(-\lambda)^{-l}F_l(\lambda)\begin{cases}1, & \text{if } 0<x<1,\\ \mathrm{e}^{i(\mu-\lambda)x} & \text{if } x\ge 1.\end{cases}+(-\mu)^{-l}F_l(\mu)\begin{cases}1, & \text{if } 0<x<1,\\ \mathrm{e}^{i(\lambda-\mu)x}, & \text{if } x\ge 1.\end{cases}$$
$$\begin{cases}(-\lambda)^{-l}F_l(\lambda), & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\mathrm{e}^{i(\mu-\lambda)x}, & \text{if } x>1.\end{cases}+\begin{cases}(-\mu)^{-l}F_l(\mu), & \text{if } 0<x<1,\\ (-\mu)^{-l}F_l(\mu)\mathrm{e}^{i(\lambda-\mu)x}, & \text{if } x>1.\end{cases}$$
$$+\begin{cases}(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu), & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\mathrm{e}^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)\mathrm{e}^{i(\lambda-\mu)x}, & \text{if } x>1.\end{cases}$$

That is

$$(-1)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}$$
$$=\begin{cases}(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu), & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\mathrm{e}^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)\mathrm{e}^{i(\lambda-\mu)x} & \text{if } x>1.\end{cases}$$

Then

$$
\begin{aligned}
&W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\
&\quad+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)\\
&=\begin{cases}(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu), & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\mathrm{e}^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)\mathrm{e}^{i(\lambda-\mu)x}, & \text{if } x>1.\end{cases}\\
&\quad+\begin{cases}0, & \text{if } 0<x<1,\\ \left(i^l\lambda-\mu\right)i^{l+1}\mathrm{e}^{i(\lambda+\mu)x}, & \text{if } x>1.\end{cases}
\end{aligned}
$$

That is

$$
\begin{aligned}
&W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)\\
&=\begin{cases}(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu), & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\mathrm{e}^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)\mathrm{e}^{i(\lambda-\mu)x}+i^{l+1}\mathrm{e}^{i(\lambda+\mu)x}(i\lambda-\mu), & \text{if } x>1.\end{cases}
\end{aligned}
\tag{136}
$$

Now let's calculate the solution

$$\phi_2(\lambda,x):=\frac{W\left(y_1(x,\mu),\phi_1(\lambda,x)\right)}{y_1(x,\mu)\left(\mu^2-\lambda^2\right)},$$

as follows:

$$
\begin{aligned}
&\phi_2(\lambda,x)\\
&=\frac{1}{y_1(x,\mu)\left(\mu^2-\lambda^2\right)}\times\left\{(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\right.\\
&\quad\left.+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)\right\}\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{y_1(x,\mu)}{y_1(x,\lambda)}\frac{1}{y_1(x,\mu)\left(\mu^2-\lambda^2\right)}\\
&\quad+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\frac{1}{y_1(x,\mu)\left(\mu^2-\lambda^2\right)}\\
&\quad+\frac{y_1(x,\mu)}{y_1(x,\lambda)}y_1'(x,\lambda)\phi_1(\lambda,x)\frac{1}{y_1(x,\mu)\left(\mu^2-\lambda^2\right)}\\
&\quad-\frac{y_1(x,\mu)}{\phi_1(\mu,x)}\phi_1(\lambda,x)\phi_1'(\mu,x)\frac{1}{y_1(x,\mu)\left(\mu^2-\lambda^2\right)}\\
&=(-\lambda)^{-l}F_l(\lambda)\frac{1}{y_1(x,\lambda)}\frac{1}{\left(\mu^2-\lambda^2\right)}\\
&\quad+(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)y_1(x,\mu)}\frac{1}{\left(\mu^2-\lambda^2\right)}\\
&\quad+\frac{y_1'(x,\lambda)\phi_1(\lambda,x)}{y_1(x,\lambda)}\frac{1}{\left(\mu^2-\lambda^2\right)}-\frac{\phi_1(\lambda,x)\phi_1'(\mu,x)}{\phi_1(\mu,x)}\frac{1}{\left(\mu^2-\lambda^2\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{y_1(x,\lambda)} \frac{(-\lambda)^{-l} F_l(\lambda)}{\left(\mu^2-\lambda^2\right)} + \frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)\, y_1(x,\mu)} \frac{(-\mu)^{-l} F_l(\mu)}{\left(\mu^2-\lambda^2\right)} \\
&\quad + \frac{y_1'(x,\lambda)\phi_1(\lambda,x)}{y_1(x,\lambda)} \frac{1}{\left(\mu^2-\lambda^2\right)} - \frac{\phi_1(\lambda,x)\phi_1'(\mu,x)}{\phi_1(\mu,x)} \frac{1}{\left(\mu^2-\lambda^2\right)} \\
&= \frac{1}{\left(\mu^2-\lambda^2\right)} \left\{ \frac{(-\lambda)^{-l} F_l(\lambda)}{y_1(x,\lambda)} + \frac{\phi_1(\lambda,x)(-\mu)^{-l} F_l(\mu)}{\phi_1(\mu,x)\, y_1(x,\mu)} \right. \\
&\quad \left. + \frac{y_1'(x,\lambda)\phi_1(\lambda,x)}{y_1(x,\lambda)} - \frac{\phi_1(\lambda,x)\phi_1'(\mu,x)}{\phi_1(\mu,x)} \right\} \\
&= \frac{1}{\left(\mu^2-\lambda^2\right)} \left\{ \frac{(-\lambda)^{-l} F_l(\lambda)}{y_1(x,\lambda)} + \frac{\phi_1(\lambda,x)(-\mu)^{-l} F_l(\mu)}{\phi_1(\mu,x)\, y_1(x,\mu)} \right. \\
&\quad \left. + \frac{y_1'(x,\lambda)}{y_1(x,\lambda)}\phi_1(\lambda,x) - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)}\phi_1(\lambda,x) \right\} \\
&= \frac{1}{\left(\mu^2-\lambda^2\right)} \left\{ \frac{(-\lambda)^{-l} F_l(\lambda)}{y_1(x,\lambda)} + \frac{(-\mu)^{-l} F_l(\mu)}{y_1(x,\mu)} \frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)} \right. \\
&\quad \left. + \left[\frac{y_1'(x,\lambda)}{y_1(x,\lambda)} - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)} \right] \phi_1(\lambda,x) \right\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\phi_2(\lambda,x) := \frac{1}{\left(\mu^2-\lambda^2\right)} &\left\{ (-\lambda)^{-l} F_l(\lambda) \frac{1}{y_1(x,\lambda)} + (-\mu)^{-l} F_l(\mu) \frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)} \right. \\
&\left. + \left[\frac{y_1'(x,\lambda)}{y_1(x,\lambda)} - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)} \right] \phi_1(\lambda,x) \right\}
\end{aligned} \tag{137}
$$

Now, since

$$
\phi_1(\lambda,x) = \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l \mathrm{e}^{i\lambda x}, & \text{if } x \ge 1. \end{cases}
$$

$$
\phi_1(\mu,x) = \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l \mathrm{e}^{i\mu x}, & \text{if } x \ge 1. \end{cases}
$$

$$
\phi_1'(\lambda,x) = \begin{cases} \dfrac{(l+1)lx^l}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^{l+1} \lambda \mathrm{e}^{i\lambda x}, & \text{if } x \ge 1. \end{cases}
$$

$$
\phi_1'(\mu,x) = \begin{cases} \dfrac{(l+1)x^l}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^{l+1} \mu \mathrm{e}^{i\mu x}, & \text{if } x \ge 1. \end{cases}
$$

$$
y_1(x,\mu) = \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l \mathrm{e}^{i\mu x}, & \text{if } x \ge 1. \end{cases}
$$

$$y_1(x,\lambda)=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^l e^{i\lambda x}, & \text{if } x\ge 1.\end{cases}$$

$$y_1'(x,\mu)=\begin{cases}\dfrac{(l+1)x^{l}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}, & \text{if } x\ge 1.\end{cases}$$

$$y_1'(x,\lambda)=\begin{cases}\dfrac{(l+1)x^{l}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\lambda e^{i\lambda x}, & \text{if } x\ge 1.\end{cases}$$

and

$$\begin{aligned}\frac{y_1'(x,\mu)}{y_1(x,\lambda)}&=\begin{cases}\dfrac{(l+1)x^{l}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}, & \text{if } x\ge 1.\end{cases}\times\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x\ge 1.\end{cases}\\
&=\begin{cases}\dfrac{x^{l+1}}{(2l+1)!!}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{i\mu x}\dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x\ge 1.\end{cases}\\
&=\begin{cases}1, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{ix(\mu-\lambda)}, & \text{if } x\ge 1.\end{cases}\end{aligned}$$

Then

$$\frac{y_1'(x,\mu)}{y_1(x,\lambda)}=\begin{cases}1, & \text{if } 0<x<1,\\ i^{l+1}\mu e^{ix(\mu-\lambda)}, & \text{if } x\ge 1.\end{cases}$$

Now

$$\begin{aligned}\frac{\phi_1'(\lambda,x)}{\phi_1(\mu,x)}&=\begin{cases}\dfrac{(l+1)lx^{l}}{(2l+1)!!}, & \text{if } 0<x<1,\\ i^{l+1}\lambda e^{i\lambda x}, & \text{if } x\ge 1.\end{cases}\times\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{1}{i^l e^{i\mu x}}, & \text{if } x\ge 1.\end{cases}\\
&=\begin{cases}\dfrac{(l+1)lx^{l}}{(2l+1)!!}\times\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{i^{l+1}\lambda e^{i\lambda x}}{i^l e^{i\mu x}}, & \text{if } x\ge 1.\end{cases}\\
&=\begin{cases}1, & \text{if } 0<x<1,\\ i\lambda e^{ix(\lambda-\mu)}, & \text{if } x\ge 1.\end{cases}\end{aligned}$$

That is

$$\frac{\phi_1'(\lambda,x)}{\phi_1(\mu,x)}=\begin{cases}1, & \text{if } 0<x<1,\\ i\lambda e^{ix(\lambda-\mu)}, & \text{if } x\ge 1.\end{cases}$$

Now we compute

$$\frac{y_1'(x,\lambda)}{y_1(x,\lambda)}-\frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)},$$

as follows

$$\frac{y_1'(x,\lambda)}{y_1(x,\lambda)} - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)}$$

$$= \begin{cases} 1, & \text{if } 0 < x < 1, \\ i^{l+1}\mu e^{ix(\mu-\lambda)}, & \text{if } x \ge 1. \end{cases} - \begin{cases} 1, & \text{if } 0 < x < 1, \\ i\lambda e^{ix(\lambda-\mu)}, & \text{if } x \ge 1. \end{cases}$$

$$= \begin{cases} 1-1, & \text{if } 0 < x < 1, \\ i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{ix(\lambda-\mu)}, & \text{if } x \ge 1. \end{cases}$$

$$= \begin{cases} 0, & \text{if } 0 < x < 1, \\ i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{ix(\lambda-\mu)}, & \text{if } x \ge 1. \end{cases}$$

$$= \begin{cases} 0, & \text{if } 0 < x < 1, \\ i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{-ix(\mu-\lambda)}, & \text{if } x \ge 1. \end{cases}$$

That is

$$\frac{y_1'(x,\lambda)}{y_1(x,\lambda)} - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)} = \begin{cases} 0, & \text{if } 0 < x < 1, \\ i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{-ix(\mu-\lambda)}, & \text{if } x \ge 1. \end{cases} \tag{138}$$

And

$$\left[\frac{y_1'(x,\lambda)}{y_1(x,\lambda)} - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)}\right]\phi_1(\lambda,x)$$

$$= \begin{cases} 0, & \text{if } 0 < x < 1, \\ i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{-ix(\mu-\lambda)}, & \text{if } x \ge 1. \end{cases} \times \begin{cases} \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ i^l e^{i\lambda x}, & \text{if } x \ge 1. \end{cases}$$

$$= \begin{cases} 0 \times \dfrac{x^{l+1}}{(2l+1)!!}, & \text{if } 0 < x < 1, \\ \left[i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{-ix(\mu-\lambda)}\right] i^l e^{i\lambda x}, & \text{if } x \ge 1. \end{cases}$$

$$= \begin{cases} 0, & \text{if } 0 < x < 1, \\ \left[i^{l+1}\mu e^{ix(\mu-\lambda)} - i\lambda e^{-ix(\mu-\lambda)}\right] i^l e^{i\lambda x}, & \text{if } x \ge 1. \end{cases}$$

$$= \begin{cases} 0, & \text{if } 0 < x < 1, \\ i^{2l+1}\mu e^{ix\mu} - i^{l+1}\lambda e^{(2\lambda-\mu)ix}, & \text{if } x \ge 1. \end{cases}$$

That is

$$\left[\frac{y_1'(x,\lambda)}{y_1(x,\lambda)} - \frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)}\right]\phi_1(\lambda,x) = \begin{cases} 0, & \text{if } 0 < x < 1, \\ i^{2l+1}\mu e^{ix\mu} - i^{l+1}\lambda e^{(2\lambda-\mu)ix}, & \text{if } x \ge 1. \end{cases} \tag{139}$$

Now we compute

$$(-\lambda)^{-l} F_l(\lambda)\frac{1}{y_1(x,\lambda)} + (-\mu)^{-l} F_l(\mu)\frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)}. \tag{140}$$

So

$$(-\lambda)^{-l} F_l(\lambda)\frac{1}{y_1(x,\lambda)} = (-\lambda)^{-l} F_l(\lambda)\begin{cases} \dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0 < x < 1, \\ \dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x \ge 1. \end{cases}$$

$$=\begin{cases}(-\lambda)^{-l}F_l(\lambda)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x\geq 1.\end{cases}$$

That is

$$(-\lambda)^{-l}F_l(\lambda)\frac{1}{y_1(x,\lambda)}=\begin{cases}(-\lambda)^{-l}F_l(\lambda)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l}F_l(\lambda)\dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x\geq 1.\end{cases}\tag{141}$$

Since

$$\frac{1}{y_1(x,\mu)}=\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{1}{i^l e^{i\mu x}}, & \text{if } x\geq 1.\end{cases}$$

and

$$\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}=\begin{cases}1, & \text{if } 0<x<1,\\ e^{i(\lambda-\mu)x}, & \text{if } x\geq 1.\end{cases}$$

it follows that

$$\begin{aligned}&(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)}\\ &=(-\mu)^{-l}F_l(\mu)\frac{1}{y_1(x,\mu)}\frac{\phi_1(\lambda,x)}{\phi_1(\mu,x)}\\ &=(-\mu)^{-l}F_l(\mu)\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{1}{i^l e^{i\mu x}}, & \text{if } x\geq 1.\end{cases}\times\begin{cases}1, & \text{if } 0<x<1,\\ e^{i(\lambda-\mu)x}, & \text{if } x\geq 1.\end{cases}\\ &=(-\mu)^{-l}F_l(\mu)\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ \dfrac{e^{i(\lambda-\mu)x}}{i^l e^{i\mu x}}, & \text{if } x\geq 1.\end{cases}\\ &=(-\mu)^{-l}F_l(\mu)\begin{cases}\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ i^{-l}e^{i(\lambda-2\mu)x}, & \text{if } x\geq 1.\end{cases}\\ &=\begin{cases}(-\mu)^{-l}F_l(\mu)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\mu)^{-l}F_l(\mu)i^{-l}e^{i(\lambda-2\mu)x}, & \text{if } x\geq 1.\end{cases}\end{aligned}$$

That is

$$(-\mu)^{-l}F_l(\mu)\frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)}=\begin{cases}(-\mu)^{-l}F_l(\mu)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\mu)^{-l}F_l(\mu)i^{-l}e^{i(\lambda-2\mu)x}, & \text{if } x\geq 1.\end{cases}\tag{142}$$

Then

$$
\begin{aligned}
&(-\lambda)^{-l} F_l(\lambda)\frac{1}{y_1(x,\lambda)}+(-\mu)^{-l} F_l(\mu)\frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)}\\
&=\begin{cases}(-\lambda)^{-l} F_l(\lambda)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda)\dfrac{1}{i^l e^{i\lambda x}}, & \text{if } x\ge 1.\end{cases}\\
&\quad+\begin{cases}(-\mu)^{-l} F_l(\mu)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}, & \text{if } x\ge 1.\end{cases}\\
&=\begin{cases}(-\lambda)^{-l} F_l(\lambda)\dfrac{(2l+1)!!}{x^{l+1}}+(-\mu)^{-l} F_l(\mu)\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda)\dfrac{1}{i^l e^{i\lambda x}}+(-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}, & \text{if } x\ge 1.\end{cases}\\
&=\begin{cases}\left[(-\lambda)^{-l} F_l(\lambda)+(-\mu)^{-l} F_l(\mu)\right]\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda) i^{-l} e^{-i\lambda x}+(-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}, & \text{if } x\ge 1.\end{cases}
\end{aligned}
$$

That is

$$
\begin{aligned}
&(-\lambda)^{-l} F_l(\lambda)\frac{1}{y_1(x,\lambda)}+(-\mu)^{-l} F_l(\mu)\frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)}\\
&=\begin{cases}\left[(-\lambda)^{-l} F_l(\lambda)+(-\mu)^{-l} F_l(\mu)\right]\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda) i^{-l} e^{-i\lambda x}+(-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}, & \text{if } x\ge 1.\end{cases}
\end{aligned}
\tag{143}
$$

Besides

$$
\begin{aligned}
\phi_2(\lambda,x) &:= \frac{1}{(\mu^2-\lambda^2)}\Bigg\{(-\lambda)^{-l} F_l(\lambda)\frac{1}{y_1(x,\lambda)}\\
&\quad+(-\mu)^{-l} F_l(\mu)\frac{\phi_1(\lambda,x)}{y_1(x,\mu)\phi_1(\mu,x)}+\left[\frac{y_1'(x,\lambda)}{y_1(x,\lambda)}-\frac{\phi_1'(\mu,x)}{\phi_1(\mu,x)}\right]\phi_1(\lambda,x)\Bigg\}\\
&=\frac{1}{(\mu^2-\lambda^2)}\begin{cases}\left[(-\lambda)^{-l} F_l(\lambda)+(-\mu)^{-l} F_l(\mu)\right]\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda) i^{-l} e^{-i\lambda x}+(-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}, & \text{if } x\ge 1.\end{cases}\\
&\quad+\begin{cases}0, & \text{if } 0<x<1,\\ i^{2l+1}\mu e^{ix\mu}-i^{l+1}\lambda e^{(2\lambda-\mu)ix}, & \text{if } x\ge 1.\end{cases}\\
&=\frac{1}{(\mu^2-\lambda^2)}\begin{cases}\left[(-\lambda)^{-l} F_l(\lambda)+(-\mu)^{-l} F_l(\mu)\right]\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda) i^{-l} e^{-i\lambda x}+(-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}+(\mu^2-\lambda^2)\left(i^{2l+1}\mu e^{ix\mu}-i^{l+1}\lambda e^{(2\lambda-\mu)ix}\right), & \text{if } x>1, \mu\ne\lambda\end{cases}
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
&\phi_2(\lambda,x)\\
&=\frac{1}{(\mu^2-\lambda^2)}\begin{cases}\left[(-\lambda)^{-l} F_l(\lambda)+(-\mu)^{-l} F_l(\mu)\right]\dfrac{(2l+1)!!}{x^{l+1}}, & \text{if } 0<x<1,\\ (-\lambda)^{-l} F_l(\lambda) i^{-l} e^{-i\lambda x}+(-\mu)^{-l} F_l(\mu) i^{-l} e^{i(\lambda-2\mu)x}+(\mu^2-\lambda^2)\left(i^{2l+1}\mu e^{ix\mu}-i^{l+1}\lambda e^{(2\lambda-\mu)ix}\right), & \text{if } x>1, \mu\ne\lambda\end{cases}
\end{aligned}
\tag{144}
$$

And

$$\frac{1}{\phi_2(\lambda,x)} = \begin{cases} \dfrac{(\mu^2-\lambda^2)x^{l+1}}{\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda \end{cases} \tag{145}$$

Now it is calculated

$$\frac{\varphi_1'}{\varphi_2} = \begin{cases} \dfrac{(l+1)lx^l}{(2l+1)!!}, & \text{if } 0<x<1, \\ i^{l+1}\lambda\mathrm{e}^{i\lambda x}, & \text{if } x\geq 1. \end{cases}$$

$$\times\begin{cases} \dfrac{(\mu^2-\lambda^2)x^{l+1}}{\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda \end{cases}$$

$$= \begin{cases} \dfrac{(l+1)lx^l}{(2l+1)!!}\times\dfrac{(\mu^2-\lambda^2)x^{l+1}}{\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)i^{l+1}\lambda\mathrm{e}^{i\lambda x}}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda \end{cases}$$

$$= \begin{cases} \dfrac{(l+1)lx^l(\mu^2-\lambda^2)x^{l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)i^{l+1}\lambda\mathrm{e}^{i\lambda x}}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda \end{cases}$$

$$= \begin{cases} \dfrac{(\mu^2-\lambda^2)l(l+1)x^{2l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)i^{l+1}\lambda\mathrm{e}^{i\lambda x}}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda \end{cases}$$

Therefore

$$\frac{\varphi_1'}{\varphi_2} = \begin{cases} \dfrac{(\mu^2-\lambda^2)l(l+1)x^{2l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1, \\ \dfrac{(\mu^2-\lambda^2)i^{l+1}\lambda\mathrm{e}^{i\lambda x}}{(-\lambda)^{-l}F_l(\lambda)i^{-l}\mathrm{e}^{-i\lambda x}+(\mu^2-\lambda^2)i^{2l+1}\mu\mathrm{e}^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}\mathrm{e}^{i(\lambda-2\mu)x}+(\lambda^2-\mu^2)i^{l+1}\lambda\mathrm{e}^{i(2\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda \end{cases} \tag{146}$$

There is another representation for

$$\frac{\varphi_1'}{\varphi_2} \text{ for } x > 1. \tag{147}$$

which is given by

$$\begin{aligned}
&\frac{\left(\mu^2-\lambda^2\right)i^{l+1}\lambda e^{i\lambda x}}{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-i\lambda x}+\left(\mu^2-\lambda^2\right)i^{2l+1}\mu e^{ix\mu}+(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(\lambda-2\mu)x}+\left(\lambda^2-\mu^2\right)i^{l+1}\lambda e^{i(2\lambda-\mu)x}} \\
&=\frac{\dfrac{\left(\mu^2-\lambda^2\right)i^{l+1}\lambda e^{i\lambda x}}{e^{i\lambda x}}}{\dfrac{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-i\lambda x}}{e^{i\lambda x}}+\dfrac{\left(\mu^2-\lambda^2\right)i^{2l+1}\mu e^{ix\mu}}{e^{i\lambda x}}+\dfrac{(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(\lambda-2\mu)x}}{e^{i\lambda x}}+\dfrac{\left(\lambda^2-\mu^2\right)i^{l+1}\lambda e^{i(2\lambda-\mu)x}}{e^{i\lambda x}}} \\
&=\frac{\dfrac{\left(\mu^2-\lambda^2\right)i^{l+1}\lambda}{1}}{\dfrac{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-2i\lambda x}}{1}+\dfrac{\left(\mu^2-\lambda^2\right)i^{2l+1}\mu e^{i(\mu-\lambda)x}}{1}}\frac{1}{+\dfrac{(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(2\lambda-2\mu)x}}{1}+\dfrac{\left(\lambda^2-\mu^2\right)i^{l+1}\lambda e^{i(\lambda-\mu)x}}{1}} \\
&=\frac{\left(\mu^2-\lambda^2\right)i^{l+1}\lambda}{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-2i\lambda x}+\left(\mu^2-\lambda^2\right)i^{2l+1}\mu e^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(2\lambda-2\mu)x}+\left(\lambda^2-\mu^2\right)i^{l+1}\lambda e^{i(\lambda-\mu)x}}
\end{aligned}$$

Therefore

$$\frac{\varphi_1'}{\varphi_2}=\begin{cases}\dfrac{\left(\mu^2-\lambda^2\right)l(l+1)x^{2l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1,\\[2ex] \dfrac{\left(\mu^2-\lambda^2\right)i^{l+1}\lambda}{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-2i\lambda x}+\left(\mu^2-\lambda^2\right)i^{2l+1}\mu e^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(2\lambda-2\mu)x}+\left(\lambda^2-\mu^2\right)i^{l+1}\lambda e^{i(\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda\end{cases} \tag{148}$$

Now if the following constants are defined

$$\begin{aligned}&\alpha_1 := (-\lambda)^{-l}F_l(\lambda)i^{-l}, \beta_1 := \left(\mu^2-\lambda^2\right)i^{2l+1}\mu, \gamma_1 := (-\mu)^{-l}F_l(\mu)i^{-l},\\ &\delta_1 := \left(\lambda^2-\mu^2\right)i^{l+1}\lambda, \epsilon_1 := \left(\mu^2-\lambda^2\right)i^{l+1}.\end{aligned} \tag{149}$$

then

$$\begin{aligned}&\frac{\left(\mu^2-\lambda^2\right)i^{l+1}\lambda}{(-\lambda)^{-l}F_l(\lambda)i^{-l}e^{-2i\lambda x}+\left(\mu^2-\lambda^2\right)i^{2l+1}\mu e^{i(\mu-\lambda)x}+(-\mu)^{-l}F_l(\mu)i^{-l}e^{i(2\lambda-2\mu)x}+\left(\lambda^2-\mu^2\right)i^{l+1}\lambda e^{i(\lambda-\mu)x}}\\ &=\frac{\epsilon_1\lambda}{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}\end{aligned} \tag{150}$$

Therefore

$$\frac{\varphi_1'}{\varphi_2}=\begin{cases}\dfrac{\left(\mu^2-\lambda^2\right)l(l+1)x^{2l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!}, & \text{if } 0<x<1,\\[2ex] \dfrac{\epsilon_1\lambda}{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}, & \text{if } x>1, \mu\neq\lambda\end{cases} \tag{151}$$

For

$$x>1, \mu\neq\lambda, \tag{152}$$

there is another representation for the term

$$\frac{\epsilon_1\lambda}{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}, \tag{153}$$

let $\bar{z}$ and $|z|$ is the conjugate complex of z and the modulo of z, respectively; then $|z|^2 = z\bar{z}$, by the properties of complex numbers. Thus

$$\begin{aligned}
&\frac{\epsilon_1\lambda}{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}\\
&=\frac{\epsilon_1\lambda}{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}\\
&\quad\times\frac{\overline{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}}{\overline{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}}\\
&=\frac{\epsilon_1\lambda\left(\overline{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}\right)}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2}\\
&=\frac{\epsilon_1\lambda\left(\overline{\alpha_1 e^{-2i\lambda x}}+\overline{\beta_1 e^{i(\mu-\lambda)x}}+\overline{\gamma_1 e^{2i(\lambda-\mu)x}}+\overline{\delta_1 e^{i(\lambda-\mu)x}}\right)}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2}\\
&=\frac{\epsilon_1\lambda\left(\overline{\alpha_1}\,\overline{e^{-2i\lambda x}}+\overline{\beta_1}\,\overline{e^{i(\mu-\lambda)x}}+\overline{\gamma_1}\,\overline{e^{2i(\lambda-\mu)x}}+\overline{\delta_1}\,\overline{e^{i(\lambda-\mu)x}}\right)}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2}\\
&=\frac{\epsilon_1\lambda\left(\overline{\alpha_1} e^{2i\lambda x}+\overline{\beta_1} e^{i(\lambda-\mu)x}+\overline{\gamma_1} e^{2i(\mu-\lambda)x}+\overline{\delta_1} e^{i(\mu-\lambda)x}\right)}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2}
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{\epsilon_1\lambda}{\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}}\\
&=\frac{\epsilon_1\lambda\left(\overline{\alpha_1} e^{2i\lambda x}+\overline{\beta_1} e^{i(\lambda-\mu)x}+\overline{\gamma_1} e^{2i(\mu-\lambda)x}+\overline{\delta_1} e^{i(\mu-\lambda)x}\right)}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2}
\end{aligned}\tag{154}$$

And let us estimate the denominator module of (154), using the Lagrange's identity.

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n}|z_k|^2\right)\left(\sum_{k=1}^{n}|w_k|^2\right)-\sum_{k<j}^{n}\left|z_k\overline{w}_j - z_j\overline{w}_k\right|^2 \tag{155}$$

See [10], Chapter 1, Section, 1.2, Exercises: 18, page 23. Then

$$\begin{aligned}
&\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2\\
&=\left(|\alpha_1|^2+|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2\right)\left(\left|e^{-2i\lambda x}\right|^2+\left|e^{i(\mu-\lambda)x}\right|^2+\left|e^{2i(\lambda-\mu)x}\right|^2+\left|e^{i(\lambda-\mu)x}\right|^2\right)\\
&\quad-\left|\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right|^2-\left|\beta_1\overline{e^{i(\mu-\lambda)x}}-\gamma_1\overline{e^{2i(\lambda-\mu)x}}\right|^2-\left|\gamma_1\overline{e^{2i(\lambda-\mu)x}}+\delta_1\overline{e^{i(\lambda-\mu)x}}\right|^2
\end{aligned}$$

$$=\left(|\alpha_1|^2+|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2\right)\left(\left|e^{-2i\lambda x}\right|^2+\left|e^{i(\mu-\lambda)x}\right|^2+\left|e^{2i(\lambda-\mu)x}\right|^2+\left|e^{i(\lambda-\mu)x}\right|^2\right)$$
$$-\left|\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right|^2-\left|\beta_1\overline{e^{i(\mu-\lambda)x}}-\gamma_1\overline{e^{2i(\lambda-\mu)x}}\right|^2-\left|\gamma_1\overline{e^{2i(\lambda-\mu)x}}-\delta_1\overline{e^{i(\lambda-\mu)x}}\right|^2$$

$$\left|\alpha_1 e^{-i\lambda x}+\beta_1 e^{ix\mu}+\gamma_1 e^{i(\lambda-2\mu)x}+\delta_1 e^{i(2\lambda-\mu)x}\right|^2$$
$$=\left(|\alpha_1|^2+|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2\right)(1+1+1+1)-\left|\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right|^2$$
$$-\left|\beta_1\overline{e^{i(\mu-\lambda)x}}-\gamma_1\overline{e^{2i(\lambda-\mu)x}}\right|^2-\left|\gamma_1\overline{e^{2i(\lambda-\mu)x}}-\delta_1\overline{e^{i(\lambda-\mu)x}}\right|^2-\left|\gamma_1\overline{e^{i(2\lambda-\mu)x}}-\delta_1\overline{e^{i(\lambda-2\mu)x}}\right|^2$$
$$=4\left(|\alpha_1|^2+|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2\right)-\left|\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right|^2$$
$$-\left|\beta_1\overline{e^{i(\mu-\lambda)x}}-\gamma_1\overline{e^{2i(\lambda-\mu)x}}\right|^2-\left|\gamma_1\overline{e^{2i(\lambda-\mu)x}}-\delta_1\overline{e^{i(\lambda-\mu)x}}\right|^2$$
$$=4\left(|\alpha_1|^2+|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2\right)-\left|\alpha_1 e^{2i\lambda x}-\beta_1 e^{i(\lambda-\mu)x}\right|^2$$
$$-\left|\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right|^2-\left|\gamma_1 e^{2i(\mu-\lambda)x}-\delta_1 e^{i(\mu-\lambda)x}\right|^2$$

That is

$$\begin{aligned}&\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2\\&=4\left(|\alpha_1|^2+|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2\right)-\left|\alpha_1 e^{2i\lambda x}-\beta_1 e^{i(\lambda-\mu)x}\right|^2\\&-\left|\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right|^2-\left|\gamma_1 e^{2i(\mu-\lambda)x}-\delta_1 e^{i(\mu-\lambda)x}\right|^2\end{aligned} \tag{156}$$

Now let us calculate each term subtracted

$$\left|\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right|^2$$
$$=\left|\alpha_1 e^{2i\lambda x}-\beta_1 e^{i(\lambda-\mu)x}\right|^2$$
$$=\left(\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right)\left(\overline{\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}}\right)$$
$$=\left(\alpha_1 e^{2i\lambda x}-\beta_1 e^{i(\lambda-\mu)x}\right)\left(\overline{\alpha_1}e^{-2i\lambda x}-\overline{\beta_1}e^{i(\mu-\lambda)x}\right)$$
$$=\alpha_1 e^{2i\lambda x}\overline{\alpha_1}e^{-2i\lambda x}-\alpha_1 e^{2i\lambda x}\overline{\beta_1}e^{i(\mu-\lambda)x}-\beta_1 e^{i(\lambda-\mu)x}\overline{\alpha_1}e^{-2i\lambda x}$$
$$+\beta_1 e^{i(\lambda-\mu)x}\overline{\beta_1}e^{i(\mu-\lambda)x}=\alpha_1\overline{\alpha_1}e^{-2i\lambda x}e^{2i\lambda x}-\alpha_1\overline{\beta_1}e^{2i\lambda x}e^{i(\mu-\lambda)x}$$
$$-\beta_1\overline{\alpha_1}e^{i(\lambda-\mu)x}e^{-2i\lambda x}+\beta_1\overline{\beta_1}e^{i(\lambda-\mu)x}e^{i(\mu-\lambda)x}$$
$$=\alpha_1\overline{\alpha_1}-\alpha_1\overline{\beta_1}e^{2i\lambda x}e^{i(\mu-\lambda)x}-\beta_1\overline{\alpha_1}e^{i(\lambda-\mu)x}e^{-2i\lambda x}+\beta_1\overline{\beta_1}$$
$$=|\alpha_1|^2-\alpha_1\overline{\beta_1}e^{2i\lambda x}e^{i(\mu-\lambda)x}-\overline{\alpha_1}\beta_1 e^{-2i\lambda x}e^{i(\lambda-\mu)x}+|\beta_1|^2$$
$$=|\alpha_1|^2-\left(\alpha_1\overline{\beta_1}e^{2i\lambda x}e^{i(\mu-\lambda)x}+\overline{\alpha_1\overline{\beta_1}e^{2i\lambda x}e^{i(\mu-\lambda)x}}\right)+|\beta_1|^2$$
$$=|\alpha_1|^2-2\operatorname{Re}\alpha_1\overline{\beta_1}e^{2i\lambda x}e^{i(\mu-\lambda)x}+|\beta_1|^2$$
$$=|\alpha_1|^2-2\operatorname{Re}\alpha_1\operatorname{Re}\overline{\beta_1}\operatorname{Re}e^{2i\lambda x}\operatorname{Re}e^{i(\mu-\lambda)x}+|\beta_1|^2$$
$$=|\alpha_1|^2-2\operatorname{Re}\alpha_1\operatorname{Re}\overline{\beta_1}\cos 2\lambda x\cos(\mu-\lambda)x+|\beta_1|^2$$

That is

$$\left|\alpha_1\overline{e^{-2i\lambda x}}-\beta_1\overline{e^{i(\mu-\lambda)x}}\right|^2=\left|\alpha_1\right|^2-2\operatorname{Re}\alpha_1\operatorname{Re}\overline{\beta_1}\cos 2\lambda x\cos(\mu-\lambda)x+\left|\beta_1\right|^2. \quad (157)$$

And

$$\begin{aligned}
&\left|\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right|^2\\
&=\left(\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right)\left(\overline{\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}}\right)\\
&=\left(\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right)\left(\overline{\beta_1 e^{i(\lambda-\mu)x}}-\overline{\gamma_1 e^{2i(\mu-\lambda)x}}\right)\\
&=\left(\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right)\left(\overline{\beta_1}\,\overline{e^{i(\lambda-\mu)x}}-\overline{\gamma_1}\,\overline{e^{2i(\mu-\lambda)x}}\right)\\
&=\left(\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right)\left(\overline{\beta_1} e^{i(\mu-\lambda)x}-\overline{\gamma_1} e^{2i(\lambda-\mu)x}\right)\\
&=\overline{\beta_1}\beta_1 e^{i(\lambda-\mu)x}e^{i(\mu-\lambda)x}-\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}\\
&\quad-\overline{\beta_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{i(\mu-\lambda)x}+\overline{\gamma_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{2i(\lambda-\mu)x}\\
&=\overline{\beta_1}\beta_1 e^{-i(\mu-\lambda)x}e^{i(\mu-\lambda)x}-\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}\\
&\quad-\overline{\beta_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{i(\mu-\lambda)x}+\overline{\gamma_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{-2i(\mu-\lambda)x}\\
&=\left|\beta_1\right|^2-\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}-\overline{\beta_1}\gamma_1 e^{i(\mu-\lambda)x}e^{2i(\mu-\lambda)x}+\left|\gamma_1\right|^2\\
&=\left|\beta_1\right|^2-\left(\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}+\overline{\beta_1}\gamma_1 e^{i(\mu-\lambda)x}e^{2i(\mu-\lambda)x}\right)+\left|\gamma_1\right|^2\\
&=\left|\beta_1\right|^2-\left(\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}+\overline{\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}}\right)+\left|\gamma_1\right|^2\\
&=\left|\beta_1\right|^2-2\operatorname{Re}\overline{\gamma_1}\beta_1 e^{i(\lambda-\mu)x}e^{2i(\lambda-\mu)x}+\left|\gamma_1\right|^2\\
&=\left|\beta_1\right|^2-2\operatorname{Re}\overline{\gamma_1}\operatorname{Re}\beta_1\operatorname{Re}e^{i(\lambda-\mu)x}\operatorname{Re}e^{2i(\lambda-\mu)x}+\left|\gamma_1\right|^2\\
&=\left|\beta_1\right|^2-2\operatorname{Re}\overline{\gamma_1}\operatorname{Re}\beta_1\cos(\lambda-\mu)x\cos 2(\lambda-\mu)+\left|\gamma_1\right|^2
\end{aligned}$$

That is

$$\begin{aligned}
&\left|\beta_1 e^{i(\lambda-\mu)x}-\gamma_1 e^{2i(\mu-\lambda)x}\right|^2\\
&=\left|\beta_1\right|^2-2\operatorname{Re}\overline{\gamma_1}\operatorname{Re}\beta_1\cos(\lambda-\mu)x\cos 2(\lambda-\mu)+\left|\gamma_1\right|^2.
\end{aligned} \quad (158)$$

And

$$\begin{aligned}
&\left|\gamma_1 e^{2i(\mu-\lambda)x}-\delta_1 e^{i(\mu-\lambda)x}\right|^2\\
&=\left(\gamma_1 e^{2i(\mu-\lambda)x}-\delta_1 e^{i(\mu-\lambda)x}\right)\left(\overline{\gamma_1 e^{2i(\mu-\lambda)x}-\delta_1 e^{i(\mu-\lambda)x}}\right)\\
&=\left(\gamma_1 e^{2i(\mu-\lambda)x}-\delta_1 e^{i(\mu-\lambda)x}\right)\left(\overline{\gamma_1}\,\overline{e^{2i(\mu-\lambda)x}}-\overline{\delta_1}\,\overline{e^{i(\mu-\lambda)x}}\right)\\
&=\left(\gamma_1 e^{2i(\mu-\lambda)x}+\delta_1 e^{i(\mu-\lambda)x}\right)\left(\overline{\gamma_1} e^{-2i(\mu-\lambda)x}+\overline{\delta_1} e^{-i(\mu-\lambda)x}\right)\\
&=\overline{\gamma_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{-2i(\mu-\lambda)x}-\overline{\delta_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}\\
&\quad-\overline{\gamma_1}\delta_1 e^{i(\mu-\lambda)x}e^{-2i(\mu-\lambda)x}+\overline{\delta_1}\delta_1 e^{i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}\\
&=\overline{\gamma_1}\gamma_1-\overline{\delta_1}\gamma_1 e^{2i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}-\overline{\gamma_1}\delta_1 e^{i(\mu-\lambda)x}e^{-2i(\mu-\lambda)x}+\overline{\delta_1}\delta_1\\
&=\left|\gamma_1\right|^2-\gamma_1\overline{\delta_1}e^{2i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}-\overline{\gamma_1}\delta_1 e^{-2i(\mu-\lambda)x}e^{i(\mu-\lambda)x}+\left|\delta_1\right|^2\\
&=\left|\gamma_1\right|^2-\left(\gamma_1\overline{\delta_1}e^{2i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}+\overline{\gamma_1\overline{\delta_1}e^{2i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}}\right)+\left|\delta_1\right|^2\\
&=\left|\gamma_1\right|^2-2\operatorname{Re}\gamma_1\overline{\delta_1}e^{2i(\mu-\lambda)x}e^{-i(\mu-\lambda)x}+\left|\delta_1\right|^2
\end{aligned}$$

$$= |\gamma_1|^2 - 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \operatorname{Re} e^{2i(\mu-\lambda)x} \operatorname{Re} e^{-i(\mu-\lambda)x} + |\delta_1|^2$$
$$= |\gamma_1|^2 - 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \cos 2(\mu-\lambda)x \cos(\mu-\lambda)x + |\delta_1|^2$$

That is

$$\begin{aligned} &\left|\gamma_1 e^{2i(\mu-\lambda)x} - \delta_1 e^{i(\mu-\lambda)x}\right|^2 \\ &= |\gamma_1|^2 - 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \cos 2(\mu-\lambda)x \cos(\mu-\lambda)x + |\delta_1|^2. \end{aligned} \tag{159}$$

Then

$$\begin{aligned} &\left|\alpha_1 e^{-2i\lambda x} + \beta_1 e^{i(\mu-\lambda)x} + \gamma_1 e^{2i(\lambda-\mu)x} + \delta_1 e^{i(\lambda-\mu)x}\right|^2 \\ &= 4\left(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2\right) - \left|\alpha_1 e^{2i\lambda x} - \beta_1 e^{i(\lambda-\mu)x}\right|^2 \\ &\quad - \left|\beta_1 e^{i(\lambda-\mu)x} - \gamma_1 e^{2i(\mu-\lambda)x}\right|^2 - \left|\gamma_1 e^{2i(\mu-\lambda)x} - \delta_1 e^{i(\mu-\lambda)x}\right|^2 \\ &= 4\left(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2\right) - |\alpha_1|^2 + 2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} \cos 2\lambda x \cos(\mu-\lambda)x \\ &\quad - |\beta_1|^2 - |\beta_1|^2 + 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 \cos(\lambda-\mu)x \cos 2(\lambda-\mu) - |\gamma_1|^2 \\ &\quad - |\gamma_1|^2 + 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \cos 2(\mu-\lambda)x \cos(\mu-\lambda)x - |\delta_1|^2 \\ &= 4\left(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2\right) - |\alpha_1|^2 - |\beta_1|^2 - |\beta_1|^2 - |\gamma_1|^2 - |\gamma_1|^2 - |\delta_1|^2 \\ &\quad + 2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} \cos 2\lambda x \cos(\mu-\lambda)x + 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 \cos(\lambda-\mu)x \cos 2(\lambda-\mu) \\ &\quad + 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \cos 2(\mu-\lambda)x \cos(\mu-\lambda)x \\ &= 4|\alpha_1|^2 + 4|\beta_1|^2 + 4|\gamma_1|^2 + 4|\delta_1|^2 - |\alpha_1|^2 - |\beta_1|^2 - |\beta_1|^2 - |\gamma_1|^2 - |\gamma_1|^2 - |\delta_1|^2 \\ &\quad + 2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} \cos 2\lambda x \cos(\mu-\lambda)x + 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 \cos(\lambda-\mu)x \cos 2(\lambda-\mu) \\ &\quad + 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \cos 2(\mu-\lambda)x \cos(\mu-\lambda)x \\ &= 3|\alpha_1|^2 + 2|\beta_1|^2 + 2|\gamma_1|^2 + 3|\delta_1|^2 \\ &\quad + 2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} \cos 2\lambda x \cos(\mu-\lambda)x + 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 \cos(\lambda-\mu)x \cos 2(\lambda-\mu) \\ &\quad + 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \cos 2(\mu-\lambda)x \cos(\mu-\lambda)x \\ &\ge 3|\alpha_1|^2 + 2|\beta_1|^2 + 2|\gamma_1|^2 + 3|\delta_1|^2 - 2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} - 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 - 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \end{aligned}$$

Therefore

$$\begin{aligned} &\left|\alpha_1 e^{-2i\lambda x} + \beta_1 e^{i(\mu-\lambda)x} + \gamma_1 e^{2i(\lambda-\mu)x} + \delta_1 e^{i(\lambda-\mu)x}\right|^2 \\ &\ge 3|\alpha_1|^2 + 2|\beta_1|^2 + 2|\gamma_1|^2 + 3|\delta_1|^2 - 2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} - 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 - 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \end{aligned} \tag{160}$$

From Equation (158)

$$\alpha_1 := (-\lambda)^{-l} F_l(\lambda) i^{-l}, \beta_1 := \left(\mu^2 - \lambda^2\right) i^{2l+1}\mu, \gamma_1 := (-\mu)^{-l} F_l(\mu) i^{-l},$$
$$\delta_1 := \left(\lambda^2 - \mu^2\right) i^{l+1}\lambda, \epsilon_1 := \left(\mu^2 - \lambda^2\right) i^{l+1}.$$

Then

$$\begin{aligned} &-2\operatorname{Re}\alpha_1 \operatorname{Re}\overline{\beta_1} - 2\operatorname{Re}\overline{\gamma_1} \operatorname{Re}\beta_1 - 2\operatorname{Re}\gamma_1 \operatorname{Re}\overline{\delta_1} \\ &= -2\operatorname{Re}\left[(-\lambda)^{-l} F_l(\lambda) i^{-l}\right] \operatorname{Re}\left[\overline{\left(\mu^2 - \lambda^2\right) i^{2l+1}\mu}\right] \\ &\quad - 2\operatorname{Re}\left[\overline{(-\mu)^{-l} F_l(\mu) i^{-l}}\right] \operatorname{Re}\left[\left(\mu^2 - \lambda^2\right) i^{2l+1}\mu\right] \end{aligned}$$

$$
\begin{aligned}
&-2\operatorname{Re}\left[(-\mu)^{-l} F_l(\mu) i^{-l}\right]\operatorname{Re}\left[\overline{\left(\lambda^2-\mu^2\right) i^{l+1}\lambda}\right] \\
&=-2\operatorname{Re}\left[(-\lambda)^{-l} F_l(\lambda) i^{-l}\right]\operatorname{Re}\left[\left(\mu^2-\lambda^2\right)\left(\overline{i^{2l+1}}\right)\mu\right] \\
&-2\operatorname{Re}\left[\overline{(-\mu)^{-l} F_l(\mu) i^{-l}}\right]\operatorname{Re}\left[\left(\mu^2-\lambda^2\right) i^{2l+1}\mu\right] \\
&-2\operatorname{Re}\left[(-\mu)^{-l} F_l(\mu) i^{-l}\right]\operatorname{Re}\left[\left(\lambda^2-\mu^2\right)\left(\overline{i^{l+1}}\right)\lambda\right] \\
&=-2\operatorname{Re}\left[(-\lambda)^{-l} F_l(\lambda) i^{-l}\right]\operatorname{Re}\left[\left(\mu^2-\lambda^2\right)\left((-i)^{2l+1}\right)\mu\right] \\
&-2\operatorname{Re}\left[\overline{(-\mu)^{-l} F_l(\mu) i^{-l}}\right]\operatorname{Re}\left[\left(\mu^2-\lambda^2\right) i^{2l+1}\mu\right] \\
&-2\operatorname{Re}\left[(-\mu)^{-l} F_l(\mu) i^{-l}\right]\operatorname{Re}\left[\left(\lambda^2-\mu^2\right)\left((-i)^{l+1}\right)\lambda\right] \\
&=-2\operatorname{Re}\left[(-\lambda)^{-l} F_l(\lambda) i^{-l}\right]\left[\operatorname{Re}\left(\mu^2-\lambda^2\right)\operatorname{Re}(-i)^{2l+1}\operatorname{Re}\mu\right] \\
&-2\operatorname{Re}\left[\overline{(-\mu)^{-l} F_l(\mu) i^{-l}}\right]\left[\operatorname{Re}\left(\mu^2-\lambda^2\right)\operatorname{Re} i^{2l+1}\operatorname{Re}\mu\right]
\end{aligned}
$$

That is

$$
\begin{aligned}
&-2\operatorname{Re}\alpha_1\operatorname{Re}\overline{\beta_1}-2\operatorname{Re}\overline{\gamma_1}\operatorname{Re}\beta_1-2\operatorname{Re}\gamma_1\operatorname{Re}\overline{\delta_1} \\
&=-2\operatorname{Re}\left[(-\lambda)^{-l} F_l(\lambda) i^{-l}\right]\left[\operatorname{Re}\left(\mu^2-\lambda^2\right)\operatorname{Re}(-i)^{2l+1}\operatorname{Re}\mu\right] \\
&-2\operatorname{Re}\left[\overline{(-\mu)^{-l} F_l(\mu) i^{-l}}\right]\left[\operatorname{Re}\left(\mu^2-\lambda^2\right)\operatorname{Re} i^{2l+1}\operatorname{Re}\mu\right]
\end{aligned} \tag{161}
$$

Since

$$
\operatorname{Re}(-i)^{2l+1}=\operatorname{Re}\mathrm{e}^{-\frac{\pi}{2}(2l+1)i}=\cos\frac{\pi}{2}(2l+1)=\cos\left(\pi l+\frac{\pi}{2}\right)=0,\forall l\in\mathbf{Z}. \tag{162}
$$

$$
\operatorname{Re} i^{2l+1}=\operatorname{Re}\mathrm{e}^{\frac{\pi}{2}(2l+1)i}=\cos\frac{\pi}{2}(2l+1)=\cos\left(\pi l+\frac{\pi}{2}\right)=0,\forall l\in\mathbf{Z}. \tag{163}
$$

and

$$
\operatorname{Re}(-i)^{l+1}=\operatorname{Re}\mathrm{e}^{-\frac{-\pi}{2}(l+1)i}=\cos\frac{\pi}{2}(l+1)=0,\ \text{for } l \text{ even}. \tag{164}
$$

it follows that

$$
-2\operatorname{Re}\alpha_1\operatorname{Re}\overline{\beta_1}-2\operatorname{Re}\overline{\gamma_1}\operatorname{Re}\beta_1-2\operatorname{Re}\gamma_1\operatorname{Re}\overline{\delta_1}=0,\ \ \forall l \text{ even}.
$$

That is

$$
\begin{aligned}
&3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2-2\operatorname{Re}\alpha_1\operatorname{Re}\overline{\beta_1}-2\operatorname{Re}\overline{\gamma_1}\operatorname{Re}\beta_1-2\operatorname{Re}\gamma_1\operatorname{Re}\overline{\delta_1} \\
&=3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2>0,\forall l \text{ even}.
\end{aligned} \tag{165}
$$

Therefore

$$
\begin{aligned}
&\left|\alpha_1\mathrm{e}^{-2i\lambda x}+\beta_1\mathrm{e}^{i(\mu-\lambda)x}+\gamma_1\mathrm{e}^{2i(\lambda-\mu)x}+\delta_1\mathrm{e}^{i(\lambda-\mu)x}\right|^2 \\
&\geq 3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2>0,\forall l \text{ even}.
\end{aligned} \tag{166}
$$

That is

$$\frac{1}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2} \leq \frac{1}{3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2}, \forall l \text{ even.} \tag{167}$$

Now it is calculated

$$\left|\frac{\varphi_1'}{\varphi_2}\right| \text{ for } x>1, \tag{168}$$

so

$$\begin{aligned}\left|\frac{\varphi_1'}{\varphi_2}\right| &= \frac{\left|\epsilon_1\lambda\left(\overline{\alpha_1}e^{2i\lambda x}+\overline{\beta_1}e^{i(\lambda-\mu)x}+\overline{\gamma_1}e^{2i(\mu-\lambda)x}+\overline{\delta_1}e^{i(\mu-\lambda)x}\right)\right|}{\left|\alpha_1 e^{-2i\lambda x}+\beta_1 e^{i(\mu-\lambda)x}+\gamma_1 e^{2i(\lambda-\mu)x}+\delta_1 e^{i(\lambda-\mu)x}\right|^2} \\ &\leq \frac{\left|\lambda\epsilon_1\lambda\overline{\alpha_1}\right|\left|e^{2i\lambda x}\right|+\left|\epsilon_1\lambda\overline{\beta_1}\right|\left|e^{i(\lambda-\mu)x}\right|+\left|\epsilon_1\lambda\overline{\gamma_1}\right|\left|e^{2i(\mu-\lambda)x}\right|+\left|\epsilon_1\lambda\overline{\delta_1}\right|\left|e^{i(\mu-\lambda)x}\right|}{3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2} \\ &= \frac{\left|\lambda\epsilon_1\lambda\overline{\alpha_1}\right|+\left|\epsilon_1\lambda\overline{\beta_1}\right|+\left|\epsilon_1\lambda\overline{\gamma_1}\right|+\left|\epsilon_1\lambda\overline{\delta_1}\right|}{3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2}\end{aligned}$$

That is

$$\left|\frac{\varphi_1'}{\varphi_2}\right| \leq \frac{\left|\lambda\epsilon_1\lambda\overline{\alpha_1}\right|+\left|\epsilon_1\lambda\overline{\beta_1}\right|+\left|\epsilon_1\lambda\overline{\gamma_1}\right|+\left|\epsilon_1\lambda\overline{\delta_1}\right|}{3|\alpha_1|^2+2|\beta_1|^2+2|\gamma_1|^2+3|\delta_1|^2}, \quad \forall l \text{ even and } x>1. \tag{179}$$

Therefore

$$\frac{\varphi_1'}{\varphi_2} \underset{x\to\infty}{=} O(1), \forall l \text{ even.} \tag{170}$$

And in fact, if it is defined

$$\frac{\left(\mu^2-\lambda^2\right)l(l+1)x^{2l+1}}{(2l+1)!!\left[(-\lambda)^{-l}F_l(\lambda)+(-\mu)^{-l}F_l(\mu)\right](2l+1)!!} := C \tag{171}$$

Therefore

$$\frac{\varphi_1'}{\varphi_2} = \begin{cases} Cx^{2l+1}, & \text{if } 0<x<1, \\ O(1), & \text{if } x>1, \forall l \text{ even.} \end{cases} \tag{172}$$