Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2019, 7, 1027-1042
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Asymptotic Periodicity in the Fecally-Orally Epidemic Model in a Heterogeneous Environment

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How to cite this paper: Tarboush, A.K. and Zhang, Z.D. (2019) Asymptotic Periodicity in the Fecally-Orally Epidemic Model in a Heterogeneous Environment. *Journal of Applied Mathematics and Physics*, 7, 1027-1042.
https://doi.org/10.4236/jamp.2019.75069

Received: April 14, 2019
Accepted: May 13, 2019
Published: May 16, 2019

Abstract

To understand the influence of seasonal periodicity and environmental heterogeneity on the transmission dynamics of an infectious disease, we consider asymptotic periodicity in the fecally-orally epidemic model in a heterogeneous environment. By using the next generation operator and the related eigenvalue problems, the basic reproduction number is introduced and shows that it plays an important role in the existence and non-existence of a positive T-periodic solution. The sufficient conditions for the existence and non-existence of a positive T-periodic solution are provided by applying upper and lower solutions method. Our results showed that the fecally-orally epidemic model in a heterogeneous environment admits at least one positive T-periodic solution if the basic reproduction number is greater than one, while no T-periodic solution exists if the basic reproduction number is less than or equal to one. By means of monotone iterative schemes, we construct the true positive solutions. The asymptotic behavior of periodic solutions is presented. To illustrate our theoretical results, some numerical simulations are given. The paper ends with some conclusions and future considerations.

Keywords

Fecally-Orally Epidemic Model, Basic Reproduction Number, Time Periodicity, Asymptotic Behavior

1. Introduction

The geographic transmission of infectious diseases is an important issue in mathematical epidemiology and various epidemic models have been proposed and analyzed by many researchers [1] [2] [3]. For instance, Capasso and Paveri-Fontana [2] formulated and studied a non-spatial model to investigate the cholera epidemic which spread in the European Mediterranean regions. To discover the impact of spatial spread of a class of bacterial and viral diseases, Capasso and Maddalena [3] proposed the following reaction-diffusion system

$$\begin{cases} u_t(x,t) - d_1 \Delta u(x,t) = -\mu u(x,t) + \alpha v(x,t), & (x,t) \in \Omega \times (0,+\infty), \\ v_t(x,t) - d_2 \Delta v(x,t) = -\gamma v(x,t) + g(u(x,t)), & (x,t) \in \Omega \times (0,+\infty), \end{cases} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ represent the spatial density of an infectious agent and an infected human population at point x in the habitat Ω and time $t \geq 0$, respectively. Here, d_1, d_2, μ, γ and α are positive constants, $\frac{1}{\mu}$ means the lifetime of the agent in the environment, $\frac{1}{\gamma}$ denotes the mean infectious period of the human infections, $\frac{1}{\alpha}$ is the multiplicative factor of the mean infectious agent to the human population, and $g(u)$ is the force of infection on the human population due to a concentration u of the infectious agent.

The dynamics of the spatially dependent model (1) and its corresponding Cauchy problem have been considered by many scholars. For example, the traveling waves' solutions were studied in [4]. Wang *et al.* [5] studied entire solutions in a time-delayed and diffusive epidemic model. In addition, Ahn *et al.* [1] discussed the corresponding free boundary problem. Recently, the corresponding strongly coupled elliptic system has been proposed in [6] to investigate the existence and non-existence of coexistence states.

In recent years, a great deal of mathematical models has been developed to study the impact of seasonal periodicity and environmental heterogeneity on the dynamics of infectious diseases [7] [8] [9]. On the other hand, the periodicity of environmental factors is realistic and highly important for the dynamics of infectious diseases.

Taking this fact into account, we extend model (1) to the following reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t} - d_1(x,t)\Delta u = -\mu(x,t)u + \alpha(x,t)v, & (x,t) \in \Omega \times (0,+\infty), \\ \frac{\partial v}{\partial t} - d_2(x,t)\Delta v = -\gamma(x,t)v + g(u), & (x,t) \in \Omega \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{2}$$

with the periodic condition

$$u(x,0) = u(x,T), v(x,0) = v(x,T), \ x \in \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathcal{R}^N$ with smooth boundary $\partial\Omega$, ν denotes the outward unit vector on $\partial\Omega$. Here, we assumed that the boundary $\partial\Omega$ is of a class $C^{2+\alpha}$ for some $0 < \alpha < 1$; and $d_1(x,t)$, $d_2(x,t)$, $\mu(x,t)$, $\alpha(x,t)$ and $\gamma(x,t)$ are all sufficiently smooth and strictly positive functions defined in $\bar{\Omega}\times[0,\infty)$ and are T-periodic.

Moreover, in order to study the attractivity of T-periodic solutions of problem (2), (3), we consider model (2) under the initial condition

$$u(x,0)=\eta_1(x), \quad v(x,0)=\eta_2(x), \quad x\in\Omega, \tag{4}$$

and the initial functions $\eta_i(x)\in C^2(\bar{\Omega})$ $(i=1,2)$ are nonnegative.

Furthermore, we make the following hypothesis on the function g

(H_1) $g\in C^1([0,\infty)), g(0), g'(z)>0, \forall z\geq 0$;

(H_2) $\frac{g(z)}{z}$ is decreasing and

$$\limsup_{z\to\infty}\frac{g(z)}{z}<\min_{\bar{\Omega}\times(0,T]}\left\{\frac{\mu(x,t)}{\alpha(x,t)}\right\}\cdot\min_{\bar{\Omega}\times(0,T]}\{\gamma(x,t)\}$$

Interest in problems (2), (3) and (2), (4) is motivated by valuable results about the periodic solution of a weakly-parabolic systems [10] [11] [12] [13], all those researchers have used the upper and lower solutions method developed by Pao [14], which is powerful and effective to derive the periodic solutions. For instance, various computation algorithms for numerical solutions of the periodic boundary problem were studied in [11]. In [12] [13], the stability and attractivity analysis, which are for quasimonotone nondecreasing and mixed quasimonotone reaction functions by the monotone iteration scheme, were given. It is important to mentioned that there are other standard approaches to derive the periodic solution of a weakly-parabolic systems, Hopf bifurcation theorem and Lyapunov functional [7] [15], numerical methods [11] [16], Poincaré index [17], Schauder fixed point theorem [18] [19], etc.

The remainder of this paper is organized as follows. In the next section, we deal with the basic reproduction number of problem (2), (3) and its properties. The existence and non-existence of T-periodic solutions of problem (2), (3) and the long time behavior of problem (2), (4) are given in Section 3. Section 4 is devoted to numerical simulations and a brief discussion. Finally, we give some conclusions and future considerations in Section 5.

2. Basic Reproduction Numbers

The focus of this section is to present the basic reproduction number and its properties for the corresponding system in Ω. It is worth emphasis that the basic reproduction number is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during its entire interval of infectiousness [20]. For spatially-independent epidemic models, which are described by ordinary differential systems, the numbers are usually calculated by the next generation matrix method [21], while for the models constructed by reaction-diffusion systems, the numbers are formulated as the spectral radius of next infection operator induced by a new infection rate matrix and an evolution operator of an infective distribution [22], and the numbers could be expressed in the term of the principal eigenvalues of relevant eigenvalue problems [17] [23].

Considering the linearized problem of (2), (3), we have

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1(x,t)\Delta u = -\mu(x,t)u + \alpha(x,t)v, & (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial v}{\partial t} - d_2(x,t)\Delta v = -\gamma(x,t)v + g'(0)u, & (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u(x,T), \quad v(x,0) = v(x,T), & \Omega. \end{cases} \tag{5}$$

Letting

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \begin{pmatrix} d_1(x,t) & 0 \\ 0 & d_2(x,t) \end{pmatrix},$$

$$F(x,t) = \begin{pmatrix} 0 & \alpha(x,t) \\ g'(0) & 0 \end{pmatrix}, \quad V(x,t) = \begin{pmatrix} \mu(x,t) & 0 \\ 0 & \gamma(x,t) \end{pmatrix},$$

then we have

$$\begin{cases} \dfrac{\partial w}{\partial t} - D\Delta w = F(x,t)w - V(x,t)w, & \Omega \times (0,+\infty), \\ \dfrac{\partial w}{\partial \nu} = 0, & \partial\Omega \times (0,+\infty). \end{cases} \tag{6}$$

Consider the following problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - D\Delta w = -V(x,t)w, & \Omega \times (0,+\infty), \\ \dfrac{\partial w}{\partial \nu} = 0, & \partial\Omega \times (0,+\infty). \end{cases} \tag{7}$$

and let $E(t,s)$ be the evolution operator of (7), then according to the standard semigroup theory there exist positive constants $K, c_0 > 0$ such that

$$\|E(t,s)\| \le K\mathrm{e}^{-c_0(t-s)}$$

for any $t \ge s, t, s \in \mathcal{R}$.

Suppose that $\hbar = (\phi,\psi)$ is the density distribution of u at the spatial location $x \in \Omega$ and time s. We use the same idea in [22] [23] to introduce the linear operator

$$L(\hbar)(t) := \int_0^\infty E(t,t-s)F(\cdot,t-s)\hbar(\cdot,t-s)\mathrm{d}s,$$

where $\hbar \in \mathcal{C}_T := \left\{\hbar : t \to w(t) \in (\mathcal{C}(\Omega))^2, w(0)(x) = w(T)(x)\right\}$. According to the definition, we can easily know that L is continuous, strong positive and compact on $\mathcal{C}_T$. We now define the basic reproduction number of system (5) by the spectral radius of L as follows

$$R_0^N := \rho(L).$$

To ensure the existence of the basic reproduction numbers, we consider now the following linear periodic-parabolic eigenvalue problem

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - d_1(x,t)\Delta\phi = \dfrac{\alpha(x,t)}{R}\psi - \mu(x,t)\phi + \lambda\phi, & \Omega\times(0,T], \\ \dfrac{\partial \psi}{\partial t} - d_2(x,t)\Delta\psi = g'(0)\phi - \gamma(x,t)\psi + \lambda\psi, & \Omega\times(0,T], \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0, & \partial\Omega\times(0,T], \\ \phi(x,0) = \phi(x,T), \quad \psi(x,0) = \psi(x,T), & \Omega, \end{cases} \tag{8}$$

where $R > 0$. Letting

$$L_R = \begin{pmatrix} \dfrac{\partial}{\partial t} - d_1(x,t)\Delta + \mu(x,t) & -\dfrac{\alpha(x,t)}{R} \\ -\dfrac{g'(0)}{R} & \dfrac{\partial}{\partial t} - d_2(x,t)\Delta + \gamma(x,t) \end{pmatrix}$$

then the corresponding abstract eigenvalue problem of (8) becomes

$$L_R \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{9}$$

in the space

$$X = \left\{ (\phi,\psi) \in \left(C^{\alpha,\frac{\alpha}{2}}\left(\bar{\Omega}\times[0,T]\right)\right)^2 : \phi,\psi \text{ are } T\text{-periodic with respect to } t \right\},$$

and the domain of the operator $\mathrm{dom}(L_R) = X_1$ is denoted by

$$X_1 = \left\{ (\phi,\psi) \in \left(C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{\Omega}\times[0,T]\right)\right)^2 : \phi = \psi = 0 \text{ on } \partial\Omega\times(0,T], \right.$$
$$\left. \phi,\psi \text{ are } T\text{-periodic in } t \right\}.$$

Since the system (9) is strongly cooperative, it follows from [24] [25] [26] that for any $R > 0$, there are a unique value $\lambda := \lambda_1(R)$, and called the principal eigenvalue, such that problem (8) admits a unique solution pair (ϕ_R, ψ_R) (subject to constant multiples) with $\phi_R > 0$ and $\psi_R > 0$ in $\Omega\times(0,T]$. That is, the solution pair $(\phi_R, \psi_R) \in X_1$ is called the principal eigenfunction. Besides, $\lambda_1(R)$ is algebraically simple and dominant, and the following result hold.

Lemma 2.1 $\lambda_1(R)$ *is continuous and strictly increasing with respect to R.*

In light of the above discussion, we can further have the following relation between the two eigenvalues.

Theorem 2.1 $sign\left(1 - R_0^N\right) = sign(\lambda_0)$, *and* λ_0 *is the principal eigenvalue of the eigenvalue problem*

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - d_1(x,t)\Delta\phi = \alpha(x,t)\psi - \mu(x,t)\phi + \lambda_0\phi, & \Omega\times((0,T], \\ \dfrac{\partial \psi}{\partial t} - d_2(x,t)\Delta\psi = g'(0)\phi - \gamma(x,t)\psi + \lambda_0\psi, & \Omega\times(0,T], \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0, & \partial\Omega\times(0,T], \\ \phi(x,0) = \phi(x,T), \quad \psi(x,0) = \psi(x,T), & \Omega, \end{cases} \tag{10}$$

and R_0^N is the unique principal eigenvalue of problem

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - d_1(x,t)\Delta\phi = \dfrac{\alpha(x,t)}{R_0^N}\psi - \mu(x,t)\phi, & \Omega\times(0,T], \\ \dfrac{\partial \psi}{\partial t} - d_2(x,t)\Delta\psi = \dfrac{g'(0)}{R_0^N}\phi - \gamma(x,t)\psi, & \Omega\times(0,T], \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0, & \partial\Omega\times(0,T], \\ \phi(x,0) = \phi(x,T), \quad \psi(x,0) = \psi(x,T), & \Omega. \end{cases} \tag{11}$$

Proof: According to (8) and (10), it is obvious that $\lambda_0 = \lambda_1(1)$. Meanwhile, it follows from the monotonicity of coefficients in (8), one can easily deduce that $\lim_{R\to 0^+}\lambda_1(R) < 0$ and $\lim_{R\to+\infty}\lambda_1(R) > 0$, therefore R_0^N is the unique positive root of the equation $\lambda_1(R) = 0$. One finally gets the desired result by the monotonicity of $\lambda_1(R)$ with respect to *R*. □

Remark 2.1 *Recalling the monotone non-increasing of* λ_1 *with respect to* $\alpha(x,t)$ *in the sense that* $\lambda_1(\alpha_1(x,t)) < \lambda_1(\alpha_2(x,t))$, *if* $\alpha_1(x,t) \ge \alpha_2(x,t)$ *and* $\alpha_1(x,t) \not\equiv \alpha_2(x,t)$ *in* Ω, *one can deduce from Theorem* 2.1 *that* R_0^N *is monotonically nondecreasing with respect to* $\alpha(x,t)$ *and* $R_0^N > 1$ *if* $\alpha(x,t)$ *is big enough.*

In what follows, we provide an explicit formula for R_0^N if all coefficients are constants.

Theorem 2.2 *Suppose that* $d_1(x,t) = d_1^*, d_2(x,t) = d_2^*$, $\alpha(x,t) = \alpha^*$ *and* $\mu(x,t) = \mu^*$. *Then the principal eigenvalue* R_0^N *for* (11), *or a threshold parameter for model* (2), (4), *is expressed by*

$$R_0^N(\Omega) = \sqrt{\frac{\alpha^* g'(0)}{\mu^*\gamma^*}}. \tag{12}$$

Proof: Let $\psi^*(x,t) \equiv 1$ in $\bar{\Omega}\times[0,+\infty)$ and

$$C^* = \frac{\alpha^* g'(0)}{\mu^*\gamma^*},$$

$$\phi^* = \frac{\alpha^*}{\sqrt{C^*}\mu^*}\psi^* \text{ in } \bar{\Omega}\times[0,+\infty).$$

Then we can check that (ϕ^*, ψ^*) is a positive solution of problem (11) with $R_0^N = \sqrt{R^*}$, and directly lead to (12) from the uniqueness of the principal eigenvalue of problem (11).

Define $f^i = \min_{\bar{\Omega}\times(0,T]} f(x,t)$ and $f^M = \max_{\bar{\Omega}\times(0,T]} f(x,t)$ for any given continuous T-periodic function *f*, we state the following estimate after combining Theorem 2.2 and the monotonicity of R_0^N with respect to all coefficients in (11).

Corollary 2.1 *The principal eigenvalue* R_0^N *for* (11) *satisfies*

$$\sqrt{\frac{g'(0)\alpha^i}{\gamma^M\mu^M}} \le R_0^N(\Omega) \le \sqrt{\frac{g'(0)\alpha^M}{\gamma^i\mu^i}}. \tag{13}$$

3. T-Periodic Solution

In this section, we discuss the existence and non-existence of a positive T-periodic solution of problem (2), (3) and the attractivity of the initial-boundary value problem (2), (4) in relation to the maximal and minimal T-periodic solutions of problem (2), (4). To begin with, the following theorem gives the non-existence of a T-periodic solution for problem (2), (3).

Theorem 3.1 *If* $R_0^N \le 1$, *problem* (2), (3) *has no positive T-periodic solution.*

Proof: If the assertion is not hold, then one can suppose that $\left(u^*(x,t), v^*(x,t)\right)$ is a positive T-periodic solution of problem (2), (3), that is, $u^*(x,t), v^*(x,t) > (0,0)$ in $\bar{\Omega}\times[0,+\infty)$ and satisfy

$$\begin{cases} \dfrac{\partial U^*}{\partial t} - d_1(x,t)\Delta u^* = \alpha(x,t)v^* - \mu(x,t)u^*, & (x,t)\in\Omega\times[0,\infty), \\ \dfrac{\partial v^*}{\partial t} - d_2(x,t)\Delta v^* = g(u^*) - \gamma(x,t)v^*, & (x,t)\in\Omega\times[0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times[0,\infty), \\ u^*(x,0) = u^*(x,T), \quad v^*(x,0) = v^*(x,T), & x\in\bar{\Omega}. \end{cases} \tag{14}$$

From (14), we can obtain that

$$\begin{cases} \dfrac{\partial U^*}{\partial t} - d_1(x,t)\Delta u^* < \alpha(x,t)v^* - \mu(x,t)u^*, & (x,t)\in\Omega\times[0,\infty), \\ \dfrac{\partial v^*}{\partial t} - d_2(x,t)\Delta v^* < g'(0)u^* - \gamma(x,t)v^*, & (x,t)\in\Omega\times[0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times[0,\infty), \\ u^*(x,0) = u^*(x,T), \quad v^*(x,0) = v^*(x,T), & x\in\bar{\Omega}. \end{cases} \tag{15}$$

On the other hand, the principal eigenvalue λ_0 in problem (10) meets

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - d_1(x,t)\Delta \phi = \alpha(x,t)\psi - \mu(x,t)\phi + \lambda_0\phi, & (x,t)\in\Omega\times[0,\infty), \\ \dfrac{\partial \psi}{\partial t} - d_2(x,t)\Delta \psi = g'(0)\phi - \gamma(x,t)\psi + \lambda_0\psi, & (x,t)\in\Omega\times[0,\infty), \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times[0,\infty), \\ \phi(x,0) = \phi(x,T), \quad \psi(x,0) = \psi(x,T), & x\in\bar{\Omega}, \end{cases} \tag{16}$$

It is follows from Lemma 2.1 that λ_0 is monotone non-increasing with respect to α, then by comparing (15) and (16) we can deduce that $\lambda_0 < 0$. Consequently, $R_0^N > 1$ by Theorem 2.1. This leads to a contradiction. □

Before approaching the existence result of a positive T-periodic solution to problem (2), (3), we need some preliminaries. Set $D = \Omega\times[0,\infty)$, $\bar{D} = \bar{\Omega}\times[0,\infty)$, $\Gamma = \partial\Omega\times[0,\infty)$, $\bar{\Omega} = \Omega\cup\partial\Omega$ and

$$S = \left\{(u,v)\in C(\bar{D}); (\hat{u},\hat{v}) \le (u,v) \le (\tilde{u},\tilde{v}), (x,t)\in\bar{D}\right\},$$

where $(\hat{u},\hat{v})$ and $(\tilde{u},\tilde{v})$ are given in the following definition. Then we have an equivalent form of problem (2), (3) as

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1(x,t)\Delta u = f_1(x,t,u,v), & (x,t) \in D, \\ \dfrac{\partial v}{\partial t} - d_1(x,t)\Delta v = f_2(x,t,u,v), & (x,t) \in D, \\ \dfrac{\partial h}{\partial \nu} = \dfrac{\partial m}{\partial \nu} = 0, & (x,t) \in \Gamma, \\ u(x,0) = u(x,T),\ v(x,0) = v(x,T), & x \in \bar{\Omega}, \end{cases} \tag{17}$$

where,

$$f_1(x,t,u,v) = -\mu(x,t)u + \alpha(x,t)v,$$
$$f_2(x,t,u,v) = -\gamma(x,t)v + g(u).$$

With regard to problem (17) we state the following definition about the upper and lower solutions.

Definition 3.1 $(\tilde{u},\tilde{v})$, $(\hat{u},\hat{v})$ *are ordered upper and lower solutions of problem* (17), *if* $(\hat{u},\hat{v}) \le (\tilde{u},\tilde{v})$ *and*

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1(x,t)\Delta u \ge f_1(x,t,u,v), & (x,t) \in D, \\ \dfrac{\partial v}{\partial t} - d_1(x,t)\Delta v \ge f_2(x,t,u,v), & (x,t) \in D, \\ \dfrac{\partial u}{\partial t} - d_1(x,t)\Delta u \le f_1(x,t,u,v), & (x,t) \in D, \\ \dfrac{\partial v}{\partial t} - d_1(x,t)\Delta v \le f_2(x,t,u,v), & (x,t) \in D, \\ \dfrac{\partial \tilde{u}}{\partial \nu} \ge 0 \ge \dfrac{\partial \hat{u}}{\partial \nu},\quad \dfrac{\partial \tilde{v}}{\partial \nu} \ge 0 \ge \dfrac{\partial \hat{v}}{\partial \nu}, & (x,t) \in \Gamma, \\ \tilde{u}(x,0) \ge \tilde{u}(x,T), \hat{u}(x,0) \le \hat{u}(x,T), & x \in \bar{\Omega}, \\ \tilde{v}(x,0) \ge \tilde{v}(x,T), \hat{v}(x,0) \le \hat{v}(x,T), & x \in \bar{\Omega}. \end{cases} \tag{18}$$

It is worth emphasizing that the upper and lower solutions in the above definition are not required to be T-periodic with respect to *t*.

Now to verify the existence of a positive T-periodic solution to problem (2), (3), it suffices to find a pair of ordered upper and lower solutions to problem (2), (3). One can seek such as in the form $(\tilde{u},\tilde{v}) = (M_1, M_2)$, $(\hat{u},\hat{v}) = (\delta\phi, \delta\psi)$ where M_i $(i=1,2)$ and δ are positive constant with δ sufficiently small, $(\phi,\psi) \equiv (\phi(x,t), \psi(x,t))$ is (normalized) positive eigenfunction corresponding to λ_0, and λ_0 is the principal eigenvalue of periodic-parabolic eigenvalue problem (10). Then it is easy to verify that $(\tilde{u},\tilde{v})$ and $(\hat{u},\hat{v})$ satisfy (18) if

$$\begin{cases} \dfrac{\partial M_1}{\partial t} - d_1(x,t)\Delta M_1 \ge \alpha(x,t)M_2 - \mu(x,t)M_1, \\ \dfrac{\partial M_2}{\partial t} - d_2(x,t)\Delta M_2 \ge g(M_1) - \gamma(x,t)M_2, \\ \dfrac{\partial(\delta\phi)}{\partial t} - d_1(x,t)\Delta(\delta\phi) \le \alpha(x,t)(\delta\psi) - \gamma_b(x,t)(\delta\phi), \\ \dfrac{\partial(\delta\psi)}{\partial t} - D_2(x,t)\Delta(\delta\psi) \le g'(\wp)(\delta\phi) - \gamma(x,t)(\delta\psi) \end{cases} \tag{19}$$

hold, where $\wp \in (0,\hat{u})$.

Recalling that $\limsup_{z\to\infty} \frac{g(z)}{z} < \min_{\bar{\Omega}\times(0,T]} \left\{ \frac{\mu(x,t)}{\alpha(x,t)} \right\} \cdot \min_{\bar{\Omega}\times(0,T]} \{\gamma(x,t)\}$, there exits constant M_0 such that $\frac{g(z)}{z} < \min_{\bar{\Omega}\times(0,T]} \left\{ \frac{\mu(x,t)}{\alpha(x,t)} \right\} \cdot \min_{\bar{\Omega}\times(0,T]} \{\gamma(x,t)\}$, for $z \ge M_0$. This implies that the first two inequalities of (19) will hold, if we take $(\tilde{u},\tilde{v}) = (M_1, M_2)$, where $M_1 = \max\{M_0, \max_{\bar{\Omega}} u(x,0), \max_{\bar{\Omega}} v(x,0)\}$, $M_2 = M_1 \cdot \max_{\bar{\Omega}\times(0,T]} \left\{ \frac{\mu(x,t)}{\alpha(x,t)} \right\}$.

Since $R_0^N > 1$, the principal eigenvalue of problem (10) is $\lambda_0 < 0$, so one can select δ small enough such that the last two inequalities of (19) hold.

Therefore, the function pair $(\tilde{u},\tilde{v}) = (M_1, M_2)$, $(\hat{u},\hat{v}) = (\delta\phi, \delta\psi)$ are ordered upper and lower solutions of problem (17), respectively.

To summarize the above conclusions, we have the following result.

Theorem 3.2 *If* $R_0^N > 1$, *then problem* (2), (3) *admits at least one positive T-periodic solution* $(u(x,t), v(x,t))$.

Remark 3.1 *Suppose that all coefficients of* (2) *are constants. The corresponding basic reproduction ratio* R_0^N *is represented by* (12). *If* α_b^* *is big enough, then problem* (2), (3) *admits at least one positive T-periodic solution. On the other hand, if* α_b^* *small enough, then problem* (2), (3) *has no positive T-periodic solution.*

We can now construct the true solutions of problem (17) by applying the monotone iterative scheme associated with the method of upper and lower solutions. Due to

$$F_1 = k_1 u + f_1, \quad F_2 = k_2 v + f_2,$$

we denote

$$k_1 = \max_{(x,t)\in\bar{\Omega}\times(0,T]} \mu(x,t), \quad k_2 = \max_{(x,t)\in\bar{\Omega}\times(0,T]} \gamma(x,t)$$

then problem (17) is equivalent to

$$\begin{cases} \frac{\partial u}{\partial t} - d_1(x,t)\Delta u + k_1 u = F_1(x,t,u,v), & (x,t)\in D, \\ \frac{\partial v}{\partial t} - d_2(x,t)\Delta v + k_2 v = F_2(x,t,u,v), & (x,t)\in D, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t)\in\Gamma, \\ u(x,0) = u(x,T),\ v(x,0) = v(x,T), & x\in\bar{\Omega}. \end{cases} \tag{20}$$

With respect to (20), F_1 and F_2 are quasimonotone nondecreasing with respect to u and v, respectively. Choose $\left(\bar{u}^{(0)}, \bar{v}^{(0)}\right) = (M_1, M_2)$ and $\left(\underline{u}^{(0)}, \underline{v}^{(0)}\right) = (\delta\phi, \delta\psi)$ as an initial iteration, we can construct a sequence $\left\{\left(u^{(n)}, v^{(n)}\right)\right\}$ from the iteration process

$$\begin{cases} \dfrac{\partial \overline{u}^{(n)}}{\partial t} - d_1(x,t)\Delta \overline{u}^{(n)} + k_1 \overline{u}^{(n)} = F_1\left(x,t,\overline{u}^{(n-1)},\overline{v}^{(n-1)}\right), & (x,t)\in D, \\ \dfrac{\partial \overline{v}^{(n)}}{\partial t} - d_2(x,t)\Delta \overline{v}^{(n)} + k_2 \overline{v}^{(n)} = F_2\left(x,t,\overline{u}^{(n-1)},\overline{v}^{(n-1)}\right), & (x,t)\in D, \\ \dfrac{\partial \underline{u}^{(n)}}{\partial t} - d_1(x,t)\Delta \underline{u}^{(n)} + k_1 \underline{u}^{(n)} = F_1\left(x,t,\underline{u}^{(n-1)},\underline{v}^{(n-1)}\right), & (x,t)\in D, \\ \dfrac{\partial \underline{v}^{(n)}}{\partial t} - d_2(x,t)\Delta \underline{v}^{(n)} + k_2 \underline{v}^{(n)} = F_2\left(x,t,\underline{u}^{(n-1)},\underline{v}^{(n-1)}\right), & (x,t)\in D, \\ \dfrac{\partial \overline{u}^{(n)}}{\partial \nu} = \dfrac{\partial \overline{v}^{(n)}}{\partial \nu} = 0, \dfrac{\partial \underline{u}^{(n)}}{\partial \nu} = \dfrac{\partial \underline{v}^{(n)}}{\partial \nu} = 0, & (x,t)\in \Gamma, \\ \overline{u}^{(n)}(x,0) = \overline{u}^{(n-1)}(x,T), \quad \overline{v}^{(n)}(x,0) = \overline{v}^{(n-1)}(x,T), & x\in\overline{\Omega}, \\ \underline{u}^{(n)}(x,0) = \underline{u}^{(n-1)}(x,T), \quad \underline{v}^{(n)}(x,0) = \underline{v}^{(n-1)}(x,T), & x\in\overline{\Omega}. \end{cases} \tag{21}$$

where $n = 1,2,\cdots$.

Then we can easily see that the sequences $\left\{\left(\overline{u}^{(n)},\overline{v}^{(n)}\right)\right\}$ and $\left\{\left(\underline{u}^{(n)},\underline{v}^{(n)}\right)\right\}$ governed by (21) are well-defined. Consequently, to show the monotone property of these sequences we have the following result.

Lemma 3.1. *The maximal and minimal sequences* $\left\{\left(\overline{u}^{(n)},\overline{v}^{(n)}\right)\right\}$ *and* $\left\{\left(\underline{u}^{(n)},\underline{v}^{(n)}\right)\right\}$ *are well-defined and possess the monotone property*

$$(\hat{u},\hat{v}) \le \left(\underline{u}^{(n)},\underline{v}^{(n)}\right) \le \left(\underline{u}^{(n+1)},\underline{v}^{(n+1)}\right) \le \left(\overline{v}^{(n+1)},\overline{v}^{(n+1)}\right) \le \left(\overline{u}^{(n)},\overline{v}^{(n)}\right) \le (\tilde{u},\tilde{v}),$$

for every $n = 1,2,\cdots$.

According to above lemma, the pointwise limits

$$\lim_{n\to\infty}\left(\overline{u}^{(n)},\overline{v}^{(n)}\right) = (\overline{u},\overline{v}), \quad \lim_{n\to\infty}\left(\underline{u}^{(n)},\underline{v}^{(n)}\right) = (\underline{u},\underline{v})$$

exist and satisfy the relation

$$(\hat{u},\hat{v}) \le \left(\underline{u}^{(n)},\underline{v}^{(n)}\right) \le (\underline{u},\underline{v}) \le (\overline{u},\overline{v}) \le \left(\overline{u}^{(n)},\overline{v}^{(n)}\right) \le (\tilde{u},\tilde{v}) \tag{22}$$

for every $n = 1,2,\cdots$.

Therefore, the maximal and minimal property of $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ is in the sense that (u,v) is any other T-periodic solution of (17) in $\langle(\hat{u},\hat{v}),(\tilde{u},\tilde{v})\rangle$, then $(\underline{u},\underline{v}) \le (u,v) \le (\overline{u},\overline{v})$ for $(x,t)\in\overline{D}$.

Next, to shows that $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ are the maximal and minimal solutions of (2), (3), respectively, we have the following theorem.

Theorem 3.3 *Let* $(\tilde{u},\tilde{v})$ *and* $(\hat{u},\hat{v})$ *be a pair of ordered upper and lower solutions of* (2), (3), *respectively, then the sequences* $\left\{\left(\overline{u}^{(n)},\overline{v}^{(n)}\right)\right\}$ *and* $\left\{\left(\underline{u}^{(n)},\underline{v}^{(n)}\right)\right\}$ *provided from* (21) *converge monotonically from above to a maximal T-periodic solution* $(\overline{u},\overline{v})$ *and from below to a minimal T-periodic solution* $(\underline{u},\underline{v})$ *in S, respectively, and satisfy relation*

$$\begin{aligned}(\overline{u},\overline{v}) &\le \left(\underline{u}^{(n)},\underline{v}^{(n)}\right) \le \left(\underline{u}^{(n+1)},\underline{v}^{(n+1)}\right) \le (\underline{u},\underline{v}) \le (\overline{u},\overline{v}) \le \left(\overline{u}^{(n+1)},\overline{v}^{(n+1)}\right) \\ &\le \left(\overline{u}^{(n)},\overline{v}^{(n)}\right) \le (\tilde{u},\tilde{v}), \text{for every } n = 1,2,\cdots\end{aligned}$$

Moreover, if $\overline{u} = \underline{u}$ or $\overline{v} = \underline{v}$ then $(\overline{u},\overline{v}) = (\underline{u},\underline{v})\left(\equiv\left(u^*,v^*\right)\right)$ and $\left(u^*,v^*\right)$

is the unique T-periodic solution of (2), (3) in *S*.

In what follows, we will study the attractivity of problem (2), (4) in relation to the maximal and minimal T-periodic solutions of problem (2), (3). The following lemma plays an important role in the establishment of attractivity result and its proof is similar to that in [12], so we omit it here.

Lemma 3.2 *let* $(u,v)(x,t;\eta)$ *be the solution of* (2), (4). *Then*

$$\left(\underline{u}^{(n)},\underline{v}^{(n)}\right)(x,t) \le (u,v)(x,t+nT;\eta) \le \left(\overline{u}^{(n)},\overline{v}^{(n)}\right)(x,t)$$

on $\overline{D}$ for every $n = 1,2,\cdots$.

With the help of above lemma and Theorem 3.1 of [12], the solution (u,v) of the problem (2) under the initial condition (4) possesses the following convergence

$$\lim_{n\to\infty}(u,v)(x,t+nT;\eta) = \begin{cases} (\underline{u},\underline{v})(x,t) & \text{if } (\hat{u},\hat{v}) \le \eta(x) \le (\underline{u},\underline{v}) \text{ in } \overline{\Omega}, \\ (\overline{u},\overline{v})(x,t) & \text{if } (\overline{u},\overline{v}) \le \eta(x) \le (\tilde{u},\tilde{v}) \text{ in } \overline{\Omega}. \end{cases} \quad (23)$$

and

$$(\underline{u},\underline{v})(x,t) \le (u,v)(x,t+nT;\eta) \le (\overline{u},\overline{v})(x,t) \quad \text{on } \overline{D} \text{ as } n \to \infty. \quad (24)$$

Consequently, the function pair $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are positive T-periodic solutions of problem (2), (3) and satisfy the following relation $(\delta\phi,\delta\psi) \le (\underline{u},\underline{v}) \le (\overline{u},\overline{v}) \le (M_1,M_2)$ on $\overline{D}$.

On one hand, for any nonnegative and nontrivial $(\eta_1(x),\eta_2(x))$ the solution (u,v) with $(\eta_1(x),\eta_2(x))$ is positive in $\overline{\Omega}$ for any $t>0$. So for any $t_0>0$, we can find $\delta>0$, such that

$$(\delta\phi,\delta\psi) \le (u(x,t_0),v(x,t_0)) \le (M_1,M_2) \quad \text{on } \overline{D}.$$

The above derivations lead to the following theorem.

Theorem 3.4 *Let* $(u(x,t;\eta_1),v(x,t;\eta_2))$ *be the solution of* (2), (4) *for* (η_1,η_2) *with* $0<\eta_i \le M_i$, $i = 1, 2$ *and let condition* $R_0^N > 1$ *be satisfied. Then we have*

1) problem (2), (3) has maximal and minimal positive T-periodic solutions $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ such that $(\underline{u},\underline{v}) \le (\overline{u},\overline{v})$ on $\overline{D}$;

2) the solution $(u(x,t;\eta_1),v(x,t;\eta_2))$ of (2), (4) possesses the convergence properties (23) and (24);

3) if $(\overline{u},\overline{v}) = (\underline{u},\underline{v}) = (u^*,v^*)$, then

$$\lim_{n\to\infty}\left(u(x,t+nT;\eta_1),v(x,t+nT;\eta_2)\right) = \left(u^*(x,t),v^*(x,t)\right), \ t>0, x\in\overline{\Omega}..$$

4. Numerical Simulation and Discussion

In this section, we provide some simulations for problem (2), (4) to illustrate our analytical results. We first fix the following diffusion coefficients and parameters:

$$d_1(x,t) = 1.5+0.5(\cos(x)+\cos(10t)), \ d_2(x,t) = 1.4+0.5(\sin(x)+\sin(10t)),$$

$$\mu(x,t) = 2+0.5(\sin(x)+\sin(10t)), \quad \gamma(x,t) = 1.5+0.5(\cos(x)+\cos(10t))$$

and $g(u)=\sqrt{u+1}-1$.

Therefore, now we will change the value of parameter $\alpha(x,t)$ and then observe the long time behavior of problem (2), (4).

Example 1: Chose big $\alpha = 20+0.2(\sin(x)+\sin(10t))$ so that

$$R_0^N \geq \sqrt{\frac{\alpha^i g'(0)}{\gamma^M \mu^M}} = \sqrt{\frac{19.6\times\frac{1}{2}}{3\times 2.5}} > 1,$$

and from Figure 1 it is easy to see that the solutions $u(x,t)$ and $v(x,t)$ of (2), (4) tends to a positive T-periodic solution.

Example 2: Take small $\alpha = 0.05+0.02(\sin(x)+\sin(10t))$, then

$$R_0^N \leq \sqrt{\frac{\alpha^M g'(0)}{\gamma^i \mu^i}} = \sqrt{\frac{0.09\times\frac{1}{2}}{1\times 0.5}} < 1,$$

hence the solutions $u(x,t)$ and $v(x,t)$ of (2), (4) decays to zero and no positive periodic solution exists to problem (2), (3) (see Figure 2).

To understand the impact of seasonal periodicity and environmental heterogeneity on the fecally-orally epidemic model, we consider a T-periodic

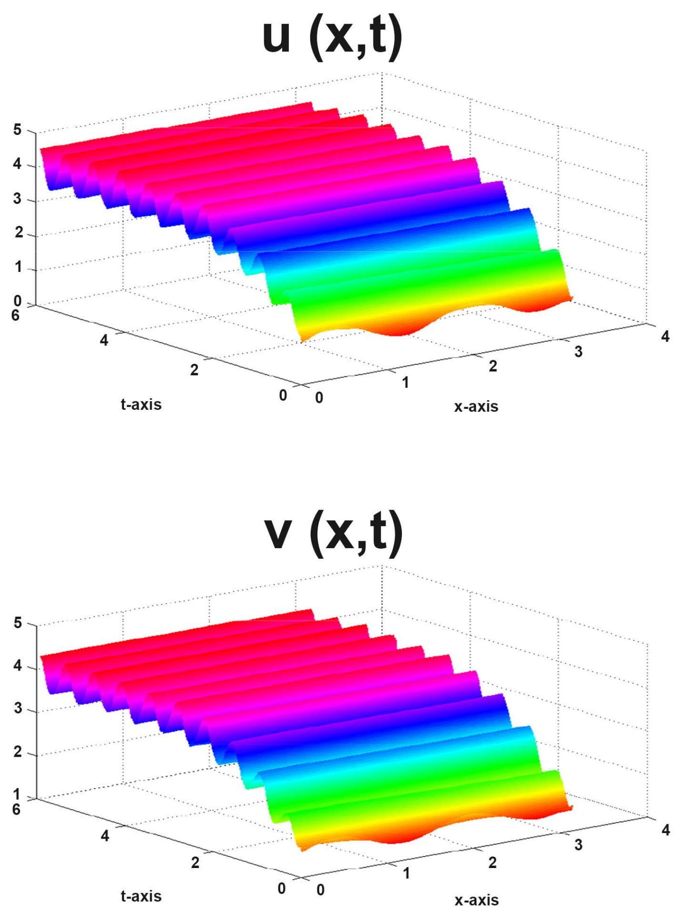

Figure 1. The solution of (2), (4) with big α presents a state of periodic oscillation.

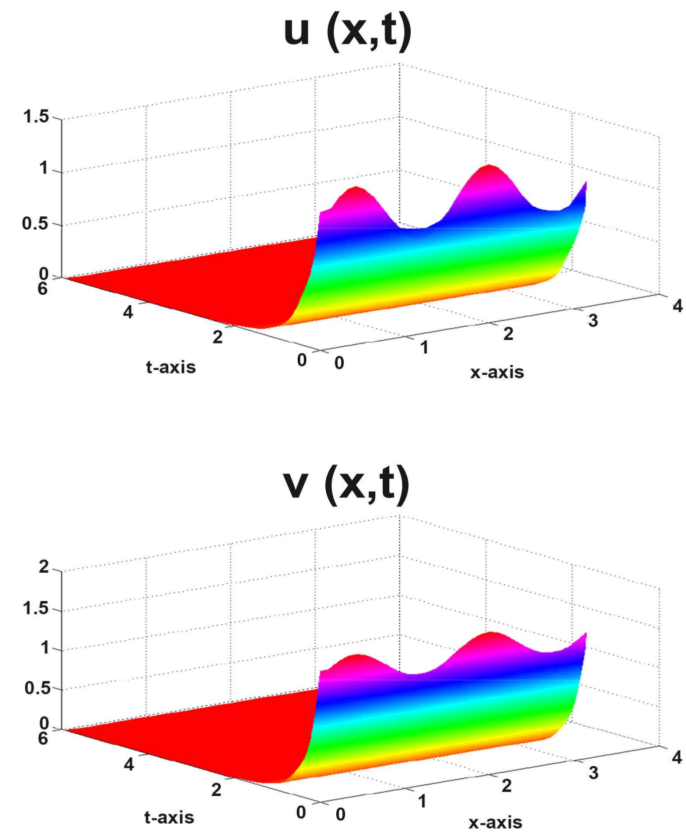

Figure 2. The solution of (2), (4) with small α decays quickly to zero.

solution of problem (2), (3) and the attractivity of problem (2), (4) in relation to the maximal and minimal T-periodic solutions of problem (2), (3). Firstly, we introduce the basic reproduction number R_0^N by using the next generation operator and associated eigenvalue problems, which is known as a threshold parameter of problem (2), (3) (Theorem 2.1). In the case that all coefficients are constants, we provide an explicit formula for R_0^N (Theorem 2.2). Secondly, the existence of a T-periodic solution of problem (2), (3) is investigated by combining the method of upper and lower solutions, the eigenvalue problems and associated monotone iterative schemes. Our results indicate that if $R_0^N > 1$ and $\alpha(x,t)$ big enough problem (2), (3) admits at least one positive T-periodic solution (Theorem 3.4, Theorem 3.5 and Figure 1), while if $R_0^N \le 1$ and $\alpha(x,t)$ sufficiently small, problem (2), (3) has no positive T-periodic solution (Theorem 3.3 and Figure 2).

Moreover, we discuss the attractivity of problem (2), (4) in relation to the maximal and minimal T-periodic solutions of problem (2), (3) by applying the monotone convergence property (Lemma 3.3 and Theorem 3.6). As result, it shown that the long time state of the solution to problem (2), (4) is between the maximal and minimal T-periodic solutions of problem (2), (3).

5. Conclusion

To better understand the impact of environmental heterogeneity and seasonal periodicity on the spatial spread of a class of bacterial and viral diseases, the fecally-orally epidemic model in heterogeneous environment has been considered. By means of next generation infection operator and associated eigenvalue problems, the spatial-temporal basic reproduction number is defined. We use the number to study the existence and non-existence of T-periodic solutions. The attractivity of our problem in relation to the maximal and minimal T-periodic solutions is discussed. We conclude that if the environment is T-periodic, then the solution of any initial-boundary system will present T-periodic phenomena gradually. Moreover, we believe that the investigation of the transmission dynamics of infectious diseases in heterogeneous environment is more close to reality than in homogeneous environment. However, here we considered the periodic solutions on specified period T. In the future, we will consider the time-periodic solutions for various possible periods. Moreover, the uniqueness of T-periodic solutions is still unclear.

Acknowledgements

We thank the Editor and the referee for their comments. The work is supported by People's Republic of China grant National Natural Science Foundation (11872189, 11472116).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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