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Application of the Improved Kudryashov Method to Solve the Fractional Nonlinear Partial Differential Equations

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Abstract

Our purpose of this paper is to apply the improved Kudryashov method for solving various types of nonlinear fractional partial differential equations. As an application, the time-space fractional Korteweg-de Vries-Burger (KdV-Burger) equation is solved using this method and we get some new travelling wave solutions. To acquire our purpose a complex transformation has been also used to reduce nonlinear fractional partial differential equations to nonlinear ordinary differential equations of integer order, in the sense of the Jumarie's modified Riemann-Liouville derivative. Afterwards, the improved Kudryashov method is implemented and we get our required reliable solutions where the results are justified by mathematical software Maple-13.

Keywords

Improved Kudryashov Method, Time-Space Fractional KdV-Burger Equation, Travelling Wave Solutions, Jumarie's Modified Riemann-Liouville Derivative

1. Introduction

Fractional differential equations have a significant role in describing the complicated nonlinear physical phenomena such as the fluid flow, viscoelasticity, signal processing, control theory, systems identification, biology, physics and other areas [1]-[6].

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, nonlinear fractional partial differential equations (FPDEs) have been attracted by the mathematician and other researchers. It is caused by both the development of the theory of fractional calculus itself. In the recent year, many analytical and numerical methods have been proposed to obtain solutions of nonlinear FPDEs, such as local fractional variational iteration method [7], local fractional adomian decomposition method [8] [9], local fractional fourier series method [10], finite element method [11], variational iteration method [12] and so on. In these methods, researchers have investigated analytical and numerical solutions and a few of them have depicted some related graphs.

The improved Kudryashov method [13] is also a similar method to these above methods and the basic principle of this method is to solve nonlinear partial differential equations analytically. This method is straight forward and easy for finding exact solutions FPDEs.

In this article, the improved Kudryashov method has been applied to find the new exact travelling wave solutions of the nonlinear time-space fractional order KdV-Burger equation, given by the following form [14]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}+\omega u \frac{\partial^{\beta} u}{\partial x^{\beta}}+\eta \frac{\partial^{2\beta} u}{\partial x^{2\beta}}+\upsilon \frac{\partial^{3\beta} u}{\partial x^{3\beta}}=0,\ t>0,\ 0<\alpha, \beta \leq 1. \tag{1.1}$$

It is applied as a nonlinear model of the propagation of waves on an elastic tube filled with a viscous fluid.

The Jumarie's modified Riemann-Liouville derivative [15] of order α, defined by the following expression

$$D_s^{\alpha} f(s)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}s} \displaystyle\int_0^s(s-\xi)^{-\alpha}(f(\xi)-f(0)) \mathrm{d} \xi, 0<\alpha<1, \\ \left(f^{(n)}(s)\right)^{\alpha-n}, n \leq \alpha<n+1, n \geq 1.\end{cases} \tag{1.2}$$

Moreover, some properties for the modified Riemann-Liouville derivative have also been given as follows

$$D_s^{\alpha} s^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} s^{r-\alpha} \tag{1.3}$$

$$D_s^{\alpha}(f(s) g(s))=f(s) D_s^{\alpha} g(s)+g(s) D_s^{\alpha} f(s) \tag{1.4}$$

$$D_s^{\alpha} f[g(s)]=f_{\alpha}'[g(s)] D_s^{\alpha} g(s)=D_s^{\alpha} f[g(s)](g'(s))^{\alpha} \tag{1.5}$$

The rest of the writing is organized as follows. In Section 2, the improved Kudryashov method has been described to find the solutions of nonlinear fractional partial differential equations with the help of fractional complex transformation. As an application, the new exact travelling wave solutions of KdV-Burger's equation have been found in Section 3. In last Section 4, the conclusion has been stated.

2. Outline of the Improved Kudryashov Method

We consider a time-space fractional nonlinear fractional partial differential equation, with independent variables x, t and dependent variable u, is given by

$$P\left(u,u_x,u_y,u_t,D_x^{\beta}u,D_y^{\gamma}u,D_t^{\alpha}u,\cdots\right),0<\alpha,\beta\le 1 \tag{2.1}$$

We use the variable transformation

$$u(x,t)=u(\xi),\xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{My^{\gamma}}{\Gamma(\gamma+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)}, \tag{2.2}$$

where K, M and L are non-zero arbitrary constants, fort transforming (2.1) to the following nonlinear fractional ordinary differential Equation (FODE) with independent variable ξ.

$$P\left(u,Ku',Mu',Lu',K^{\beta}D_{\xi}^{\beta},M^{\gamma}D_{\xi}^{\gamma}u,L^{\alpha}D_{\xi}^{\alpha}u,\cdots\right) \tag{2.3}$$

We seek for the exact solution of Equation (2.3) in the following form:

$$u(\xi)=\frac{\sum_{i=0}^{M}a_iQ(\xi)^i}{\sum_{j=0}^{N}b_jQ(\xi)^j}, \tag{2.4}$$

where $a_i,b_j\left(i=1,2,3,\cdots,M \text{ and } j=1,2,3,\cdots,N\right)$ are unknown constants and $Q(\xi)$ are the following functions

$$Q(\xi)=\frac{1}{\sqrt{1+c_1e^{2\xi}}}\quad\text{or,}\quad Q(\xi)=\frac{-1}{\sqrt{1+c_1e^{2\xi}}} \tag{2.5}$$

Here the above functions $Q(\xi)$ satisfy to the first order differential equation

$$\frac{dQ}{d\xi}=\lambda Q^3-Q,\lambda\ne 0 \tag{2.6}$$

To calculate the necessary number of derivatives of function $u(\xi)$, Equation (2.6) is necessary. We can obtain the positive integers M and N by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Equation (2.3)

We substitute $u(\xi)$ and its various derivatives in Equation (2.3). Then we collect all terms with the same powers of function $Q(\xi)$ and equate the resulting expression to zero. We obtain algebraic systems of equations. Solving this system, we get values for the unknown parameters.

Finally we put these values of unknown parameters and use the solutions of Equation (2.6) for constructing the travelling wave solutions of the nonlinear evolution Equation (2.1).

3. Application of the Method

In this section, the improved kudryshov method has been used to construct the exact solutions for nonlinear space-time fractional KdV-Burger equation given in (1.1).

Here we use the fractional complex transform

$$u(x,t)=u(\xi),\xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)} \tag{3.1}$$

where K and L are constants, permits to reduce the Equation (1.1) into an ODE.

After integrating Equation (1.1) once, we have the following form

$$Lu+\frac{1}{2}\omega Ku^2+\eta K^2u'+\upsilon K^3u''=0 \tag{3.2}$$

taking the integrating constant as zero.

Considering the homogeneous balance between u'' and u^2 in Equation (3.2), we obtain $M=N+4$. Suppose $N=1$ and then $M=5$.

Thus the travelling wave solution takes the following form:

$$u(\xi)=\frac{a_0+a_1Q+a_2Q^2+a_3Q^3+a_4Q^4+a_5Q^5}{b_0+b_1Q} \tag{3.3}$$

where a_0,a_1,a_2,a_3,a_4,a_5 and b_0,b_1 are unknown constants.

Substituting Equation (3.3) into Equation (3.2) and taking into account relation Equation (2.6), we get a polynomial of $Q(\xi)$. Collecting all the terms with the same power of $Q(\xi)$ together and equating each coefficient to zero, we can obtain a system of algebraic equations. Solving the resulting system by using Maple-13, we get the following sets of values of unknown constants and the corresponding solutions.

Case-1: $K=\frac{1}{10}\frac{\eta}{\upsilon}$, $L=\frac{3}{125}\frac{\eta^3}{\upsilon^2}$, $a_0=0$, $a_1=0$, $a_2=0$, $a_3=0$, $a_4=-\frac{12}{25}\frac{b_0\lambda^2\eta^2}{w\upsilon}$, $a_5=-\frac{12}{25}\frac{b_1\lambda^2\eta^2}{w\upsilon}$, $b_0=b_0$, $b_1=b_1$

The travelling wave solution of Equation (1.1) is:

$$\begin{aligned}u_1(x,t)&=\frac{-12\lambda^2\eta^2}{25w\upsilon\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{Kx^\beta}{\Gamma(\beta+1)}+\frac{Lt^\alpha}{\Gamma(\alpha+1)}\right\}}\right]^2}\\&=\frac{-12\lambda^2\eta^2}{25w\upsilon\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{1}{10}\frac{\eta}{\upsilon}\frac{x^\beta}{\Gamma(\beta+1)}+\frac{3}{125}\frac{\eta^3}{\upsilon^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}}\right]^2}\end{aligned} \tag{3.4}$$

And for example, one of the solitary wave solutions is:

$$u_1(x,t)=\frac{48}{25\left(2+\mathrm{e}^{-0.2256758334x^{0.5}+0.0541620002t^{0.5}}\right)^2},$$

when $\lambda=2,\eta=1,\omega=1,\upsilon=-1,c_1=1,\alpha=0.5,\beta=0.5$.

Case-2: $K=-\frac{1}{10}\frac{\eta}{\upsilon}$, $L=\frac{3}{125}\frac{\eta^3}{\upsilon^2}$, $a_0=0$, $a_1=0$, $a_2=\frac{24}{25}\frac{b_0\eta^2\lambda}{w\upsilon}$, $a_3=\frac{24}{25}\frac{b_1\eta^2\lambda}{w\upsilon}$, $a_4=-\frac{12}{25}\frac{b_0\lambda^2\eta^2}{w\upsilon}$, $a_5=-\frac{12}{25}\frac{b_1\lambda^2\eta^2}{w\upsilon}$, $b_0=b_0$, $b_1=b_1$

The travelling wave solution of Equation (1.1) is:

$$u_2(x,t)=\frac{24}{25}\frac{\eta^2\lambda}{w\upsilon}\frac{1}{\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{-\eta}{10\upsilon}\frac{x^\beta}{\Gamma(\beta+1)}+\frac{3}{125}\frac{\eta^3}{\upsilon^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}}\right]}-\frac{12}{25}\frac{\lambda^2\eta^2}{w\upsilon}\frac{1}{\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{-\eta}{10\upsilon}\frac{x^\beta}{\Gamma(\beta+1)}+\frac{3}{125}\frac{\eta^3}{\upsilon^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}}\right]^2} \tag{3.5}$$

And for example, one of the solitary wave solutions and its corresponding graph is:

$$u_2(x,t)=\frac{36}{25\left(3+\mathrm{e}^{0.111422509x^{0.3}-0.0135405t^{0.5}}\right)}-\frac{108}{25\left(3+\mathrm{e}^{0.111422509x^{0.3}-0.0135405t^{0.5}}\right)^2},$$

when $\lambda=3,\eta=-1,\omega=1,\upsilon=2,c_1=1,\alpha=0.5,\beta=0.3$.

Case-3: $K=\frac{1}{10}\frac{\eta}{\upsilon}$, $L=-\frac{3}{125}\frac{\eta^3}{\upsilon^2}$, $a_0=\frac{12}{25}\frac{b_0\eta^2}{w\upsilon}$, $a_1=\frac{12}{25}\frac{b_1\eta^2}{w\upsilon}$, $a_2=0$, $a_3=0$, $a_4=-\frac{12}{25}\frac{b_0\lambda^2\eta^2}{w\upsilon}$, $a_5=-\frac{12}{25}\frac{b_1\lambda^2\eta^2}{w\upsilon}$, $b_0=b_0$, $b_1=b_1$

The travelling wave solution of Equation (1.1) is:

$$u_3(x,t)=\frac{12}{25}\frac{\eta^2}{w\upsilon}-\frac{12}{25}\frac{\lambda^2\eta^2}{w\upsilon}\frac{1}{\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{1}{10}\frac{\eta}{\upsilon}\frac{x^\beta}{\Gamma(\beta+1)}-\frac{3}{125}\frac{\eta^3}{\upsilon^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}}\right]^2} \tag{3.6}$$

And for example, one of the solitary wave solutions and its corresponding graph is:

$$u_3(x,t)=\frac{12}{25}+\frac{48}{25\left(-2+\mathrm{e}^{-0.2256758334x^{0.5}-0.0541620002t^{0.5}}\right)^2}$$

when $\lambda=-2,\eta=1,\omega=1,\upsilon=1,c_1=1,\alpha=0.5,\beta=0.5$.

Case-4: $K=-\frac{1}{10}\frac{\eta}{\upsilon}$, $L=-\frac{3}{125}\frac{\eta^3}{\upsilon^2}$, $a_0=-\frac{12}{25}\frac{b_0\eta^2}{w\upsilon}$, $a_1=-\frac{12}{25}\frac{b_1\eta^2}{w\upsilon}$, $a_2=\frac{24}{25}\frac{b_0\eta^2\lambda}{w\upsilon}$, $a_3=\frac{24}{25}\frac{b_1\eta^2\lambda}{w\upsilon}$, $a_4=-\frac{12}{25}\frac{b_0\lambda^2\eta^2}{w\upsilon}$, $a_5=-\frac{12}{25}\frac{b_1\lambda^2\eta^2}{w\upsilon}$, $b_0=b_0$, $b_1=b_1$.

The travelling wave solution of Equation (1.1) is:

$$u_4(x,t)=-\frac{12}{25}\frac{\eta^2}{w\upsilon}+\frac{24}{25}\frac{\eta^2\lambda}{w\upsilon}\frac{1}{\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{-\eta}{10\upsilon}\frac{x^\beta}{\Gamma(\beta+1)}-\frac{3}{125}\frac{\eta^3}{\upsilon^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}}\right]}-\frac{12}{25}\frac{\lambda^2\eta^2}{w\upsilon}\frac{1}{\left[\lambda+c_1\mathrm{e}^{2\left\{\frac{-\eta}{10\upsilon}\frac{x^\beta}{\Gamma(\beta+1)}-\frac{3}{125}\frac{\eta^3}{\upsilon^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}}\right]^2} \tag{3.7}$$

And for example, one of the solitary wave solutions and its corresponding graph is:

$$u_4(x,t)=\frac{6}{25}+\frac{36}{25\left(-3+e^{0.111422509x^{0.3}+0.0135405t^{0.5}}\right)}+\frac{108}{25\left(-3+e^{0.111422509x^{0.3}+0.0135405t^{0.5}}\right)^2}$$

when $\lambda=3,\eta=-1,\omega=1,\upsilon=2,c_1=1,\alpha=0.5,\beta=0.3$.

In the above, Equations (3.4), (3.5), (3.6) and (3.7) yield the new types of travelling wave solutions of (1.1) whereas Figures 1-4 give their respective velocity profiles. These four figures are sketched by choosing particular values of arbitrary constants involving in these equations and such types of solutions are called also solitary wave solutions.

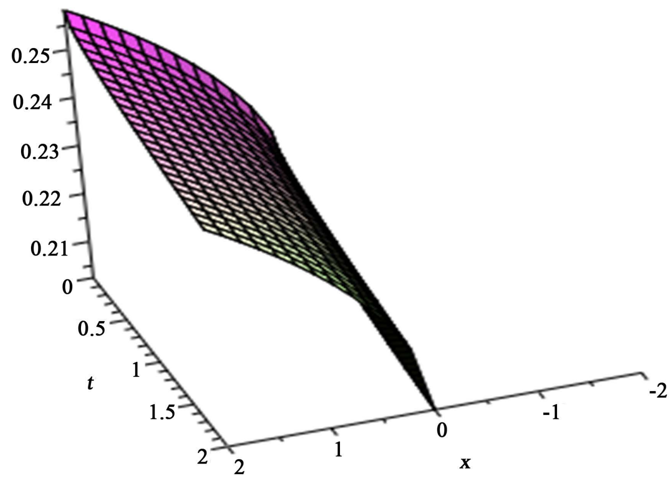

Figure 1. Profile of $u_1(x,t)$.

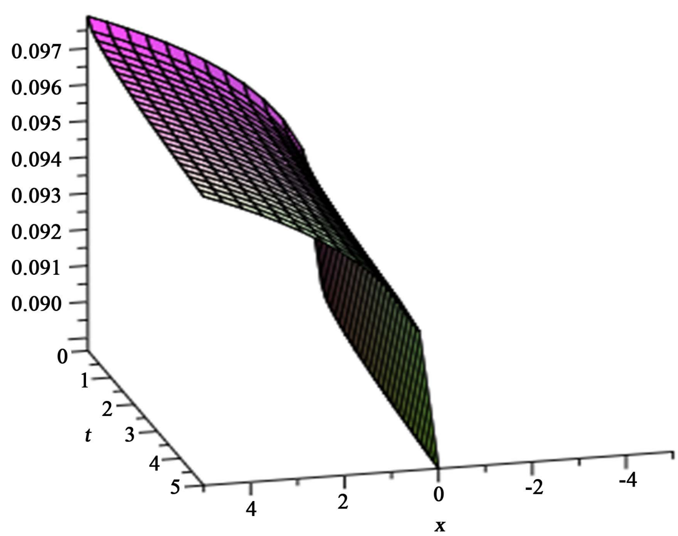

Figure 2. Profile of $u_2(x,t)$.

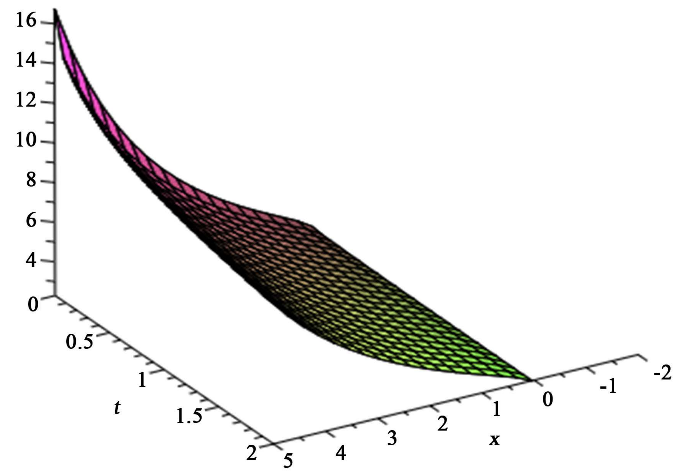

Figure 3. Profile of $u_3(x,t)$.

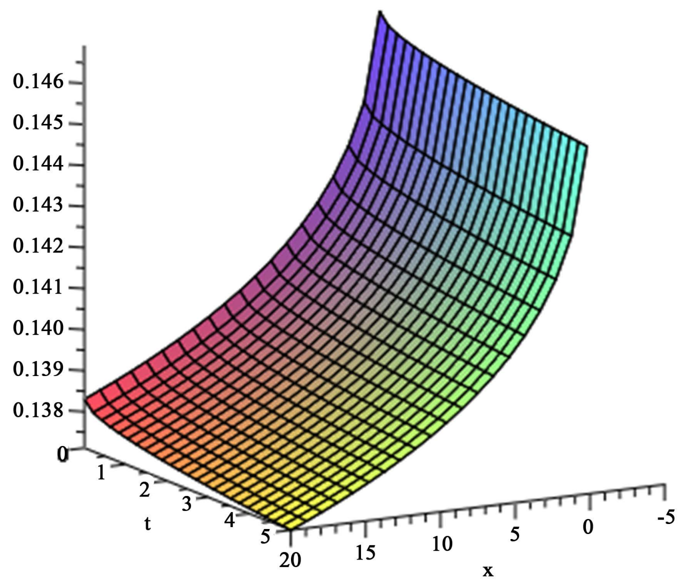

Figure 4. Profile of $u_4(x,t)$.

Implication of the Maple-13: The mathematical software Maple-13 is used for obtaining a system of algebraic equations and finding unknown constants by solving this system. Justification of the results and finally figures are also sketching by this software.

4. Conclusion

The improved Kudryashov method has been explored successfully to solve the nonlinear fractional partial differential equation using the sense of Jumarie's modified Riemann-Liouville derivative and the fractional complex transformation. As a result, some new types of exact travelling wave solutions for the space-time fractional KdV-Burger equation have been found and these solutions are verified. It can be concluded that this method is very simple, reliable and can be used to solve any higher fractional order nonlinear partial differential equations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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