Drawing Kerr

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Abstract

The Kerr metric is analyzed using strictly geometrical elements. A 4-dimensional surface representing the geometry of the Kerr model is embedded into a 5-dimensional flat space. The field strengths of the model are explicitly worked out and understanding of the theory is supported by numerous figures. The structure of the field equations is analyzed.

Keywords

Kerr Geometry, Geometrical Interpretation, Kerr Surface

1. Introduction

The metric of a rotating stellar object was found by Kerr [1] in 1963, albeit in a form that was hard to process. Two years later the metric was rewritten with the help of elliptical-hyperbolic coordinates by Boyer and Lindquist [2]. Enderlein [3] has given a good representation of the elliptical-hyperbolic coordinate system and Krasinski [4] has looked more deeply at the problem. We have analyzed the Kerr model in a series of papers, with a clear presentation in [5] and [6]. Nevertheless, we believe that the Kerr model is still too little understood. Thus, we have decided to present a very detailed presentation, supplemented by numerous drawings.

2. Basics of the Kerr Metric

The Kerr metric in the form of Boyer-Lindquist

$$ds^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 + \frac{\sin^2 \vartheta}{\rho^2} \left[(r^2 + a^2) d\varphi - a dt \right]^2 - \frac{\Delta}{\rho^2} \left[dt - a \sin^2 \vartheta d\varphi \right]^2 \quad (2.1)$$

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \vartheta$$

is nowadays the standard form of the Kerr metric used in the literature. r, ϑ , and φ are quasi-polar coordinates and t the coordinate time. However, we ar-

gue that the elements in this metric do not disclose the geometric structure of the model. Therefore we have put the metric into the form

$$ds^2 = dx^1{}^2 + dx^2{}^2 + \left[\alpha_R dx^3 + i\alpha_R \omega \sigma dx^4 \right]^2 + a_S^2 \left[-i\alpha_R \omega \sigma dx^3 + \alpha_R dx^4 \right]^2 \quad (2.2)$$

$$dx^1 = \alpha_S a_R dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \quad dx^4 = idt = \rho_S di\psi$$

With the definitions

$$\alpha_S = \frac{A}{\delta}, \quad a_S = \frac{\delta}{A}, \quad A^2 = r^2 + a^2, \quad \delta^2 = r^2 + a^2 - 2Mr,$$

$$\Lambda^2 = r^2 + a^2 \cos^2 \vartheta, \quad \sigma = A \sin \vartheta, \quad \omega = a/A^2, \quad (2.3)$$

$$a_R^2 = 1 - \omega^2 \sigma^2 = \frac{\Lambda^2}{A^2}, \quad \alpha_R = 1/a_R, \quad \rho_S = a_R A \sqrt{\frac{2r}{M} \frac{r^2 + a^2}{r^2 - a^2}}$$

we will realize that a geometric meaning can be assigned to the quantities of the metric (2.2). We will discuss this in the following.

First, we want to make clear that the line element

$$ds^2 = \frac{\Lambda^2}{A^2} dr^2 + \Lambda^2 d\vartheta^2 + A^2 \sin^2 \vartheta d\varphi^2 \quad (2.4)$$

can be interpreted as a line element on an oblate ellipsoid of revolution represented in quasipolar coordinates $x^\alpha = \{r, \vartheta, \varphi\}$ in a 3-dimensional flat space. The connection to Cartesian coordinates $x^{\alpha'}, \alpha' = 1', 2', 3'$ is given by

$$x^{3'} = A \sin \vartheta \cos \varphi$$

$$x^{2'} = A \sin \vartheta \sin \varphi \quad (2.5)$$

$$x^{1'} = r \cos \vartheta$$

The quasipolar coordinate net is given by confocal ellipsoids, hyperbolae as their orthogonal trajectories, and circles as horizontal slices of the ellipsoids of revolution. The coordinate surfaces are given by the equation of the ellipsoids and hyperbolae

$$\frac{(x^{2'})^2 + (x^{3'})^2}{A^2} + \frac{(x^{1'})^2}{r^2} = 1, \quad \frac{(x^{2'})^2 + (x^{3'})^2}{a^2 \sin^2 \vartheta} - \frac{(x^{1'})^2}{a^2 \cos^2 \vartheta} = 1$$

Enderlein [3] has depicted this coordinate net as shown in **Figure 1**.

Figure 1. Elliptic-hyperbolic coordinate system by Enderlein.

A slice of the ellipsoid of revolution should make understandable the position of the coordinate lines and is shown in **Figure 2**.

Figure 2. The elliptic-parabolic coordinate system.

In order to work out the meaning of the quantities in (2.2), we first deal in detail with the ellipses by reducing the problem to two dimensions

$$\begin{aligned}x^{1'} &= r \cos \vartheta \\x^{2'} &= A \sin \vartheta\end{aligned}\tag{2.7}$$

Evidently r and A are the minor and major axes of the confocal ellipses. Although the coordinate r is not the radial direction of the system, the historical notation is maintained. The meaning of the quasi-polar angle ϑ can be found in **Figure 3**.

Figure 3. Basic elements of an ellipse.

Further properties of the ellipses can be made accessible. If one draws the focal rays s_1 and s_2 from the foci of an ellipse to any point of the ellipse, one finds, with **Figure 4**, the relations

$$s_1^2 = x^{1'2} + (x^{2'} + a)^2, \quad s_2^2 = x^{1'2} + (x^{2'} - a)^2$$

Figure 4. Focal rays and eccentricity.

and then

$$s_1 = A + a \sin \vartheta, \quad s_2 = A - a \sin \vartheta$$

$$s_1 s_2 = A^2 - a^2 \sin^2 \vartheta = r^2 + a^2 \cos^2 \vartheta$$

Thus, the fundamental quantities of the ellipses can be represented as the geometric and the arithmetic means of the focal rays

$$\Lambda = \sqrt{s_1 s_2}, \quad A = \frac{s_1 + s_2}{2} \quad (2.8)$$

The factor ahead of dr^2 in (2.4) has to be examined in more detail. At $\vartheta = 0$, i.e. at the minor axes of the ellipses, this factor $(r^2 + a^2 \cos^2 \vartheta) / (r^2 + a^2)$ is equal to 1, so that the radial part of the metric is reduced to dr^2 . At $\vartheta = \pi/2$ the factor is r^2 / A^2 . It follows that the radial arc element has the value dA at this position. Thus, the quantity Λ/A describes the varying distance of two neighboring confocal ellipses; at the minor axis this distance is dr and at the major axis it is dA . This proves that several quantities of the geometry show this behavior, so that it is sufficient to compute these quantities at one of the minor axes, and then to multiply them by this factor, which we call *elliptical factor*. However, these quantities offer a further surprise. With

$$a_R^2 = \frac{\Lambda^2}{A^2} = 1 - \frac{a^2}{A^2} \sin^2 \vartheta = 1 - \omega^2 \sigma^2, \quad \omega = \frac{a}{A^2} \quad (2.9)$$

one recognizes that within these relations are buried the angular velocity ω , and the orbital speed $\omega\sigma$ of a potentially rotating model. The BL system al-

ready makes available the fundamental quantities for a rotating system. Not surprisingly, the Kerr geometry is built on an elliptical system.

From the equations of the ellipses and hyperbolae one gets the nonvanishing components of the curvature vectors

$$\rho_\alpha^E = \{\rho_E, 0, 0\}, \quad \rho_\alpha^H = \{0, \rho_H, 0\}, \quad \rho_E = \frac{\Lambda^3}{rA}, \quad \rho_H = -\frac{\Lambda^3}{a^2 \sin \vartheta \cos \vartheta} \quad (2.10)$$

$\alpha = 1, 2, 3$ is a tetrad index. The two vectors are perpendicular and are in each case tangent to the other family of curves. In this way the term radial is defined in this geometry. “Radial” refers to the directions which are specified by the tangents of the hyperbolae.

Facing the 3rd dimension, $\sigma = A \sin \vartheta$ is the curvature radius of the circles, the parallels of the ellipsoid of revolution. Finally, one has the *curvatures* of the coordinate lines

$$B_\alpha = \left\{ \frac{1}{\rho_E}, 0, 0 \right\}, \quad N_\alpha = \left\{ 0, \frac{1}{\rho_H}, 0 \right\}, \quad C_\alpha = \left\{ 0, 0, \frac{1}{\sigma} \right\} \quad (2.11)$$

These quantities and their derivatives constitute the curvature equations of the system and satisfy the “field equations” $R_{\alpha\beta} = 0$, where $R_{\alpha\beta}$ is the 3-dimensional Ricci for the parabolic-hyperbolic system.

3. The Kerr Surface

So far, we have associated geometrical meaning to the quantities of the flat metric (2.4), written in elliptic-hyperbolic coordinates. Now we return to the metric (2.2), but we omit the timelike part. To explain the factor $\alpha_s = A/\delta$ in the radial arc element, we extend (2.5) with the extra dimension $x^{0'}$

$$\begin{aligned} x^{0'} &= -\int \tan \varepsilon dr \\ x^{1'} &= r \cos \vartheta \\ x^{2'} &= A \sin \vartheta \cos \varphi \\ x^{3'} &= A \sin \vartheta \sin \varphi \end{aligned} \quad (3.1)$$

We will show that (3.1) provides an embedding of a surface into a 5-dimensional flat space, which we will discuss in more detail.

We define the angle ε by

$$\sin \varepsilon = -\frac{r}{A} \sqrt{\frac{2M}{r}}, \quad \cos \varepsilon = \frac{\delta}{A}, \quad \tan \varepsilon = -\frac{\sqrt{2Mr}}{\delta}, \quad (3.2)$$

where ε has the orientation cw. Evidently $\tan \varepsilon$ tends to infinity for $\delta = 0$. The solution to r of this equation is

$$r_H = M + \sqrt{M^2 - a^2} \quad (3.3)$$

r_H settling the event horizon of the Kerr model. Finally, the integral in (3.1) reads as

$$x^{0'} = \int_{r_H}^r \sqrt{\frac{2Mr}{r^2 + a^2 - 2Mr}} dr \quad (3.4)$$

The solution does not have a closed form; however, the integral can be evaluated numerically. If one suppresses the φ -dimension the surface appears as shown in **Figure 5**.

Figure 5. The surface with suppressed φ -dimension.

The surface looks like a “funnel”. Its horizontals are ellipses. The ellipse fixed by r_H is the ellipse at the waist of the funnel and is the “end” of the geometry. $\delta = 0$ marks the limit of the model. The projection of the funnel onto the base planes shows ellipses crossed by hyperbolae. Thus, the integral lines (3.4) have a 1st and 2nd curvature. We note that the ansatz (3.1) still does not provide the Kerr metric. We have to add further ingredients to gain the metric (2.2). But **Figure 5** shows a lot of the properties that the genuine Kerr surface will have.

This surface is closely related to the Schwarzschild surface, *i.e.* Flamm’s paraboloid. For $a = 0$ the metric (2.2) is reduced to the Schwarzschild metric and the elliptical coordinate system breaks down to the spherical polar system. The integral lines are parabolae and their projections onto the base plane are straight lines.

If one extends to the φ -dimension and suppresses the ϑ -dimension, the surface builds up on concentric circles and is even more closely related to Flamm’s paraboloid.

We must bear in mind that dr is the increase of the minor axes of the ellipses on the base plane. In contrast,

$$dx^\perp = a_R dr \quad (3.5)$$

is the distance of two neighboring ellipses and depends on the angle ϑ . If one pulls up elliptical cylinders on two of such ellipses, one can see that their cutting curves with the surface do not lie on the horizontal slices of the surface. During a circulation on the surface the points of the cutting curve oscillate. From

$$dx_{hol}^{0'} = -\tan \varepsilon dr = -\tan \varepsilon \alpha_R(r, \vartheta) dx^\perp = -\tan \chi dx^\perp \quad (3.6)$$

one gets the augment of the height of the surface into the extra dimension.

It is evident that the tangent related to the auxiliary angle χ which points to the next higher point of the surface depends on the angle ϑ . A short calculation would show that one does not arrive at the desired seed metric by using (3.6) either. For our model only the horizontal elliptical slices of the surface are of importance. If one follows the normal vector of the surface along an ellipse, one will discover that this vector also oscillates on its way, because the walls of the surface are round about differently scarped. In order to be able to use the surface, it has to be equipped with an additional structure. On the minor axes of the ellipses the elliptical factor is $a_R = 1$ and the geometry is Schwarzschild-like. That is where we start: we define a *rigging vector* in such a way that it coincides with the normal vector at this position, and that it always encloses the same angle with the base plane during its circulation. Then this rigging vector is no longer vertical to the surface and its vertical planes are no longer tangent to the surface. The family of all of these planes—and if one adds the φ -dimension, the family of the 3-dimensional hyperplanes—represents our graphic space, if we assume that we live in such a world. Those hyperplanes are anholonomic, as will be shown.

Now we replace the holonomic differential (3.6) by an anholonomic one

$$dx_{anh}^{0'} = -\tan \varepsilon a_R(r, \vartheta) dr = -\tan \varepsilon dx^1 \quad (3.7)$$

that is no longer integrable. We call the family of hyperplanes that are orthogonal to these differentials and that are no longer V_3 -forming *physical surface*. It is the area of all possible physical observations.

However, the structure of the $[r, \varphi]$ -part of the surface is simple. On this patch no elliptical properties and hence no anholonomies arise. It corresponds to Flamm's paraboloid of the Schwarzschild geometry. Sharp [7] has searched for such a surface for $\vartheta = \pi/2$. However, he did not start from the seed metric, but incorporated rotational parts of the Kerr metric into his computations. Hence, his result differs substantially from ours. From

$$ds^2 = dx^{0^2} + dx^{1^2} + dx^{2^2} + dx^{3^2}$$

one indeed obtains with

$$ds^2 = (\tan^2 \varepsilon + 1) \frac{\Lambda^2}{A^2} dr^2 + \Lambda^2 d\vartheta^2 + A^2 \sin^2 \vartheta d\varphi^2$$

the desired seed metric, which serves as a basis for the actual Kerr metric. From the definition $a_S = \delta/A$ in (2.3) and the introduction of $\cos \varepsilon = \delta/A$, and with the definition of the elliptical factor a_R we obtain the spatial part of the Kerr metric

$$ds^2 = \alpha_S a_R dr^2 + \Lambda^2 d\vartheta^2 + A^2 \sin^2 \vartheta d\varphi^2 \quad (3.8)$$

and the surface related to this metric we will call *Kerr surface*.

The anholonomic construction will be examined in more detail. **Figure 6** shows the constellation of the holonomic and anholonomic vectors, namely the two “radial” vectors, the vector¹ dx_{hol}^1 in the tangent planes of the surface and

the associated vector dx_{anh}^1 in the anholonomic hyperplanes, which have the same projections onto the basic plane.

Figure 6. Holonomic and anholonomic differentials.

Having explained the structure of the Kerr metric, we return to Equation (3.2) and recognize that the ascent of the integral curve is $-\tan \varepsilon = \infty$ for $\delta = 0$, *i.e.* at $\varepsilon = \pi/2$. Thus, the integral lines are normal to the base plane at r_H . For $\varepsilon = 0$ the ascent is $\tan \varepsilon = 0$ and the geometry will be flat in the infinite.

We also read from (3.2) that for vanishing eccentricity a of the ellipses, *i.e.* $a = 0$, we get with $A = r$

$$\sin \varepsilon = -\sqrt{\frac{2M}{r}}, \quad \cos \varepsilon = \sqrt{1 - \frac{2M}{r}}, \quad \frac{1}{\cos \varepsilon} = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

Thus, in this case $\sin \varepsilon$ is the velocity of a freely falling observer in the Schwarzschild model and $1/\cos \varepsilon$ the related Lorentz factor.

We have good reasons to interpret

$$v_s = \sin \varepsilon = -\frac{r}{A} \sqrt{\frac{2M}{r}}, \quad \alpha_s = \frac{1}{\sqrt{1-v_s^2}} = \frac{A}{\delta} \quad (3.9)$$

as velocity of a freely falling observer in the exterior field of a rotating stellar object and α_s as the correlated Lorentz factor of this motion. r/A is the ratio of the axes of the ellipses. This means that the velocity of a freely falling observer depends on position in relation to the ellipses.

At the waist of the Kerr surface ($\delta=0$) is $v_R=-1$, *i.e.* the observer has (asymptotically) reached the velocity of light, measured with his proper time independently of his former position on the ellipse. Since we do not assume that velocities higher than the velocity of light can occur, we have to realize that the

¹All the other components of the vector except the mentioned one vanish in the local reference system.

waist of the Kerr surface is not only a geometrical limit, but also a physical limit of the model. No object can cross the event horizon r_H .

In addition, we have to bear in mind that a radial motion in the fields of rotating objects in free fall is not possible. Since frame dragging acts on observers, the motion of an observer will get a circular component. Due to Einstein's law of composition of velocities the velocity of an observer will asymptotically reach the velocity of light before reaching the event horizon. The limit in this case is the ergosphere, as we have shown in [5] [6]. Our point of view is supported by similar circumstances in the Schwarzschild model. In contrast to the claims of Misner, Thorne, and Wheeler [8], we have shown in several papers and in [5] [6] that an observer starting from an arbitrary position can reach the Schwarzschild radius only asymptotically in infinite proper time. We have also shown [9] that the surface of a collapsing star, described by the Schwarzschild interior solution, collapses eternally, reaching the inner horizon of the model asymptotically in infinite proper time. Thus the final state of a collapsing non-rotating stellar object is an ECO (Eternally Collapsing Object) [10].

Thus, if we believe that this geometrical description of a rotating star is a good description of Nature, we have to dismiss the possibility of the formation of rotating black holes. We have supplemented the Kerr solution with an interior solution [11] [12], which has the property of developing into the interior Schwarzschild solution for $a = 0$. However, we have made no attempt to implement a collapse for this model and we have not found any effort in the literature in this regard. If such an approach is possible, a RECO (Rotating Eternally Collapsing Object) would be expected.

4. Curvatures of the Elliptic-Hyperbolic Geometry

So far we have shown that the Kerr model is based on an elliptic-hyperbolic system, endowed with an integral surface with elliptical horizontals, which could be envisaged as an elliptically deformed Flamm's paraboloid. The Boyer-Lindquist coordinate system, with its curved coordinate lines, contributes to Einstein's field equations, which have little to do with the physical content of the model but are incorporated in the connexion coefficients of the physical quantities and must be treated for this reason.

Still suppressing the timelike part of the seed metric, we have to deal with the curvature vectors

$$\begin{aligned} \rho_a^S &= \{\rho_S, 0, 0, 0\}, \quad \rho_a^E = \{0, \rho_E, 0, 0\}, \\ \rho_a^H &= \{0, 0, \rho_H, 0\}, \quad \rho_a^C = \{0, 0, 0, \rho_C\}, \quad a = 0, 1, 2, 3 \end{aligned} \quad (4.1)$$

ρ_E and ρ_H are the components of the curvature vectors of the ellipses and hyperbolae. They have been mentioned in (2.10). $\rho_C = \sigma = A \sin \vartheta$ is the radius of the circles of the parallels of the ellipsoids of revolution. ρ_S is the curvature radius of the integral lines and was calculated in [5] [6]².

The inverse quantities of (4.1) are the curvatures in vector form. Their tetrad

²The interested reader should have at hand one of these monographs.

components related to BL-coordinates are

$$\begin{aligned} M_a &= \{1, 0, 0, 0\} \frac{1}{\rho_S}, \quad B_a = \{\sin \varepsilon, \cos \varepsilon, 0, 0\} \frac{1}{\rho_E}, \quad N_a = \{0, 0, 1, 0\} \frac{1}{\rho_H}, \\ C_a &= \{\sin \varepsilon \sin \theta, \cos \varepsilon \sin \theta, \cos \theta, 0\} \frac{1}{\rho_C}, \end{aligned} \quad (4.2)$$

where

$$\sin \theta = \frac{r}{\Lambda} \sin \vartheta, \quad \cos \theta = \frac{A}{\Lambda} \cos \vartheta. \quad (4.3)$$

All the quantities in (4.2) obey the structure

$$\frac{d}{dr} \frac{1}{r} + \frac{1}{r^2} = 0.$$

The “field equations” for B , N and C refer to the elliptic-hyperbolic system and drop out from Einstein’s field equations. But we cannot omit these quantities, because we need them for the covariant derivative of the physical quantities which we will discuss later on.

We start with the detailed discussion of these quantities. The quantity B is related to the curvature radii ρ_E of the ellipses. We know that the curvature B is normal to the ellipses. Thus, we introduce an auxiliary reference system $a'' = 0'', 1'', 2''$ and suppress the φ -dimension for the sake of simplifying the problem. In this system B has only one component

$$B_{a''} = \{0, 1, 0\} \frac{1}{\rho_E}, \quad (4.4)$$

as seen in **Figure 7**. For the sake of simplicity, the holonomic Kerr surface is used for some of the quantities for the drawings.

Figure 7. Vertical splitting of B_1 .

It is evident that $B_{1''}$ can be split into two components with respect to the BL system. One component (B_1) is tangent to the integral curve and the other one (B_0) is normal to the curve, pointing in the 0-direction, *i.e.* the local extradimension. Thus, one has

$$B_a = \{\sin \varepsilon, \cos \varepsilon, 0\} \frac{1}{\rho_E}. \quad (4.5)$$

$B_1 = \frac{1}{\rho_E} \cos \varepsilon$ is the very component an observer can measure in his physical space, B_0 is hidden to him.

Furthermore, we ask for the components of B in the Cartesian coordinate system of the embedding space. We obtain

$$B_{a'} = \{0, \cos \theta, \sin \theta\} \frac{1}{\rho_E}. \quad (4.6)$$

The components of $B_{a'}$ are depicted in **Figure 8**:

Figure 8. Horizontal splitting of $B_{1''}$.

From the figure it can be seen that θ_E is the angle of ascent of the curvature radius of the ellipse and can be calculated by (4.3), setting $\theta_E = \theta$. With $\{\rho_E, \theta_E\}$ one can describe the elliptical system instead of $\{r, \vartheta\}$. Alternatively, one can describe the hyperbolic system with $\{\rho_H, \theta_H\}$, where $\theta_H = \pi/2 - \theta_E$, bearing in mind that the ellipses and hyperbolae are orthogonal trajectories.

Now we turn to the quantity $\mathcal{N}$, which is related to the curvature radii ρ_H of the hyperbolae. We find the hyperbolae in the horizontals of the Kerr surface.

Again, the auxiliary reference system is chosen in such a way that the $1''$ -direction is tangent to the hyperbolae and the $2''$ -direction is tangent to the ellipses. Thus, the curvature N has only one component in this system

$$N_{a''} = \{0, 0, 1\} \frac{1}{\rho_H} \quad (4.7)$$

as can be seen in **Figure 9**.

Figure 9. Horizontal splitting of N .

In the Cartesian reference system N has the components

$$N_{a'} = \{0, -\sin \theta, \cos \theta\} \frac{1}{\rho_H} = \{0, \cos \theta_H, -\sin \theta_H\} \left(-\frac{1}{\rho_H} \right), \quad (4.8)$$

recalling that ρ_H is a negative quantity and that $\theta_H = \pi/2 - \theta$. In the BL system one has

$$N_a = \{0, 0, 1\} \frac{1}{\rho_H}. \quad (4.9)$$

Next, we discuss the quantity C , which is related to the curvature radii of the circular parallels of the ellipsoids of revolution, *i.e.* $\rho_C = \sigma = A \sin \vartheta$. Thus, C has only one component in the Cartesian system lying in the $2'$ -direction

$$C_{a'} = \{0, 0, 1\} \frac{1}{\sigma}. \quad (4.10)$$

But two components occur in the auxiliary system a''

$$C_{a''} = \{0, \sin \theta, \cos \theta\} \frac{1}{\sigma} \quad (4.11)$$

as seen from **Figure 10**.

Figure 10. Horizontal splitting of C .

Finally, the component $C_{1''}$ has a projection onto the tangent of the integral curve and onto the local extradimension

$$C_a = \{\sin \varepsilon \sin \theta, \cos \varepsilon \sin \theta, \cos \theta\} \frac{1}{\sigma}. \quad (4.12)$$

This is depicted in **Figure 11**.

Figure 11. Vertical splitting of C .

Recall that the angle ε is cw and thus C_0 is pointing into the opposite direction of the local extradimension x^0 .

The quantity M still needs to be discussed. It does not belong to the elliptic-hyperbolic system, but to the integral surface. M is related to the curvature radii of the integral lines and has the components

$$\begin{aligned} M_a &= \{1, 0, 0\} \frac{1}{\rho_s}, \quad M_{a'} = \{\cos \varepsilon, \sin \varepsilon \cos \theta, \sin \varepsilon \sin \theta\} \frac{1}{\rho_s}, \\ M_{a''} &= \{\cos \varepsilon, \sin \varepsilon, 0\} \frac{1}{\rho_s} \end{aligned} \quad (4.13)$$

Figure 12. Vertical splitting of M .

M is situated in the direction of the local extradimension as shown in **Figure 12**. The quantity is only important for the use of 5-dimensional covariant derivatives. Since we do not use this formalism in this paper, we will not discuss this quantity in detail.

So far we have intuitively derived various representations of the fundamental quantities by drawing figures. For the interested reader we note the transformation matrices

$$\begin{aligned} \Lambda_{a''}^a &= \begin{pmatrix} \cos \varepsilon & \sin \varepsilon \\ -\sin \varepsilon & \cos \varepsilon \\ & & 1 \end{pmatrix}, \quad \Lambda_{a'}^{a''} = \begin{pmatrix} 1 & & \\ & \cos \theta & \sin \theta \\ & -\sin \theta & \cos \theta \end{pmatrix} \\ \Lambda_{a'}^a &= \begin{pmatrix} \cos \varepsilon & \sin \varepsilon \cos \theta & \sin \varepsilon \sin \theta \\ -\sin \varepsilon & \cos \varepsilon \cos \theta & \cos \varepsilon \sin \theta \\ & -\sin \theta & \cos \theta \end{pmatrix} \end{aligned} \quad (4.14)$$

With the help of these one can calculate all the expressions of this section.

Now we are prepared to discuss the full seed metric for the Kerr model. One has

$$ds^2 = \alpha_s^2 a_R^2 dr^2 + \Lambda^2 d\mathcal{G}^2 + A^2 \sin^2 \mathcal{G} d\varphi^2 + a_s^2 \rho_s^2 di\psi^2. \quad (4.15)$$

Therein dx^4 is evidently defined by

$$dx^4 = idt = \rho_s di\psi. \quad (4.16)$$

The metric is written in the original Hilbert notation with index 4, *i.e.* (++++), and the timelike element is interpreted as an arc on a pseudo circle, shown as a pseudoreal representation in **Figure 13**.

Figure 13. The geometrical definition of time.

We have to bear in mind that when using a real and an imaginary axis the pseudo circle cannot be drawn or even imagined. The time we measure with our clocks is the real accessory number of an imaginary angle and the flow of time is the arc of a pseudo circle. The infinite past and the infinite future are the points on the 45° axes in the figure.

Sometimes another pseudoreal representation is used, a hyperbola instead of a circle. This has the advantage that the infinite can be better visualized, but the hyperbola shows a position-dependent curvature, while the pseudo circle has a constant curvature, including the infinite. Therefore a pseudo circle is also called *hyperbola of constant curvature*. Unfortunately the pseudoreal representation with hyperbolae misleads some authors to take the hyperbolae literally, but this hinders a geometric explanation of the Kerr metric with the help of an embedding.

We call the factor a_s in (4.15) *gravitational factor*. Its derivative leads us to the force of gravity $E_1 = \frac{1}{\rho_s} \tan \varepsilon$ and is a member of the Ricci-rotation coefficients of the seed model. From (2.3) we see that if the rotational parameter is $a = 0$ the elliptical factor is $a_R = 1$, and also $A = r$. Thus, taking ρ_s from (2.3) we obtain $\rho_s = \sqrt{\frac{2r^3}{M}}$, the curvature radius of Flamm's paraboloid of the Schwarz-

schild geometry. In this case, $E_1 = \frac{1}{\rho_s} \tan \varepsilon$ will be the gravitational force of the Schwarzschild model. Since the angle ε is cw, the gravitational force is pointing inwards.

Now we analyze the geometrical meaning of this quantity. In the 5-dimensional embedding space it has the components

$$\begin{aligned} E_a &= \left\{ -\frac{1}{\rho_s}, \frac{1}{\rho_s} \tan \varepsilon, 0 \right\} = \{ \cos \varepsilon, -\sin \varepsilon, 0 \} \left(-\frac{1}{\rho_s \cos \varepsilon} \right), \\ E_{a'} &= \{ 1, 0, 0 \} \left(-\frac{1}{\rho_s \cos \varepsilon} \right) \end{aligned} \quad (4.17)$$

From **Figure 14**

Figure 14. The gravitational force.

we can see that $\rho_s \cos \varepsilon$ are the projections of the curvature radii ρ_s into the extradimension $0'$ of the embedding space. E_0 is the inverse of $\rho_s \cos \varepsilon$. It is split into the components with respect to the local BL reference system a . E_1 is tangent to the integral curves. It is the very quantity a physical observer can experience. The second component is hidden to him.

Having explained the fundamental quantities of the seed metric with curvatures, we reformulate the metric with the help of curvature radii. For the flat elliptic-hyperbolic system we have

$$ds^2 = \rho_H^2 d\theta_H^2 + \rho_E^2 d\theta_E^2 + \rho_C^2 d\varphi^2$$

and for the seed metric with $\rho_w = \rho_s \cos \varepsilon$ we have

$$ds^2 = \rho_s^2 d\varepsilon^2 + \rho_E^2 d\theta_E^2 + \rho_C^2 d\varphi^2 + \rho_{\psi}^2 di\psi^2.$$

5. The Field Equations

All the quantities B, N, C, E discussed in the previous section are members of the Ricci-rotation coefficients and satisfy the subequations of Einstein's field equations. For a detailed discussion we again refer to [5] [6]. Here we restrict ourselves to noting some relations. For the elliptic-hyperbolic basic system we obtain

$$B_{1|1} + B_1 B_1 = \Omega^2, \quad N_{2|2} + N_2 N_2 = -\Omega^2.$$

The occurrence of the quantity Ω is the second surprise of the elliptic-hyperbolic system. It is a second-rank tensor describing rotational effects which we will meet with the genuine Kerr metric. Evidently the sum of the above equations vanishes. In a covariant form the curvature equations of the elliptic-hyperbolic system can be written as

$$\left[N^s_{\parallel s} + N^s N_s \right] + \left[B^s_{\parallel s} + B^s B_s \right] = 0, \quad \left[C^s_{\parallel s} + C^s C_s \right] = 0, \quad s = 1, 2, 3, 4. \quad (5.1)$$

Thus, these subequations of the Ricci drop out from Einstein's field equations. The only quantity of physical interest is the force of gravity. It is contained in Einstein's field equations with the structure

$$E^s_{\parallel s} - E^s E_s, \quad (5.2)$$

i.e. the field of gravity is coupled to itself, due to the non-linearity of Einstein's field equations.

To get the Equations (5.1) and (5.2) we have skipped the tedious procedure of *dimensional reduction*, *i.e.* getting rid of all 0-components and all 0-derivatives of the fundamental quantities, and we have restricted ourselves to the flat basic system, the shadow on the horizontals of the Kerr surface. Thus, we are left with 4-dimensional quantities and their *graded derivatives* [5] [6]. We will only briefly remark on how the quantity Ω comes into the theory. We have extensively used coordinate systems with ellipses and their curvature vectors, which represent a kind of polar system. But we have to bear in mind that the pole of this system is not fixed, but moves on the evolute of the ellipses, if the curvature vector moves on the ellipses. Thus, the change of ρ_E and of the related curvature B has two contributions, the one by motion of the tip of the curvature vector on the evolute, the other one by motion of the tail of the curvature vector on the evolute. The latter produces the term Ω^2 . The same holds for ρ_H and the related curvature N .

The seed metric does not provide a vacuum solution, it is an auxiliary metric, a forerunner to better explain the Kerr metric. Thus, we will not continue with this model but will turn to study the rotational effects in the next section.

6. Rotation

Starting with the seed metric (4.15), we define an anholonomic transformation

$$\begin{aligned}
\Lambda_{3'}^{3'} &= \alpha_R, & \Lambda_{4'}^{3'} &= i\alpha_R\omega, & \Lambda_{3'}^{4'} &= -i\alpha_R\omega\sigma^2, & \Lambda_{4'}^{4'} &= \alpha_R \\
\Lambda_{3'}^3 &= \alpha_R, & \Lambda_{4'}^3 &= -i\alpha_R\omega, & \Lambda_{3'}^4 &= i\alpha_R\omega\sigma^2, & \Lambda_{4'}^4 &= \alpha_R.
\end{aligned} \tag{6.1}$$

$$\omega = \frac{a}{A^2}, \quad \omega\sigma = \frac{a}{A} \sin \vartheta$$

Operating on the coordinate indices of the tetrads, we get the genuine Kerr metric

$$\begin{aligned}
ds^2 &= dx^{1^2} + dx^{2^2} + \left[\alpha_R dx^3 + i\alpha_R \omega \sigma dx^4 \right]^2 + a_s^2 \left[-i\alpha_R \omega \sigma dx^3 + \alpha_R dx^4 \right]^2 \\
dx^4 &= \rho_s di\psi = idt
\end{aligned} \tag{6.2}$$

ω is the angular velocity and $\omega\sigma$ the orbital velocity of an observer subjected to frame dragging by the field of a rotating source. The 4-bein system obtained in this way is named system C after Carter and is one of the preferred reference systems attributed to the Kerr model. The coordinate system is oblique-angled, the Carter tetrads are mutually perpendicular by definition.

In contrast, if we perform a Lorentz transformation

$$L_{3'}^{3'} = \alpha_R, \quad L_{4'}^{3'} = i\alpha_R\omega\sigma, \quad L_{3'}^{4'} = -i\alpha_R\omega\sigma, \quad L_{4'}^{4'} = \alpha_R \tag{6.3}$$

operating on the tetrad indices of the seed metric, we obtain instead of

$$ds^2 = \alpha_s^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + dx^{3^2} + a_s^2 dx^{4^2}, \quad dx^3 = A \sin \vartheta d\varphi \tag{6.4}$$

the metric

$$ds^2 = \alpha_s^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \left[\alpha_R dx^3 + i\alpha_R \omega \sigma dx^4 \right]^2 + \left[-i\alpha_R \omega \sigma dx^3 + \alpha_R a_s dx^4 \right]^2 \tag{6.5}$$

which differs from the genuine Kerr metric (6.2) only by the position of the gravitational factor a_s concerning the last brackets. Although the two metrics are very similar, they have a quite different physical interpretation. While in (6.2) the rotation is inherent, the metric (6.5) is still static, but observers are rotating around the source producing the exterior field. We use the similarities of the metrics to make clearer the structure of the rotational effects of the Kerr metric.

It will be shown that the metrics (6.2) and (6.5) exhibit the same fundamental rotational structures. Since it turns out that (6.5) is much easier to treat, we make use of this property, and we will compare the results for both metrics at the end of the section.

Evidently, it is sufficient to consider the [3,4]-piece of the metric. We make a further simplification: we put $a_s = 1/\alpha_s = 1$, *i.e.* we switch off the gravitational force. Thus, we are left only with rotational effects and we get

$$ds^2 = a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \left[\alpha_R dx^3 + i\alpha_R \omega \sigma dx^4 \right]^2 + \left[-i\alpha_R \omega \sigma dx^3 + \alpha_R dx^4 \right]^2, \tag{6.6}$$

the metric for the elliptic-hyperbolic base system, the shadow of the Kerr system on the parallels, a flat system with rotating observers. The Ricci-rotation coefficients of this metric contain new forces derived from the orbital velocity $\omega\sigma$ and the associated Lorentz factor α_R . From (2.9) we know that the angular velocity ω depends on r . It is decreasing outwards and zero at infinity. Thus, one

has a *differential rotation law*, a property of the model being a condition for a physically functional rotating model.

From the Lorentz factor α_R from the orbital motion we get with

$$\frac{1}{\alpha_R} \alpha_{R|\alpha} = F_\alpha + D_\alpha \quad (6.7)$$

the relations

$$F_\alpha = \alpha_R^2 \omega^2 \sigma \sigma_\alpha, \quad D_\alpha = \alpha_R^2 \omega \omega_{|\alpha} \sigma^2. \quad (6.8)$$

Evidently the first quantity in the above formulae is the relativistic generalization of the centrifugal force; it is normal to the rotation axis. The second emerges from the differential law of rotation $\omega = \omega(r)$. It has only one component and is pointing inwards. From the orbital velocity we derive

$$H_{\alpha\beta} = 2i\alpha_R^2 \omega \sigma_{[\alpha} c_{\beta]}, \quad D_{\alpha\beta} = i\alpha_R^2 \omega_{|\alpha} \sigma c_{\beta}, \quad c_\beta = \{0, 0, 1\}. \quad (6.9)$$

H is antisymmetric and is the relativistic generalization of the Coriolis field strength. The quantity D is a consequence of the differential law of rotation and is asymmetric ($D_{3\beta} = 0$). Both quantities are summarized to

$$\Omega_{\beta\alpha} = -H_{\alpha\beta} - D_{\alpha\beta}. \quad (6.10)$$

By the decomposition of Ω into an antisymmetric and a symmetric part

$$\Omega_{\beta\alpha} = -[H_{\alpha\beta} + D_{(\alpha\beta)}] - D_{(\alpha\beta)} \quad (6.11)$$

one obtains the total rotational field strength and the deformation field strength. Ω is the very quantity we met at the end of Section 5 when considering the motions of the curvature vectors on the evolutes of the ellipses and hyperbolae. The symmetric part represents the shears

$$u_{(\alpha||\beta)} = -D_{(\alpha\beta)}, \quad u_m = \{0, 0, 0, 1\} \quad (6.12)$$

that should be understood as the observers sliding past each other on account of the different speeds on neighboring circular paths, whereby shears of the surrounding volume elements arise. The new quantities satisfy the relations

$$u_{m||n} u^n = F_m, \quad u_{\alpha||\beta} = \Omega_{\alpha\beta}, \quad \alpha = 1, 2, 3, \quad m = 1, 2, 3, 4, \quad (6.13)$$

where the double strokes indicate the ordinary covariant derivative in tetrad representation.

From Einstein's field equations $R_{mn} \equiv 0$ one obtains equations of Maxwell type

$$\begin{aligned} F^m_{||m} - F^m F_m - \Omega^{mn} \Omega_{nm} &= 0 \\ \Omega^{mn}_{||m} + 2\Omega^{[nm]} F_m &= 0 \end{aligned} \quad (6.14)$$

The centrifugal force is coupled to the field energy, which is composed of quadratic terms. The quantity Ω is coupled to the Poynting vector $2\Omega^{[nm]} F_m$. Further, the relations

$$\begin{aligned} F_{[m|n]} + D_{[m|n]} &= 0, \quad F_{[m|n]} + 2\Omega_{3[m}\Omega_{n]3} = 0 \\ \Omega_{[mn||s]} + \Omega_{[mn}F_{s]} &= 0 \end{aligned} \quad (6.15)$$

are satisfied. They are Maxwell-like as well. The conservation laws

$$\frac{\partial}{\partial t} (F^m F_m + \Omega^{mn} \Omega_{mn}) = 0, \quad (2\Omega^{[nm]} F_m)_{||n} = 0 \quad (6.16)$$

are valid. In the formulae above we used the 4th graded derivative [5] [6]. It corresponds to the spatial covariant derivative.

If we drop the restriction $a_s = 1$ we obtain the gravitational force E_m entering Einstein's field equations with the structure (5.2). In the above equations E will accompany the centrifugal force F_m , which has the opposite orientation but not the same direction.

Now we compare the simplified equations with the one of the genuine Kerr metric

$$E_{||s}^s + F_{||s}^s - \Omega_C^{rs} \Omega_{sr}^C = 0 \quad (6.17)$$

$$F_{[m|n]} + D_{[m|n]} = 0, \quad F_{[m|n]} = 2\Omega_{3[n}\Omega_{n]3}, \quad E_{[m|n]} = 0 \quad (6.18)$$

$$\Omega_{[mn||s]}^C = \Omega_{[mn}^C D_{s]} - \Omega_{[mn}^C E_{s]} = 0 \quad (6.19)$$

$$[E_C^s E_s^C + \Omega_C^{sr} \Omega_{rs}^C]_{||4} = 0, \quad [2\Omega_C^{ms} E_s^C]_{||m} = 0, \quad E_n^C = E_n + F_n \quad (6.20)$$

The tag C indicates the quantities of the Carter system, where some of the new quantities differ from the quantities of the simplified system by a factor. In the first brackets one finds terms quadratic in the field strengths. They represent the field energy and are conserved. The second brackets contain the divergence-free Poynting vector. We recognize that the simplified rotational piece of the Kerr model is very close to the genuine Kerr theory.

For a better understanding of the above equations we make a further simplification by putting $\omega = \text{const.}$. In this case the quantities D_α and $D_{\alpha\beta}$, referring to the differential rotation law, vanish. Only the centrifugal force F_α and the vorticity $H_{\alpha\beta}$ remain. The field equations for these quantities are Maxwell-like.

First, we define the axial vector

$$H^\alpha = -\frac{i}{2} \varepsilon^{\alpha\beta\gamma} H_{\beta\gamma}, \quad H^{\alpha\beta} = i \varepsilon^{\alpha\beta\gamma} H_\gamma \quad (6.21)$$

and we obtain

$$H^\alpha = \alpha_R^2 \omega \tau^\alpha, \quad \tau^\alpha = \{\cos \theta, \sin \theta, 0\}. \quad (6.22)$$

Having transformed this vector into Cartesian coordinates we have

$$H^{\alpha'} = \alpha_R^2 \omega \tau^{\alpha'}, \quad \tau^{\alpha'} = \{1, 0, 0\}. \quad (6.23)$$

Since $\tau^{\alpha'}$ is a unit vector lying in the 1'-direction, it is parallel to the rotation axis of the ellipsoids as can be seen from **Figure 15**.

Figure 15. The axial unit vector.

Thus, one can write in vector notation

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \mathbf{F}^2 + 2\mathbf{H}^2, \quad \operatorname{rot} \mathbf{F} = 0 \\ \operatorname{div} \mathbf{H} &= 0, \quad \operatorname{rot} \mathbf{H} = \mathbf{F} \times 2\mathbf{H} \end{aligned} \quad (6.24)$$

wherein div is the 3-dimensional covariant divergence and rot the 3-dimensional covariant rotator. The centrifugal force $\mathbf{F}$ has as a source the field energy, the Coriolis force $\mathbf{H}$ is coupled to the Poynting vector. The conservation laws take the form

$$\frac{\partial}{\partial t}(\mathbf{F}^2 + 2\mathbf{H}^2) = 0, \quad \operatorname{div}(\mathbf{F} \times 2\mathbf{H}) = 0. \quad (6.25)$$

Evidently, the field equations have a similar structure to the Maxwell equations of electrodynamics. The genuine Kerr Equations (6.14)-(6.20) also have almost the same structure. The similarity of gravitation and electrodynamics was first discovered by Lense and Thirring [13] and Thirring [14] [15] [16] in weak field approximation and treated in general form by Hund [17]. In the last decades this problem was investigated by many authors and called gravito-electromagnetism (GEM).

7. Kerr Interior Solution

Several authors have tried to complement the Kerr solution with an interior model. Although the results have been unsatisfactory to date, the search for a solution is still ongoing. We [11] [12] have proposed an interior solution which goes over into the Schwarzschild solution by setting the rotational parameter a to zero. For this model we did not solve Einstein's field equations but instead constructed the model in terms of geometrical methods.

The interior metric

$$\begin{aligned}
ds^2 &= dx^1{}^2 + dx^2{}^2 + \left[\alpha_R dx^3 + i\alpha_R \omega \sigma dx^4 \right]^2 + a_T^2 \left[-i\alpha_R \omega \sigma dx^3 + \alpha_R dx^4 \right]^2 \\
dx^1 &= \alpha_I a_R dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \\
dx^4 &= i dt, \quad \alpha_R = \frac{1}{a_R}, \quad a_R^2 = 1 - \omega^2 \sigma^2
\end{aligned} \tag{7.1}$$

has the same form as the exterior (2.2). It differs by the geometrical factor

$$\alpha_I = \frac{1}{a_I}, \quad a_I^2 = 1 - \frac{r^2}{\mathbb{R}^2} \tag{7.2}$$

and the gravitational factor

$$\begin{aligned}
a_T &= \frac{1}{2} \left[(1 + 2\Phi_g^2) \cos \eta_g - \cos \eta \right] \Phi_g^{-2} \\
\Phi_g^2 &= \frac{r_g^2 + a^2}{r_g^2 - a^2}, \quad \cos \eta_g = \sqrt{1 - \frac{r_g^2}{\mathbb{R}^2}}, \quad \cos \eta = \sqrt{1 - \frac{r^2}{\mathbb{R}^2}} = a_I
\end{aligned} \tag{7.3}$$

$\mathbb{R}$ and the rotational parameter a are constants. All quantities with the subscript g are the constant values of the variables at the boundary surface matching the exterior solution. Evidently, for $a = 0$, $\Phi_g^2 = 1$ the gravitational factor has the form of the Schwarzschild interior solution. The spacelike piece of this model is a cap of a sphere with radius $\mathbb{R}$ and the aperture angle η_g . The cap matches Flamm's paraboloid at $r_g = \mathbb{R} \sin \eta_g$.

The embedding for the spacelike piece of the Kerr model is

$$\begin{aligned}
x^{0'} &= \pm \sqrt{\mathbb{R}^2 - r^2} \\
x^{1'} &= r \cos \vartheta \\
x^{2'} &= A \sin \vartheta \cos \varphi \\
x^{3'} &= A \sin \vartheta \sin \varphi
\end{aligned} \tag{7.4}$$

depicted in **Figure 16**.

Figure 16. The interior Kerr geometry.

From this surface a band has to be cut off and the remaining surface has to be matched horizontally to the auxiliary surface of the Kerr metric. $\mathcal{R}$ is the radius of the circular arc at the minor axes of the ellipses. All individual ‘radial’ curves have hyperbolic contributions in their properties. For $r=0, A=a$ the horizontal ellipses reduce to a distance which is clamped by the common foci of the ellipses. If one now adds the third dimension, these points rotate through φ . For $\vartheta=\pi/2$ a circle emerges. The radius of curvature of the ellipses is zero on this circle and the assigned field strengths are infinitely large. This is the Kerr ring singularity.

Since the junction condition is satisfied, both solutions, the interior and the exterior, match, as can be seen in **Figure 17**. The Kerr interior shows centrifugal, Coriolis, and gravitational forces, which can be geometrically explained as we have done with the forces of the exterior solution. The interior has a complicated stress-energy-momentum tensor, consisting of gravitational energy, current, and stresses. All that is treated in [5] [6] in detail.

Figure 17. The complete Kerr solution.

8. Summary

We have revisited the Kerr model with the methods of tetrads. These are orthogonal local reference systems. The components of the field quantities represented in these systems are measurable quantities and have a clear geometrical or physical meaning. We have visualized the curvature of space with surfaces and have demonstrated how these quantities emanate from geometrical structures with several drawings.

In addition, we have separated the quantities B , C , and C describing the curvature of the elliptic-hyperbolic system from the physical quantities E, F, Ω , and D describing the physical content of the model. We have shown that the field equations of these physical quantities satisfy Maxwell-like equations. Thus, there is hope for a better understanding of the Kerr model.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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