Existence of Random Attractor Family for a Class of Nonlinear Higher-Order Kirchhoff Equations

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Abstract

The existence of random attractor family for a class of nonlinear nonlocal higher-order Kirchhoff partial differential equations with additive white noise is studied. The weak solution of the equation is established by the Ornstein-Uhlenbeck process to deal with the random term, and a bounded random absorption set is obtained. And then, the existence of the random attractor family is proved by the isomorphism mapping method.

Keywords

Stochastic Dynamical Systems, Random Attractor Family, White Noise, Ornstein-Uhlenbeck Process, Tempered Sets

1. Introduction

In this paper, we consider the following of nonlinear strongly damped stochastic Kirchhoff equations with additive white noise:

$$u_{tt} + M\left(\left\|D^m u\right\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(x, u_t) = q(x)\dot{W}, \tag{1}$$

with the Dirichlet boundary condition

$$u(x,t) = 0, \ \frac{\partial^i u}{\partial v^i} = 0, \ i = 1, 2, \cdots, m-1, \ x \in \partial\Omega, \ t > 0, \tag{2}$$

and the initial value conditions

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega \subset \mathrm{R}^n. \tag{3}$$

where $m > 1$ is a positive integer, β is a normal number, Ω is a bounded region with smooth boundary in R^n, M is a general real-valued function, $g(x, u_t)$ is a nonlinear nonlocal source term, and $q(x)z(\theta_t\omega)$ is a random

term. The assumptions about M and g will be given later.

Xintao Li and Lu Xu [1] have studied the following stochastic delay discrete wave equation

$$u_{itt} + Au_i + u_{it} - f(u_{it}) = \alpha\dot{\omega}_1(t),\tag{4}$$

$$u_\tau = u(s+\tau),\ s\in[-\tau,0],\ \tau>0.\tag{5}$$

The existence of random attractors for this equation is proved by means of tail-cutting technique and energy estimation under appropriate dissipative conditions.

Ailing Ban [2] have considered the following of stochastic wave equations

$$du_t + \left(-\alpha\Delta u_t + \beta u_t - K\Delta u + f(u)\right)dt = g(x)dW(t),\tag{6}$$

with the Dirichlet boundary condition

$$u(x,t)\big|_{x\in\partial U} = 0,\tag{7}$$

and the initial value conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x).\tag{8}$$

where $u = u(x,t)$ is the real function on $U\times[0,+\infty)$, $\alpha>0$ is the strong damping coefficient, $\beta>0$ is the damping coefficient, and $K>0$ is the dissipation coefficient. In this paper, they mainly discuss the asymptotic behavior of strongly damped stochastic wave equation with critical growth index. By using the weighted norm, they prove that for any positive strong damping coefficient and dissipation coefficient, there is a compact attractor for the stochastic dynamical system determined by the solution of the equation.

Caidi Zhao, Shengfan Zhou [3] studied the sufficient conditions for the existence of global random attractors for a class of binary systems and their applications

$$\ddot{u}_m + \dot{u}_m + (Au)_m + \lambda_m u_m + \beta\sin u_m = f_m + a_m\dot{w}_m,\ m\in Z, t>0,\tag{9}$$

$$\dot{u}_m(0) = u_{1m,0},\quad u_m(0) = u_{m,0}.\tag{10}$$

They first give some sufficient conditions for the existence of global random attractor for general stochastic dynamical systems, and then use these sufficient conditions to give a simple method for finding the global random attractors of the upper bound of Kolmogorov ε-entropy. Finally, these results are applied to stochastic Sin-Gordon equation.

Guoguang Lin, Ling Chen and Wei Wang [4] have studied the existence of random attractors for higher-order nonlinear strongly damped Kirchhoff equations

$$du_t + \left[(-\Delta)^m u_t + \phi\left(\left\|D^m u\right\|^2\right)(-\Delta)^m u + g(u)\right]dt$$
$$= f(x)dt + q(x)dW(t),\ x\in\Omega, m>1,\tag{11}$$

$$u(x,t) = 0,\ \frac{\partial^i u}{\partial v^i} = 0, i = 1,2,\cdots,m-1,\ x\in\partial\Omega, t>0,\tag{12}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega \subset \mathrm{R}^n. \tag{13}$$

They mainly use the Ornstein-Uhlenbech process to deal with the stochastic term of Equation (11), thus obtain the global well-posedness of the solution, and then prove the existence of the global random attractor.

As we all know, attractors have absorptivity and invariance, and have a clear description of the long-term behavior and the asymptotic stability of the solution of the equation. Because the long-term behavior of the system develops within the overall attractor, and then on this compact set, through the study of the overall behavior characteristics of the system, we can find the most common rules of the system and the basic information of future development. In real life, the evolution of many problems will be disturbed by some uncertain factors. At this time, the deterministic dynamic system can no longer describe these problems. Therefore, it is necessary to study the attractors of stochastic equations with additive noise terms.

In recent years, stochastic attractors for stochastic nonlinear equations with white noise have been favored by many scholars, and many scholars have done a lot of research on these problems and obtained good results. Xiaoming Fan, Donghong Cai and Jianjun Ye [5] studied stochastic attractors for dissipative KdV equations with multiplicative noise; Fuqi Yin, Shengfan Zhou, Hongyan Li and Hongjuan Hao [6] [7] by introducing weighted norm and orthogonal decomposition of linear operators corresponding to the first-order evolution equation with respect to time, the existence of stochastic attractors for stochastic Sine-Gordon equation with strong damping is proved. More research on stochastic Kirchhoff equation with white noise is detailed in reference [8] [9] [10] [11] [12].

The structure of this paper is as follows: in Section 2, some basic assumptions and knowledge of dynamical system required in this paper are introduced; in Section 3, the existence of random attractor family subfamilies is proved by using the isomorphism mapping method.

2. Basic Hypothesis and Elementary Knowledge

In this section, some symbols, definitions and assumptions about Kirchhoff type stress term $M(s)$ and nonlinear nonlocal source term $g(x, u_t)$ are given. In addition, some basic definitions of stochastic dynamical systems are also introduced.

For narrative convenience, we introduce the following symbols:

$$\nabla = D, \ H = L^2(\Omega), \ H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega),$$

$$H_0^{m+k}(\Omega) = H^{m+k}(\Omega) \cap H_0^1(\Omega), \ E_k = H_0^{m+k}(\Omega) \times H_0^k(\Omega), (k = 0, 1, 2, \cdots, m).$$

And definition

$$(y_1, y_2) = (D^{m+k}u_1, D^{m+k}u_2) + (D^k v_1, D^k v_2), \ \forall y_i = (u_i, v_i) \in E_k, \ i = 1, 2.$$

It is assumed that the Kirchhoff type stress term $M(s)$ and the nonlinear

non-local source term $g(x,u_t)$ satisfy the following conditions, respectively:

A1) $M \in C^2(\Omega)$; and $1+\varepsilon < \delta_0 \leq M(s) \leq \delta_1$, where δ_0, δ_1 is a constant;

A2) $g(x,\cdot) \in C^1(\Omega)$ is Lipschitz continuous and satisfies

i) $g(x,0) = 0$ for any $x \in \mathrm{R}$;

ii) There exists a constant $L_g > 0$, such that for any $x \in \mathrm{R}$, have

$$\left\| D^k \left(g(x,u_t) - g(x,v_t) \right) \right\| \leq L_g \left\| D^k (u_t - v_t) \right\|.$$

The following will introduce some basic knowledge about random attractor.

Let $\left(B(\mathrm{R}^+) \times F \times B(X), B_k(\omega) \subset D(\omega) \right)$ be a probabilistic space and define a family of transformations of the sum and ergodic of a family of measures preserving $\{ \theta_t, t \in \mathrm{R} \}$

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t),$$

Then $\left(\Omega, F, P, (\theta_t)_{t \in \mathrm{R}} \right)$ is an orbiting metric dynamical system.

Let $\left(X, \|\cdot\|_X \right)$ be a complete separable metric space and $B(X)$ be a Borel σ-algebra on.

Definition 1 (Following as [12]) Let $\left(\Omega, F, P, (\theta_t)_{t \in \mathrm{R}} \right)$ be a metric dynamical system, suppose that the mapping

$$S : \mathrm{R}^+ \times \Omega \times X \mapsto X, \quad (t,w,x) \mapsto S(t,w,x),$$

is $\left(B(\mathrm{R}^+) \times F \times B(X), B(X) \right)$-measurable mapping and satisfies the following properties:

1) The mapping $S(t,\omega) := S(t,\omega,\cdot)$ satisfies

$$S(0,\omega) = id, \quad S(t+s,\omega) = S(t,\theta_s\omega) \circ S(s,\omega);$$

for any $s,t \geq 0, \omega \in \Omega$.

2) $(t,w,x) \mapsto S(t,w,x)$ is continuous, for any $\omega \in \Omega$.

Then S is a continuous stochastic dynamical system on $\left(\Omega, F, P, (\theta_t)_{t \in \mathrm{R}} \right)$.

Definition 2 (Following as [12]) It is said that random set $B(\omega) \subset X$ is tempered, for $\omega \in \Omega$, $\beta \geq 0$, we have

$$\lim_{|s| \to \infty} \inf e^{-\beta s} d\left(B(\theta_{-s}\omega) \right) = 0,$$

where $d(B) = \sup_{x \in B} \|x\|_X$, for any $x \in X$.

Definition 3 (Following as [12]) Let $D(\omega)$ be the set of all random sets on X, and random set $B_k(\omega)$ is called the suction collection on $D(\omega)$, if for any $B_k(\omega) \subset D(\omega)$ and $P-a.e.\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that

$$S(t,\theta_{-t}\omega)\left(B(\theta_{-t}\omega) \right) \subset B_0(\omega).$$

Definition 4 (Following as [12]) Random set $A(\omega)$ is called the random attractor of continuous stochastic dynamical system $S(t)$ on X, if random set $A(\omega)$ satisfies the following conditions:

1) $A(\omega)$ is a random compact set;

2) $A(\omega)$ is the invariant set $D(\omega)$, that is, for any $t > 0$, we have $S(t,\omega)A(\omega) = A(\theta_t\omega)$;

3) $A(\omega)$ attracts all sets on $D(\omega)$, that is, for any $B(\omega) \in D(\omega)$ and

$P - a.e. \omega \in \Omega$, with the following limit:

$$\lim_{t \to \infty} d\left(S(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)\right) = 0,$$

where $d(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_H$ is Hausdorff half distance. (There $A, B \subseteq H$).

Definition 5 (Following as [12]) Let random set $B_k(\omega) \subset D(\omega)$ be a random suction set for stochastic dynamical system $(S(t, \omega))_{t \geq 0}$, and random set $B_k(\omega)$ satisfies the following conditions:

1) Random set $B_k(\omega)$ is a closed set on Hilbert space X;

2) For $P - a.e. \omega \in \Omega$, random set $B_k(\omega)$ satisfies for any sequence $x_n \in S(t_n, \theta_{-t_n}\omega)B_0(\theta_{-t_n}\omega)$, there is a convergence subsequence in space X, when $t_n \to +\infty$. Then stochastic dynamical system $(S(t, \omega))_{t \geq 0}$ has a unique global attractor

$$A_k(\omega) = \bigcap_{\tau \geq t_k(\omega)} \overline{\bigcup_{t \geq \tau} S(t, \theta_{-t}\omega)B_0(\theta_{-t}\omega)}.$$

The Ornstein-Uhlenbeck process [12] is given as follows:

Let $z(\theta_t\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \theta_t\omega(\tau) d\tau$, where $t \in R$. It can be seen that for any $t \geq 0$, the stochastic process $z(\theta_t\omega)$ satisfies the Ito equation

$$dz + \alpha z dt = dW(t).$$

According to the nature of the O-U process, there exists a probability measure P, θ_t-invariant set, and the above stochastic process

$$z(\theta_t\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \theta_t\omega(\tau) d\tau$$

satisfies the following properties:

1) The mapping $S \to z(\theta_s\omega)$ is a continuous mapping, for any given $\omega \in \Omega_0$;

2) The random variable $\|z(\omega)\|$ is tempered;

3) There exist a tempered set $r(\omega) > 0$, such that

$$\|z(\theta_t\omega)\| + \|z(\theta_t\omega)\|^2 \leq r(\theta_t\omega) \leq r(\omega)e^{\frac{\alpha}{2}|t|};$$

4) $\lim_{t \to \infty} \frac{1}{t} \int_0^t |z(\theta_t\omega)|^2 d\tau = \frac{1}{2\alpha}$;

5) $\lim_{t \to \infty} \frac{1}{t} \int_0^t |z(\theta_t\omega)| d\tau = \frac{1}{\sqrt{\pi\alpha}}$.

3. Existence of Random Attractor Family

In this section, we mainly consider the existence of random attractor family of problem (1)-(3). At first, Young inequality and Holder inequality are used to prove the positive definiteness of operator L; and then the weak solution of the equation is established by Ornstein-Uhlenbeck process to deal with the random term, thus a bounded random absorption collection is obtained. Finally, the existence of random attractor family of this problem is proved by isomorphism mapping method.

The problem (1)-(3) can be rewritten to

$$\begin{cases} du = u_t dt, \\ du_t + \left[M\left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) A^m u + \beta A^m u_t + g(x, u_t) \right] dt = q(x) dW(t), t \in [0, +\infty), \ (14) \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega. \end{cases}$$

where $A = -\Delta$.

Let $\varphi = (u, v)^T$, $v = u_t + \varepsilon u$, then the question (14) can be simplified to

$$\begin{cases} d\varphi + L\varphi dt = F(\theta_t \omega, \varphi), \\ \varphi_0(\omega) = (u_0, u_1 + \varepsilon u_0)^T. \end{cases} \quad (15)$$

where

$$L = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right) A^m + \varepsilon^2 \right) I & (\beta A^m - \varepsilon) I \end{pmatrix}, \quad F(\theta_t \omega, \varphi) = \begin{pmatrix} 0 \\ -g(x, u_t) + q(x) dW(t) \end{pmatrix}.$$

Let $y = v - q(x) z(\theta_t \omega)$, Then the question (14) may read as follows:

$$\begin{cases} \psi_t + L\psi = \overline{F}(\theta_t \omega, \psi), \\ \psi_0(\omega) = (u_0, u_1 + \varepsilon u_0 - q(x)\delta(\theta_t \omega))^T. \end{cases} \quad (16)$$

where $\psi = (u, z)^T$,

$$L = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right)^m + \varepsilon^2 \right) I & (\beta A^m - \varepsilon) I \end{pmatrix}, \quad \overline{F}(\theta_t \omega, \psi) = \begin{pmatrix} q(x) z(\theta_t \omega) \\ -g(x, u_t) + (\varepsilon + 1 - \beta A^m) q(x) z(\theta_t \omega) \end{pmatrix}.$$

Lemma 1 Let $E_k = H_0^{m+k}(\Omega) \times H_0^k(\Omega), (k = 0, 1, 2, \cdots, m)$, for any

$y = (y_1, y_2)^T \in E_k$, if $0 < \varepsilon \le \dfrac{2}{\beta - 1 + \lambda_1^{-m}}$, we have

$$(Ly, y)_{E_k} \ge k_1 \|y\|_{E_k}^2 + k_2 \left\| D^{m+k} y_2 \right\|^2 \ge k_1 \|y\|_{E_k}^2 + k_3 \left\| D^k y_2 \right\|^2,$$

where $k_1 = \min\left\{ \varepsilon - \dfrac{\beta - 1}{2}\varepsilon^2 - \dfrac{\varepsilon^2}{2}\lambda_1^{-m}, \dfrac{\beta\lambda_1^m}{4} - \dfrac{\varepsilon^2}{2} - \varepsilon \right\}, k_2 = \dfrac{\beta + 1}{2}, k_3 = k_2 \lambda_1^m$.

Proof: For any $y = (y_1, y_2)^T$, we have

$$Ly = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right) A^m + \varepsilon^2 \right) I & (\beta A^m - \varepsilon) I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon y_1 - y_2 \\ \left(\left(M\left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right) A^m + \varepsilon^2 \right) y_1 + (\beta A^m - \varepsilon) y_2 \end{pmatrix}. \quad (17)$$

$$(Ly, y)_{E_k}$$
$$= \left(D^{m+k}(\varepsilon y_1 - y_2), D^{m+k} y_1 \right)$$

$$+ \left(D^k \left(\left(\left(M \left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right) A^m + \varepsilon^2 \right) y_1 + \left(\beta A^m - \varepsilon \right) y_2 \right), D^k y_2 \right)$$

$$= \varepsilon \left\| D^{m+k} y_1 \right\|^2 - \left(D^{m+k} y_2, D^{m+k} y_1 \right) + \left(M \left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right) \left(A^m y_1, A^k y_2 \right) \quad (18)$$

$$+ \varepsilon^2 \left(D^k y_1, D^k y_2 \right) + \beta \left\| D^{m+k} y_2 \right\|^2 - \varepsilon \left\| D^k y_2 \right\|^2,$$

From hypothesis (A1), we have

$$\left(M \left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \beta\varepsilon \right) \left(A^m y_1, A^k y_2 \right)$$

$$= \left(M \left(\left\| A^{\frac{m}{2}} u \right\|^2 \right) - \varepsilon \right) \left(D^{m+k} y_1, D^{m+k} y_2 \right) + \left(\varepsilon - \beta\varepsilon \right) \left(D^{m+k} y_1, D^{m+k} y_2 \right) \quad (19)$$

$$\geq \left(D^{m+k} y_1, D^{m+k} y_2 \right) - \frac{\beta-1}{2} \varepsilon^2 \left\| D^{m+k} y_1 \right\|^2 - \frac{\beta-1}{2} \left\| D^{m+k} y_2 \right\|^2,$$

So

$$(Ly, y)_{E_k}$$

$$\geq \varepsilon \left\| D^{m+k} y_1 \right\|^2 - \frac{\beta-1}{2} \varepsilon^2 \left\| D^{m+k} y_1 \right\|^2 - \frac{\beta-1}{2} \left\| D^{m+k} y_2 \right\|^2$$

$$+ \varepsilon^2 \left(D^k y_1, D^k y_2 \right) + \beta \left\| D^{m+k} y_2 \right\|^2 - \varepsilon \left\| D^k y_2 \right\|^2$$

$$\geq \varepsilon \left\| D^{m+k} y_1 \right\|^2 - \frac{\beta-1}{2} \varepsilon^2 \left\| D^{m+k} y_1 \right\|^2 - \frac{\beta-1}{2} \left\| D^{m+k} y_2 \right\|^2$$

$$- \frac{\varepsilon^2}{2} \lambda_1^{-m} \left\| D^{m+k} y_1 \right\|^2 - \frac{\varepsilon^2}{2} \left\| D^k y_2 \right\|^2 + \beta \left\| D^{m+k} y_2 \right\|^2 - \varepsilon \left\| D^k y_2 \right\|^2$$

$$\geq \left(\varepsilon - \frac{\beta-1}{2} \varepsilon^2 - \frac{\varepsilon^2}{2} \lambda_1^{-m} \right) \left\| D^{m+k} y_1 \right\|^2 + \left(\frac{\beta\lambda_1^m}{4} - \frac{\varepsilon^2}{2} - \varepsilon \right) \left\| D^{m+k} y_2 \right\|^2$$

$$+ \frac{\beta+1}{2} \left\| D^k y_2 \right\|^2, \quad (20)$$

where $0 < \varepsilon \leq \dfrac{2}{\beta - 1 + \lambda_1^{-m}}$.

Choose $k_1 = \min \left\{ \varepsilon - \dfrac{\beta-1}{2} \varepsilon^2 - \dfrac{\varepsilon^2}{2} \lambda_1^{-m}, \dfrac{\beta\lambda_1^m}{4} - \dfrac{\varepsilon^2}{2} - \varepsilon \right\}, k_2 = \dfrac{\beta+1}{2}, k_3 = k_2 \lambda_1^m$,

then we have

$$(Ly, y)_{E_k} \geq k_1 \left\| y \right\|_{E_k}^2 + k_2 \left\| D^{m+k} y_2 \right\|^2 \geq k_1 \left\| y \right\|_{E_k}^2 + k_3 \left\| D^k y_2 \right\|^2. \quad (21)$$

Therefore, Lemma 1 is proved.

Lemma 2 Let φ is a solution of the problem (15), then there is a bounded random compact set $\tilde{B}_{0k}(\omega) \in D(E_k)$, such that for any random set $B_k(\omega) \in D(E_k)$, existence a random variable $T_{B_k(\omega)} > 0$, so that

$$\varphi(t, \theta_t \omega) B_k(\theta_{-t} \omega) \subset \tilde{B}_{0k}(\omega), \quad \forall t \geq T_{B_k(\omega)}, \omega \in \Omega.$$

Proof: Let ψ is a solution of the problem (16), taking inner product of two

sides of the Equation (15) is obtained by using $\psi = (u, y)^{\mathrm{T}} \in E_k$ in E_k, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2 + (L\psi, \psi)_{E_k} = (\bar{F}(\theta_t\omega, \psi), \psi), \tag{22}$$

where

$$\begin{aligned}
&(\bar{F}(\theta_t\omega, \psi), \psi) \\
&= (D^{m+k}q(x)z(\theta_t\omega), D^{m+k}u) \\
&\quad + (D^k(-g(x, u_t) + (\varepsilon + 1 - \beta A^m)q(x)z(\theta_t\omega)), D^k z).
\end{aligned} \tag{23}$$

From Lemma 1, we have

$$(L\psi, \psi)_{E_k} \geq k_1\|\psi\|_{E_k}^2 + k_3\|D^k y\|^2, \tag{24}$$

According to Holder inequality, Young inequality and Poincare inequality, we have

$$(D^{m+k}q(x)z(\theta_t\omega), D^{m+k}u) \leq \frac{1}{2}\|D^{m+k}u\|^2 + \frac{1}{2}\|D^{m+k}q(x)\|^2|z(\theta_t\omega)|^2, \tag{25}$$

$$(\varepsilon D^k q(x)z(\theta_t\omega), D^k y) \leq \frac{k_3}{4}\|D^k y\|^2 + \frac{\varepsilon^2}{k_3}\|D^k q(x)\|^2|z(\theta_t\omega)|^2, \tag{26}$$

$$\begin{aligned}
&((1 - \beta A^m)D^k q(x)z(\theta_t\omega), D^k y) \\
&\leq \frac{k_3}{4}\|D^k y\|^2 + \frac{1}{k_3}\|D^k q(x)\|^2|z(\theta_t\omega)|^2 + \frac{\beta^2}{k_3}\left\|A^{m+\frac{k}{2}}q(x)\right\|^2|z(\theta_t\omega)|^2,
\end{aligned} \tag{27}$$

from assumption (A2), we have

$$\begin{aligned}
&(-D^k g(x, u_t), D^k y) \\
&\leq \|D^k(g(x, u_t) - g(x, 0))\|\|D^k y\| \leq L_g\|D^k u_t\|\|D^k y\| \\
&= L_g\|D^k(y - \varepsilon u + q(x)z(\theta_t\omega))\|\|D^k y\| \\
&\leq L_g(\|D^k y\| + \varepsilon\|D^k u\| + \|D^k q(x)\||z(\theta_t\omega)|)\|D^k y\| \\
&\leq \left(L_g + \frac{k_3}{4}\right)\|D^k y\|^2 + \frac{\varepsilon^2 L_g^2}{k_3}\lambda_1^{-m}\|D^{m+k}u\|^2 + \frac{L_g^2}{k_3}\|D^k q(x)\|^2|z(\theta_t\omega)|^2.
\end{aligned} \tag{28}$$

Combining (22)-(28) yields, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2 + 2k_1\|\psi\|_{E_k}^2 \\
&\leq C_1 + \left(\frac{2(L_g^2 + \varepsilon^2 + 1)}{k_3}\|D^k q(x)\|^2 + \|D^{m+k}q(x)\|^2 + \frac{2\beta^2}{k_3}\left\|A^{m+\frac{k}{2}}q(x)\right\|^2\right)|z(\theta_t\omega)|^2.
\end{aligned} \tag{29}$$

Taking $N_1 = \frac{2(L_g^2 + \varepsilon^2 + 1)}{k_3}\|D^k q(x)\|^2 + \|D^{m+k}q(x)\|^2 + \frac{2\beta^2}{k_3}\left\|A^{m+\frac{k}{2}}q(x)\right\|^2$,

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2 + 2k_1\|\psi\|_{E_k}^2 \leq C_1 + N_1|z(\theta_t\omega)|^2. \tag{30}$$

From Gronwal inequality $P_{a.e.w} \in \Omega$, then

$$\left\| \psi \left(t, \omega \right) \right\|_{E_k}^2 \le \mathrm{e}^{-2k_1 t} \left\| \psi \left(0, \omega \right) \right\|_{E_k}^2 + \int_0^t \mathrm{e}^{-2k_1(t-s)} \left(C_1 + N_1 \left| z \left(\theta_s \omega \right) \right|^2 \right) \mathrm{d}s. \tag{31}$$

And because $z \left(\theta_t \omega \right)$ is tempered, and $z \left(\theta_t \omega \right)$ is continuous about t, so according to reference [3], we can get a temper random variable $r_1 : \Omega \to R^+$, so that for any $t \in R, \omega \in \Omega$, we have

$$\left| z \left(\theta_t \omega \right) \right|^2 \le r_1 \left(\theta_t \omega \right) \le \mathrm{e}^{k_1 t} r_1 \left(\omega \right). \tag{32}$$

Replace ω in Equation (30) with $\theta_{-t} \omega$, we can obtain that

$$\left\| \psi \left(t, \theta_{-t} \omega \right) \right\|_{E_k}^2 \le \mathrm{e}^{-2k_1 t} \left\| \psi \left(0, \theta_{-t} \omega \right) \right\|_{E_k}^2 + \int_0^t \mathrm{e}^{-2k_1(t-s)} \left(C_1 + N_1 \left| z \left(\theta_{s-t} \omega \right) \right|^2 \right) \mathrm{d}s, \tag{33}$$

Available from (32)

$$\begin{aligned} &\int_0^t \mathrm{e}^{-2k_1(t-s)} \left(C_1 + N_1 \left| z \left(\theta_{s-t} \omega \right) \right|^2 \right) \mathrm{d}s \\ &= \int_{-t}^0 \mathrm{e}^{2k_1 \tau} \left(C_1 + N_1 \left| z \left(\theta_\tau \omega \right) \right|^2 \right) \mathrm{d}\tau \le \frac{C_1}{2k_1} + \frac{1}{3k_1} N_1 r_1 \left(\omega \right). \end{aligned} \tag{34}$$

Therefore

$$\left\| \psi \left(t, \theta_{-t} \omega \right) \right\|_{E_k}^2 \le \frac{C_1}{2k_1} + \frac{1}{3k_1} N_1 r_1 \left(\omega \right). \tag{35}$$

Because $\varphi \left(\theta_{-t} \omega \right) \in B_k \left(\theta_{-t} \omega \right)$ is tempered, and $\left| z \left(\theta_{-t} \omega \right) \right|$ is also tempered, so we can let

$$R_0^2 \left(\omega \right) = \frac{C_1}{2k_1} + \frac{1}{3k_1} N_1 r_1 \left(\omega \right). \tag{36}$$

then $R_0^2 \left(\omega \right)$ is also tempered, put $\hat{B}_{0k} = \left\{ \xi \in E_k \left| \left\| \psi \right\|_{E_k} \le R_0 \left(\omega \right) \right. \right\}$ is a random absorb set, and because of

$$\begin{aligned} &\tilde{S} \left(t, \theta_{-t} \omega \right) \psi \left(0, \theta_{-t} \omega \right) \\ &= \varphi \left(t, \theta_{-t} \omega \right) \left(\psi \left(0, \theta_{-t} \omega \right) + \left(0, q(x) z \left(\theta_{-t} \omega \right) \right)^{\mathrm{T}} \right) - \left(0, q(x) z \left(\theta_{-t} \omega \right) \right)^{\mathrm{T}}, \end{aligned} \tag{37}$$

So let

$$\tilde{B}_{0k} \left(\omega \right) = \left\{ \varphi \in E_k \left| \left\| \varphi \right\|_{E_k} \le R_0 \left(\omega \right) + \left\| D^k q(x) \delta \left(\omega \right) \right\| = \overline{R}_0 \left(\omega \right) \right. \right\}, \tag{38}$$

then $\tilde{B}_{0k} \left(\omega \right)$ is a random absorb set of $\varphi \left(t, \omega \right)$, and $\tilde{B}_{0k} \left(\omega \right) \in D \left(E_k \right)$.

Thus, the whole proof is complete.

It is shown below that there exists a compact suction collection for stochastic dynamical system $S \left(t, \omega \right)$.

Lemma 3 When $k = m$, for any $B_m \left(\omega \right) \in D \left(E_m \right)$, let $\varphi \left(t \right)$ is a solution of the Equation (15) with the initial value $\varphi_0 = \left(u_0, u_1 + \varepsilon u_0 \right)^{\mathrm{T}} \in B_m$, and it can decompose $\varphi = \varphi_1 + \varphi_2$, where φ_1, φ_2 satisfy

$$\begin{cases} \mathrm{d}\varphi_1 + L\varphi_1 \mathrm{d}t = 0, \\ \varphi_{10} \left(\omega \right) = \left(u_0, u_1 + \varepsilon u_0 \right)^{\mathrm{T}}. \end{cases} \tag{39}$$

$$\begin{cases} \mathrm{d}\varphi_2 + L\varphi_2 \mathrm{d}t = F \left(\omega, \varphi \right), \\ \varphi_{20} \left(\omega \right) = 0. \end{cases} \tag{40}$$

then $\left\|\varphi_1\left(t,\theta_{-t}\omega\right)\right\|_{E_m}^2 \to 0, \left(t \to \infty\right)$, for any $\varphi_0\left(\theta_{-t}\omega\right) \in B_m\left(\theta_{-t}\omega\right)$, there exists a temper random radius $R_1\left(\omega\right)$, such that

$$\left\|\varphi_2\left(t,\theta_{-t}\omega\right)\right\|_{E_m} \leq R_1\left(\omega\right), \quad \text{for any} \quad \omega \in \Omega.$$

Proof: When $k = m$, let
$\psi = \psi_1 + \psi_2 = \left(u_1, u_{1t} + \varepsilon u_1\right)^{\text{T}} + \left(u_2, u_{2t} + \varepsilon u_2 - q\left(x\right)z\left(\theta_t\omega\right)\right)^{\text{T}}$ is a solution of Equation (16), then according to Equation (39) and Equation (40), we know ψ_1, ψ_2 meet separately

$$\begin{cases} \psi_{2t} + L\psi_1 = 0, \\ \psi_{10} = \left(u_0, u_1 + \varepsilon u_0 - q\left(x\right)z\left(\theta_t\omega\right)\right)^{\text{T}}. \end{cases} \tag{41}$$

$$\begin{cases} \psi_{2t} + L\psi_2 = \overline{F}\left(\theta_t\omega, \psi_2\right), \\ \psi_{20} = 0. \end{cases} \tag{42}$$

Taking inner product Equation (41) with $\psi_1 = \left(u_1, u_{1t} + \varepsilon u_1\right)^{\text{T}}$ in E_m, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\psi_1\right\|_{E_m}^2 + \left(L\psi_1, \psi_1\right)_{E_m} = 0, \tag{43}$$

From Lemma 1 and Gronwall inequality, we have

$$\left\|\psi_1\left(t,\omega\right)\right\|_{E_m}^2 \leq \mathrm{e}^{-2k_1 t}\left\|\psi_1\left(0,\omega\right)\right\|_{E_m}^2, \tag{44}$$

substituting ω by $\theta_{-t}\omega$ in (43), and because $z\left(\theta_{-t}\omega\right) \in B_m$ is tempered, then

$$\left\|\psi_1\left(t,\theta_{-t}\omega\right)\right\|_{E_m}^2 \leq \mathrm{e}^{-2k_1 t}\left\|\psi_0\left(\theta_{-t}\omega\right)\right\|_{E_m}^2 \to 0, \left(t \to \infty\right), \forall \psi_0\left(\theta_{-t}\omega\right) \in B_m. \tag{45}$$

Taking inner product (42) with $\psi_2 = \left(u_2, u_{2t} + \varepsilon u_2 - q\left(x\right)z\left(\theta_t\omega\right)\right)^{\text{T}}$ in E_m, and from Lemma 1 and Lemma 2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\psi_2\right\|_{E_m}^2 + 2k_1\left\|\psi_2\right\|_{E_m}^2 \leq C_1 + N_2\left|z\left(\theta_t\omega\right)\right|^2, \tag{46}$$

where $N_2 = \dfrac{2\left(L_g^2 + \varepsilon^2 + 1\right)}{k_3}\left\|D^m q\left(x\right)\right\|^2 + \left\|D^{2m} q\left(x\right)\right\|^2 + \dfrac{2\beta^2}{k_3}\left\|A^{\frac{3m}{2}} q\left(x\right)\right\|^2$.

Substituting ω by $\theta_{-t}\omega$ in (46) and from Gronwall's Inequality and (32), we have

$$\begin{aligned} \left\|\psi_2\left(t,\theta_{-t}\omega\right)\right\|_{E_m}^2 &\leq \mathrm{e}^{-2k_1 t}\left\|\psi_2\left(0,\theta_{-t}\omega\right)\right\|_{E_m}^2 + \int_0^t \mathrm{e}^{-2k_1(t-s)}\left(C_2 + N_2\left|z\left(\theta_{s-t}\omega\right)\right|^2\right)\mathrm{d}s \\ &\leq \frac{C_2}{2k_1} + \frac{1}{3k_1}N_2 r_1\left(\omega\right). \end{aligned} \tag{47}$$

So there exists a temper random radius

$$R_1^2\left(\omega\right) = \frac{C_2}{2k_1} + \frac{1}{3k_1}N_2 r_1\left(\omega\right), \tag{48}$$

such that

$$\left\|\varphi_2\left(t,\theta_{-t}\omega\right)\right\|_{E_m} \leq R_1\left(\omega\right), \quad \text{for any} \quad \omega \in \Omega. \tag{49}$$

This completes the Proof of Lemma 3.

Lemma 4 The stochastic dynamical system $\left\{S\left(t,\omega\right), t \geq 0\right\}$, while

$t = 0, P_{a.e.w} \in \Omega$ determined by Equation (15) has a compact attracting set $K(\omega) \subset E_k$.

Proof: Let $K(\omega)$ be a closed sphere in space E_k with a radius of $R_1(\omega)$. According to embedding relation $E_k \subset E_0$, then $K(\omega)$ is a compact set in E_k, for any temper random set $B_k(\omega) \in E_k$, for $\forall \psi(t, \theta_{-t}\omega) \in B_k$, according to Lemma 3, $\psi_2 = \psi - \psi_1 \in K(\omega)$, so for any $\forall t \geq T_{B_k(\omega)} > 0$, we have

$$
\begin{aligned}
&d_{E_k}\left(S(t, \theta_{-t}\omega)B_k(\theta_{-t}\omega), K(\omega)\right) \\
&= \inf_{\vartheta(t) \in K(w)} \left\| \psi(t, \theta_{-t}\omega) - \vartheta(t) \right\|_{E_k}^2 \leq \left\| \psi_1(t, \theta_{-t}\omega) \right\|_{E_k}^2 \\
&\leq e^{-2k_1 t} \left\| \psi_{10}(t, \theta_{-t}\omega) \right\|_{E_k}^2 \to 0, (t \to \infty).
\end{aligned}
\tag{50}
$$

So, the whole proof is complete.

According to Lemma 1 to Lemma 4, there are the following theorems

Theorem 1 The stochastic dynamical system $\{S(t, \omega), t \geq 0\}$ has a random attractor family $A_k(\omega) \subset K(\omega) \subset E_k$, for any $\omega \in \Omega$, and there exists a temper random set $K(\omega)$, such that

$$
A_k(\omega) = \bigcap_{t \geq 0, \tau \geq t} \overline{\bigcup S\left(t, \theta_{-\tau}\omega, K(\theta_{-\tau}\omega)\right)},
$$

and

$$
S(t, \omega)A_k(\omega) = A_k(\theta_t\omega).
$$

4. Conclusion

In this paper, starting from the positive definiteness of the operator, the weak solution of the equation established by O-U process is used to deal with the stochastic term, and a bounded stochastic absorption set is obtained, thus tempered random set is obtained. Then, the isomorphic mapping method is used to prove that the stochastic dynamical system $S(t, \omega)$ has a attractor family $A_k (k = 1, 2, \cdots, m)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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