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Quantization of Hubble's Law and the Emergence of Classical Localization in a Quantum World

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Abstract

We apply a canonical transformation Hubble's law to turn it into a quantum equation and derive its solutions in a homogenous universe (assumptions analogous to the FLRW universe). The eigenfunctions of Hubble's law are also stationary states (eigenfunctions of the Hamiltonian). The study of these solutions reveals many striking results, including: 1) By enforcing boundary conditions at the cosmic horizon, we derive a fundamental lower limit to the uncertainty in any rest mass (or measurement thereof) $\delta m_{\min} = 2\pi H_0 \hbar^2 c^{-2}$. This implies a lower limit also to the mean particle mass which we call the mass quantum $\bar{m}_H \equiv \delta m_{\min}/2 = 5.1\times 10^{-68}$ kg. 2) We postulate that particles with finite mass can be regarded as a composition of a large number of mass quanta and deduce a relation between mass uncertainty δm_0 and mass m_0: $\delta m_0 \doteq \sqrt{m_0/\bar{m}_H}\,\delta m$. 3) This uncertainty leads naturally to localization of the composite mass, with the radius of localization proportional to the inverse square root of mass $r_{loc} = \sqrt{\pi\hbar (m_0 H_0)^{-1}}$. We associate this localization with the classical localization of a massive particle. 4) We derive an expression for the critical mass where there is a crossover from quantum behavior to classical behavior $m_{\text{crossover}} = (16/9)^{1/5}\,\pi\rho^{2/5}\hbar^{3/5}H_0^{-3/5}$, where ρ is the material mass density. The classical sizes derived in 4) are consistent with empirical results for our universe. We note the theory described here has no free parameters, hence it points to a new fundamental equation of the universe, essentially defining the mass quantum. It is a pure quantum theory that does not invoke general relativity at any stage, and the derivation uses mathematics accessible to an upper level undergraduate student in physics.

Keywords

Quantum Cosmology, Hubbles Law, Canonical Quantization, Quantum Theory

1. Introduction

Why do ordinary objects appear to follow classical space-time trajectories but very small objects follow quantum (wavelike) trajectories? Here we derive a quantitative theory that provides one answer to this question. Generally, this paper is motivated by topics of great current interest at the very foundations of physics: Is quantum theory (QT) applicable not only at microscopic length scales but also at large, even cosmological length scales? Does the expansion of the universe have any implications for microscopic particles?

Regarding length scale, in a recent review article [1] Legget writes, "as pointed out by Schrödinger in 1935 in his famous 'cat' paper [2], there is no good reason to accept (the) division of the world into a microscopic regime where QT reigns and a macroscopic one governed by classical physics." On the other hand, [3] reminds us, "it is safe to state, in any case, that quantum superpositions of truly massive, complex objects are *terra incognita*". Whether or not QT prevails at all length scales is now a topic of intense interest, motivating experimental tests of quantum mechanics at mesoscopic scales [1] [3] [4] [5]. In this paper, we take the rule that QT applies at all length scales as a premise of the theory.

Whether or not the Hubble expansion has an impact at microscopic scales is another area of controversy. According to Nobelist Leon Lederman, "The expansion of the universe doesn't actually affect the spaces between particles. The universe's expansion is not a force that will rip particles, molecules, or even objects apart [6]." Meanwhile, theories of "phantom energy" predict precisely the opposite [7]. This paper takes the latter view as a premise of the theory, which we paraphrase, in a universe where the Hubble parameter $H_0 > 0$, the stationary states of all massive free particles are modified compared to stationary states in flat space $(H_0 = 0)$. Some other works assuming cosmological expansion affecting quantum systems are found in [8] [9] [10]. Outside the domain of general relativity, C. L. Herzenberg has developed a theory that predicts classical localization from the universe' expansion [11] [12]. The mechanism of Herzenberg's theory is unrelated to the theory presented here.

The single-particle theory presented here approximates a more general quantum field theory whose derivation we leave for the future. We consider spin-zero bosons and quasi-particles on any length scale, from elementary Higgs bosons, to composite bosons such as the ground state ^{1}H atom, to very large spin zero quasiparticles such as nonrotating stars.

Another important feature is what this theory leaves out. This is a pure quantum theory that makes no reference to General Relativity (GR) or even the concept of gravity. Again, we view this work as an approximation of a complete theory including GR that can be addressed in future work. We emphasize that our theory does not change or overturn GR; it simply ignores it. At first this might appear to be inconsistent with our use of Hubble's natural law in the derivation. But Hubble's law (redshift increases with distance) is an empirical result from astronomical observations, and its validity is independent of whatever theory is used to explain it. We are therefore justified in our use of Hubble's law without reference to gravity. However, we shall sometimes borrow colloquial terminology from GR (e.g. "flat space") when it improves flow and creates no ambiguities.

To the best of our knowledge, this derivation of free particle states in expanding space is new. To make this paper accessible to the widest possible audience, we try to keep the mathematics and framework as simple as possible. This single-particle derivation is accessible to the student with only an undergraduate level understanding of QT[1] and avoids potential conflicts between QT and GR as those theories are known to be incompatible.

A unique feature of this derivation is how the stationary states are derived by enforcing consistency with observational evidence, without resorting to Hamiltonian or Lagrangian mechanics. As we shall see, this is a useful trick since we do not know how to model expanding space with an exact pseudo-potential energy function. Instead, the results of our theory permit derivation of an exact pseudopotential that, when applied to flat space, results in the same stationary states as for an expanding universe.

Outline of the Rest of the Paper

In Section 2, we apply the canonical substitutions to turn Hubble's law into a quantum operator equation, and then present its eigenfunctions. We follow a straightforward path to determination of the time part of the wavefunctions, their energies and momenta. For example, we find that two co-moving observers may measure different values for the energy and momentum for the same particle. We then show the correspondence between our stationary states and the stationary states in flat space, *i.e.* plane waves.

In part 3, we apply the boundary condition that the wavefunctions must vanish at the cosmic horizon. This boundary condition places a lower limit on the uncertainty of the particle rest mass, and implies the existence of a mass quantum, which is the smallest particle rest mass allowed by the theory. The mass quantum and its uncertainty point to a simple method to estimate the mass uncertainty δm_0 for an arbitrary rest mass m_0. We see how the mass uncertainty leads to a natural localization of the particle, which we identify with (a lower limit to) the classical localization of the particle. For macroscopic objects, localization radii are very small compared to the object size, and so their trajectories take on classical behavior. For convenience of reference we call this new theory, "localization theory".

[1]Besides new physics, we suggest that this derivation is a topic suitable for undergraduate or graduate courses in QT as an example of how to bring one of the major principles of cosmology into QT.

We derive an expression for the critical mass where we predict a crossover from quantum behavior to classical behavior, and it depends on material density. These results are in broad agreement with the observed universe.

In the discussion section, we show the close relationship between localization theory and the very successful theory of quantum decoherence as competing explanations for classical localization. We note that in the non-relativistic regime, our results agree with a particular version of decoherence theory called "spontaneous quantum collapse."

Finally, we consider how localization theory may impact gravitational attraction and measurements of the gravitational constant G in some limiting situations. Here and at various points in the paper we make predictions for the outcomes of a variety of real experiments that could support or conflict with the theory. The main part of the paper ends with some concluding remarks.

We move several of the less critical sections of the derivation into appendices. In Appendix 2, we list a set of criticisms of the derivation and provide some initial responses. We consider normalization, completeness and orthogonality of our wavefunctions. In Appendix 3 we outline the derivation of a pseudopotential that can be used in flat space calculations which has stationary states have the same form as those in expanding space.

Appendix 1 presents an alternative derivation for the value of the mass quantum that takes a very different approach than that in the main text. The coarse agreement between the two derivations lends further support to our conclusion that the classical localization arises from the finite size and finite age of our universe.

2. Quantization of Hubble's Law

Hubble's natural law, $\boldsymbol{v}(\boldsymbol{r}) = H_0 \boldsymbol{r}$, is an interpretation consistent with electromagnetic observations of distant galaxies. It states that distant bodies are receding from any observer with a radial velocity equal to the distance of separation multiplied by the Hubble constant H_0 (neglecting proper motion). Although H_0 is known to vary slowly with time, we approximate it with a constant $H_0 = 2.2 \times 10^{-18}$ Hz (except in Appendix 1). To describe the expansion of the universe, this paper uses only Hubble's natural law, and does not appeal to the theory of general relativity (GR).

To begin, we consider finite set of noninteracting classical test particles[2] (mass density~0, temperature = 0, pressure = 0) in a large empty void between galaxies.[3] The condition for a stationary particle configuration is evidently: If each test particle p has an inwardly directed velocity

$$\boldsymbol{v}_p = -H_0 \boldsymbol{r}_p, \tag{1}$$

[2]Test particles have such small mass that the gravitational influence of those particles can be neglected. For a cloud of test particles, we require that the cloud's density is small enough so that the gravitational effect of the particle swarm is also negligibly small.

[3]Although we do not refer to general relativity, our assumptions are quite similar to those for a Friedmann-Lemaître-Robertson-Walker Universe. We shall ignore any the gravitational effects caused by inhomogeneities in the rest of the mass in the universe.

then the Hubble expansion is cancelled exactly and the particle configuration is time independent. This condition for stationarity is key to the derivation that follows.

We quantize Equation (1) for a single particle (instead of a set) with arbitrary rest mass m_0, where it's probability density is spread out over a large region in the void. The vector $\boldsymbol{r}_p$ is now interpreted as the center of mass position for the wavefunction. We multiply through Equation (1) by the relativistic mass γm_0 (Lorentz factor $\gamma^{-1} \equiv \sqrt{1-v^2c^{-2}}$) and rearrange to obtain

$$0 = \boldsymbol{p}_r + \gamma H_0 m_0 \boldsymbol{r} . \tag{2}$$

where $\boldsymbol{r} \to (\boldsymbol{r} - \boldsymbol{r}_p)$.

We make the canonical substitution $\boldsymbol{p}_r = -i\hbar\nabla_r$, and multiply on the right by the particle wavefunction $\Psi(r,t)$ to obtain the quantum equivalent of Equation (2)

$$0 = \left[-i\hbar\nabla_r + \frac{\gamma m_0 H_0 r}{\hbar}\right]\Psi(r,t) = \left[-i\hbar\nabla_r + \frac{m_0 c}{\hbar}\frac{H_0 c^{-1} r}{\sqrt{1-\left(H_0 c^{-1} r\right)^2}}\right]\Psi(r,t) \tag{3}$$

In (3) we have put the particle center of mass at the origin ($\boldsymbol{r}_p = 0$) and replaced the velocity in γ using Equation (1). Equation (3) is the quantum version of Hubble's law, and it is consistent with special relativity.

2.1. Solution

Equation (3) is a first-order differential equation. It was constructed using the assumption that its solutions are stationary states, which implies that the wavefunction is separable and for some time-independent function $\omega(\boldsymbol{r})$

$$\Psi(r,t) = \mathrm{e}^{-i\omega(r)t}\psi(r). \tag{4}$$

Our path to Equation (4) is novel because at no point did we employ Hamiltonian mechanics. Direct substitution into Equation (3) verifies the solution

$$\psi(r) = A\mathrm{e}^{-i\phi}\exp\left[i\frac{m_0 c r_H}{\hbar}\left(1-\frac{r^2}{r_H^2}\right)^{1/2}\right] \tag{5}$$

where the Hubble distance $r_H \equiv H_0^{-1}c$ and the normalization constant $A = \sqrt{3r_H^{-3}}$. For convenience, we set the global phase offset $\phi = m_0 c r_H \hbar^{-1}$ so that the wavefunction is real at $r = 0$. To be a stationary state we must choose the positive value of the square root for $\left(1 - r^2/r_H^2\right)^{1/2}$. Put another way, there is no solution where the momentum points away from the origin. That there is no sign ambiguity arises directly from the fact that Equation (3) is a *first-order* differential equation.

The distance r_H is also called the "cosmic horizon," because classical particles beyond this distance are moving with speed greater than *c*, hence may have no interaction with a particle centered at the origin. The cosmic horizon is a dividing line between two regions of the wavefunction. Inside the horizon the wavefunction has the expected form of an oscillating exponential (**Figure 1(a)**, blue

solid). As $r \to r_H$, the particle momentum tends to infinity. This does not imply that the wavefunction undergoes an infinite number of phase cycles while approaching the horizon, as is evident in **Figure 1(a)**. Instead, the total number of phase cycles $N_{\text{cycles}} \equiv m_0 c r_H (2\pi\hbar)^{-1}$ executed between $0 \le r \le r_H$ is a finite number proportional to the rest mass. For a hydrogen atom,
$N_{\text{cycles,H atom}} = 1.0 \times 10^{41}$.

In **Figure 1(b)**, our analysis shows the surprising fact that a quantum particle can tunnel a short distance beyond the cosmic horizon with an evanescent decay. In the evanescent region the particle has imaginary velocity, while a classical particle in the same region would have a velocity greater than the speed of light.

In section 3 we deal with the momentum singularity at $r \to r_H$ by requiring the wavefunction vanishes at the cosmic horizon and in the evanescent region. However, under certain extreme initial conditions at the time of the big bang, the evanescent solution could be nonzero. For the sake of conciseness, we do not consider such cases here.

2.2. Energies of Stationary States

For an observer at any position $\boldsymbol{r}$, the stationary state of a particle centered at

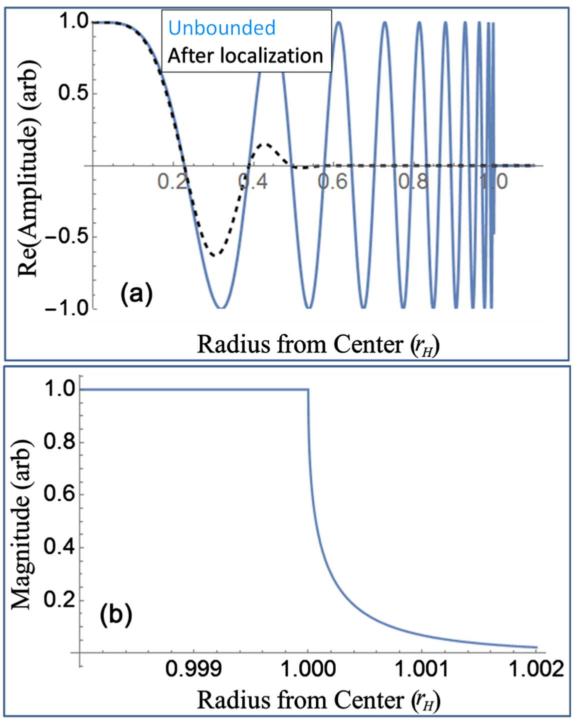

Figure 1. Example plots of the real part of the amplitude (a)-blue solid line and magnitude (b)-blue solid line of Equation (5). The horizontal axis is plotted in units of r_H. These plots assume a tiny particle mass with $(m_0 c r_H \hbar^{-1}) = 60$. The black dashed line in (a) shows the corresponding wavefunction after application of boundary conditions. The boundary conditions require the wavefunction to be localized at the origin, as discussed in Section 3.

$\boldsymbol{r}_p$ including time dependence is, from Equation (4)

$$\Psi(\boldsymbol{r},t) \equiv \psi(\boldsymbol{r}-\boldsymbol{r}_p)\mathrm{e}^{-i\omega(r-r_p)t}. \tag{6}$$

We now compute the value of ω that is consistent with our spatial wavefunctions. Einstein's equation from Special Relativity (SR) states

$$\hbar^2\omega^2(\boldsymbol{r}-\boldsymbol{r}_p) = m_0^2c^4 + \hbar^2k^2(\boldsymbol{r}-\boldsymbol{r}_p). \tag{7}$$

where the expression for the momentum $\hbar\boldsymbol{k}$ can be read directly from Equation (3)

$$k(\boldsymbol{r}-\boldsymbol{r}_p) = \frac{m_0c}{\hbar}\frac{(\boldsymbol{r}-\boldsymbol{r}_p)/r_H}{\sqrt{1-\left(|\boldsymbol{r}-\boldsymbol{r}_p|/r_H\right)^2}}. \tag{8}$$

It is important to notice that SR is a local theory, and Equation (7) applies only if the energy, momentum, and mass are measured at the same space-time position or event. In flat space, stationary states have the same momentum everywhere, and all observers measure the same particle energy. Here, the stationary states have a momentum ($\hbar\boldsymbol{k} = \hbar\boldsymbol{k}(\boldsymbol{r})$) that varies depending on the observer's relative position. The same is true for the energy. For example, consider the stationary state $\Psi(r,t)$ with both the particle and the observer at $r=0$. Here $k(r\to 0)=0$ and we recover the Compton frequency

$$\omega(r\to 0) = \frac{m_0c^2}{\hbar} \tag{9}$$

We consider a second observer at distance r' with zero velocity relative to the first.[4] the second observer measures a different energy:

$$\omega(r') = \frac{m_0c^2}{\hbar}\sqrt{1+\frac{\hbar}{m_0^2c^2}k(r')^2}. \tag{10}$$

This leads to an important result of this paper: though the states $\Psi(r<r_H,t)$ are stationary by construction hence have a *time-independent* energy, their apparent energy depends on observer position even for co-moving observers. A similar result is obtained in the classical case and the position dependence of energy derives from the locality of SR and not specifically from QT.

2.3. Comparison to Free Particle States in Flat Space

In flat space vacuum, the stationary states are plane waves

$$\psi_{PW}(\boldsymbol{r},t) \propto \exp[i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t]. \tag{11}$$

In this section we compare and draw analogies between these and our stationary states $\Psi(\boldsymbol{r},t)$ for expanding space. Firstly, both ψ_{PW} and Ψ oscillate periodically with time (*c.f.* Equation (4)), everywhere. In this section we approximate $\Psi(r,t)$ in a small region with extent $a \ll r_H$, and see that $\Psi(r,t)$

[4]The zero-velocity condition is defined by and can be checked by the two observers through the exchange of a laser beam with known frequency. Zero velocity implies the laser has the same photon energy for both observers.

reduces to the form of a plane wave.

Consider a cubical volume of space with side a and centered at the position $\boldsymbol{r}_0$. The distance a is assumed to be but small compared to r_H: $0 < a \ll r_H$. We define the offset coordinate $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0$ with $0 \le \delta r \le a$. Substituting into Equation (8), expanding in a Taylor's series about $\delta\boldsymbol{r} = 0$, and keeping only the lowest order term in $\delta\boldsymbol{r}$ we find

$$\boldsymbol{k}(\boldsymbol{r}_0) \cong \frac{m_0 c}{\hbar} \frac{\boldsymbol{r}_0 / r_H}{\sqrt{1 - (\boldsymbol{r}_0 / r_H)^2}}. \tag{12}$$

Equation (12) is *independent of observer position* within the cube and defines a 1:1 correspondence between the stationary states $\Psi(\boldsymbol{r}, t)$ in expanding space and $\psi_{PW}(\boldsymbol{r}, t)$ in flat space; the former converges to the latter as $H_0 \to 0$.

Equation (12) is helpful for applications where we wish to propagate a given initial state forward in time. Suppose at $t = 0$ we are given an arbitrary wavefunction $\phi(r, 0)$ with support only a local region of space. To propagate this wavefunction forward in time, we perform the following transformations at $t = 0$:

$$\begin{aligned} \phi(\boldsymbol{r}, 0) = \int_{-\infty}^{\infty} \tilde{\phi}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{k} &\xrightarrow{k_{PW} \to k(r_0)} \int_{-\infty}^{\infty} \tilde{\phi}(\boldsymbol{k}(\boldsymbol{r}_0)) \mathrm{e}^{i\boldsymbol{k}(\boldsymbol{r}_0)\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{k} \\ &\xrightarrow{\mathrm{e}^{ikr} \to \psi(r)} \int_{r_0=0}^{r_H} \tilde{\phi}(\boldsymbol{r}_0) \Psi(\boldsymbol{r} - \boldsymbol{r}_0, 0) \mathrm{d}\boldsymbol{r}_0 \end{aligned}. \tag{13}$$

The expansion on the right hand side can be propagated forward using the time dependence in Equation (6). Notice that this transformation takes an integral over momentum space and replaces it will an integral over real space. This is possible because there is a 1:1 correspondence between a specific momentum in flat space and the position of the center of mass in expanding space.

3. The Emergence of Spontaneous Localization

3.1. Quantization of Mass Uncertainty $\sigma_{\min}$

The crucial element to complete this theory is to apply the boundary condition that the wavefunction must go to zero at the cosmic horizon. This condition is satisfied in the usual way: by constructing linear combinations of the unbounded wavefunctions that have the desired property. For example, the wavefunctions for a particle in a box are constructed by combining pairs of oppositely directed travelling waves into standing waves, enforcing nodes at the box edges.

Similarly, we demand that $\Psi(\boldsymbol{r}, t)$ must go to zero at the cosmic horizon. If this were not the case, then an observer approaching the cosmic horizon (from the inside) would measure a divergent momentum and energy, $p, E \to \infty$.[5] The behavior of the unbounded states Equation (5) at the cosmic horizon ($r \to r_H$)

[5]This assumption is equivalent to enforcing a soft upper limit to particle kinetic energy. This is exactly what is observed for the most energetic particles (i.e. cosmic rays) in our universe. Although other processes such as scattering from cosmic microwave background photons place even more stringent limits on the maximum observed kinetic energy.

gives

$$\psi\left(r \to r_H\right) = A\mathrm{e}^{-i\phi} = A\exp\left[-\frac{m_0 c r_H}{\hbar}\right] \tag{14}$$

The only non-constant factor in the exponential is the rest mass. Thus, an integral over a range of rest masses achieves our boundary condition

$$\psi\left(r, m_0\right) \to \int_0^\infty a\left(m\right)\psi\left(r, m\right)\mathrm{d}m, \tag{15}$$

where $a\left(m\right)$ represents the particle probability distribution as a function of rest mass. Because the mean rest mass $\bar{m}_0$ of elementary particles are fixed, for such particles we infer that $a\left(m\right)$ has a strong narrow peak at $m = m_0$. For the sake of this discussion, the detailed functional form of $a\left(m\right)$ is important to the exact value of δm result within a factor of order 1, and this factor is not important for the main conclusions of this paper. As an example we choose $a\left(m\right)$ to be a top hat function centered at m_0 and having width δm:

$$a\left(m\right) = \frac{1}{\delta m} \quad \text{for } \left[\left(m_0 - \delta m/2\right) < m < \left(m_0 + \delta m/2\right)\right], \tag{16}$$

and zero otherwise. Putting (16) into (15) and setting the result equal to zero at the cosmic horizon we obtain a quantization condition on the distribution parameter δm

$$\delta m = \frac{2\pi\hbar}{c r_H} n, \quad n = 1, 2, 3, \cdots \tag{17}$$

which has a minimum value $\delta m_{\min}$ when $n = 1$. Later we will need the standard deviation of the distribution in (16), which is easily computed: $\sigma_{\min} \equiv \sqrt{1/12}\,\delta m_{\min}$.

3.2. The Mass Quantum $\bar{m}_H$ and Its Uncertainty

Because the universe has finite extent, Equation (17) shows that the mass distribution width is $\delta m_{\min}$. This does not necessarily imply that the actual rest mass is quantized, as postulated below. But it does imply that no mass may be smaller than a minimum $\bar{m}_H$, dependent on the Hubble parameter. If we try to create a particle with nominal zero mass, its mass is still uniformly distributed over the range $\delta m_{\min}$, and the mean value of the smallest mass is

$$\bar{m}_H \equiv \delta m_{\min}/2 = \frac{\pi\hbar H_0}{c^2} = 5.1 \times 10^{-68}\ \text{kg} \tag{18}$$

We shall call $\bar{m}_H$ the mass quantum and identify its "uncertainty" with the standard deviation $\sigma_H = \sqrt{1/12}\,\delta m_{\min} = 3^{-1/2}\,\bar{m}_H$. *This is perhaps the most important result of this paper.*

To further motivate the concept of a minimum mass, we turn to a semiclassical argument based on quantization of angular momentum. Consider a classical particle with relativistic mass $\gamma m_0 \equiv m_{SR}$ executing circular motion with radius $r_{\max}$. We set the angular momentum of this particle equal to the minimum, Planck's value: $m_{SR} v r_{\max} = h$. Given the maximum velocity is *c*, the minimum value of the mass is $m_{SR,\min} = h\left(cr_{\max}\right)^{-1}$. Letting $r_{\max} \to r_H$, we recover Equation (18) to within a factor of two. This argument reinforces the result of Equation (17). We see that the two ingredients for the existence of a mass quantum are 1) the quantization of angular momentum and 2) that the accessible universe has finite volume.

3.3. Uncertainty in Larger Masses

In the next step, we introduce a new theory postulate:

Postulate: every mass m_0 and its uncertainty can be modeled as the superposition of many mass quanta and their uncertainties.

We do not propose that particles are composed of many very small particles, only that mass is quantized. However, because the quanta are bosons, there is no conflict with multiple quanta residing in the same place at the same time.

We model any mass m_0 as a superposition of $m_0/\overline{m}_H$ individual quanta all with the same center of mass. What uncertainty is associated with m_0? To model uncertainty, we treat the mass of any individual quantum m_q as a random variable uniformly distributed in the range ($0 \le m_q \le \delta m_{\min}$) with mean value $\overline{m}_H$. The central limit theorem [13] then provides an estimate of the uncertainty in m_0 [6]

$$\sigma_{m_0} = \sqrt{N_{quanta}}\,\sigma_H = 3^{-1/2}\sqrt{m_0\,\overline{m}_H}\,. \tag{19}$$

When $N_{quanta} \gg 1$, the central limit theorem also gives us the shape of the probability distribution around the nominal mass m_0,

$$P_{m_0}\left(m\right) \propto \pi^{-1/2}\exp\left[-\left(m-m_0\right)^2/\sigma_{m_0}^2\right] \tag{20}$$

Equations (19) and (20) are another important result. They derive from the postulate that large masses can be treated as a superposition of mass quanta with known uncertainty.

3.4. Greater Localization for Larger Uncertainties

Because the value of σ_H is inversely proportional to the maximum allowed (Hubble) radius r_H, it may be intuited that masses with larger uncertainty will be confined to smaller radii, hence smaller volumes. Inverting Equation (17) and replacing $\delta m \to \sqrt{12}\sigma$, the radius of localization r_{loc} for a mass m_0 takes a particularly simple form[7]

[6]We note that for Equation (19) to be consistent with Equation (17), the number of quanta must be a perfect square, $N_{quanta} = n^2$, where n is an integer. This has no practical significance because ordinary masses are comprised of a very large number of quanta so a nearby perfect square value will always be well within the mass uncertainty.

[7]It is notable that Herzenberg (Herzenberg 2006, 2009) derived a very similar localization distance with the same functional form as presented here:

$$r_{loc,\text{Herzenberg}} \approx \sqrt{\frac{\pi\hbar}{2m_0 H_0}},$$

despite the fact that Herzenberg used a derivation quite distinct from ours. This "coincidence" may be explained, since the geometry of expanding space is essentially the same in both papers.

$$r_{loc}\left(m_0\right) \equiv \frac{2\pi\hbar}{\sqrt{12}\sigma_{m_0} c} = \sqrt{\frac{\pi\hbar}{m_0 H_0}} \tag{21}$$

As expected, the localization radius becomes infinite as $H_0 \to 0$, and it is inversely proportional to the square root of the rest mass. This means that the particle wavefunction is confined to a volume $\Omega\left(m_0\right)$ approximately equal to that of a sphere with radius r_{loc}: $\Omega\left(m_0\right) \cong \frac{4}{3}\pi r_{loc}^3$. Below, we identify $\Omega\left(m_0\right)$ with the classical localization volume of macroscopic particles, from raindrops to stars. An example of the wavefunction post localization is found in **Figure 1(a)**, indicated by a black dashed curve. To reveal the localization behavior in more detail, **Figure 2** plots the envelope function multiplying the wavefunction in **Figure 1** after the boundary condition has been met.

Statement (21) is quite general, and can be recovered exactly by reversing the derivation leading to Equation (17). *Equation* (21) *has no free parameters*, meaning that if it does not approximate observed reality, then the only option is to abandon it. There is no ambiguity. Comparisons of r_{loc} (column 5) and estimates of the classical radius (column 4) for representative masses are given in **Table 1**.

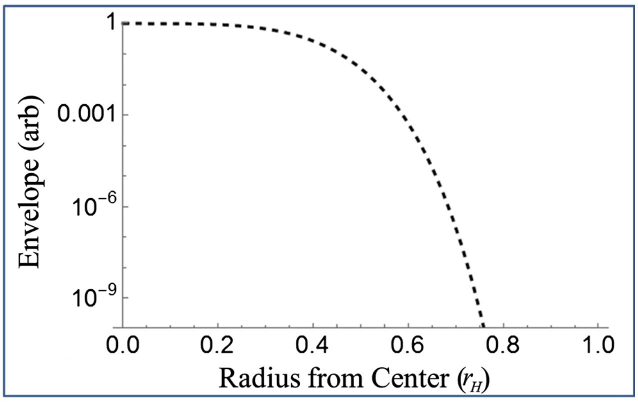

Figure 2. Plot of the enveloping function that multiplies the wavefunction in **Figure 1** after localization has been applied. The curve at small radius has a Gaussian-like shape.

Table 1. Estimates of the classical radius and localization parameter r_{Loc} for various (elementary or composite) bosons.

Particle type	Mass m_0 (kg)	Density (kg/m^3)	Classical Radius (meters)	r_{Loc} in steady-state universe $H_0 = 2.2\times10^{-18}$ s (meters)
Mass quantum $\bar{m}_H \equiv \frac{\pi\hbar H_0}{c^2}$	5.1×10^{-68}	-	-	1.4×10^{26}
"Dark matter particle" see text	3×10^{-58}	-	-	(200,000 LY) 1.9×10^{21}
Hydrogen atom	1.67×10^{-27}	-	-	750,000
Higgs boson	2.2×10^{-25}	-	-	65,000
Mass for $r_{loc} = 1$ cm (Lead)	9.5×10^{-12}	11,340	5.8×10^{-7}	0.01
1 kg water	1	1000	0.062	3.08×10^{-8}
Total mass of universe	1.99×10^{30}	3×10^{-28}	7.0×10^{8}	2.2×10^{-23}

Assuming a spherical shape, we can invert Equation (21) to express the critical mass for crossover from quantum to classical behavior as a function of the material density ρ

$$m_{0,\text{crossover}} = (16/9)^{1/5}\,\pi\rho^{2/5}\hbar^{3/5}H_0^{-3/5} \tag{22}$$

The values of critical mass for a variety of materials are shown in Table 2.

4. Discussion and Conclusions

Ask a physicist about the size of an atom and she would probably describe the boundaries of the diffuse electron cloud surrounding the nucleus. But this is only half of the story. From another perspective, the atom is a point-like composite particle whose center of mass is described by another, different wavefunction. To obtain a complete description of the spatial distribution of just one of the bound electrons, we must convolve the atom's center of mass wavefunction with the internal wavefunction of the electron about the nucleus. This paper focuses on the often-neglected center of mass wavefunction, specifically for a point-like composite boson with arbitrary mass.

This theory offers one explanation for why large objects appear to be perpetually localized in space. At this point we take a step back and consider the successes of localization theory.

We have quantized Hubble's law and computed the stationary state wavefunctions in a steady-state universe that obeys Hubble's law. We characterized those states for their position-dependent momentum and energy. We show how the limits of a finite universe give rise to a fundamental uncertainty in particle masses. We make quantitative predictions about spontaneous localization of massive particles which we associate with classical localization. We perform these tasks with a pure quantum theory that does not rely on general relativity. Localization theory is purely based on standard quantum mechanics and has no free parameters.

In further support of our theory that mass is quantized, we offer a second derivation of the mass quantum based on the Heisenberg uncertainty principle (Appendix 1).

Table 2. Estimates of the critical mass and critical radius for various materials. The critical mass is crossover point from quantum behavior (for smaller mass) and classical behavior (for larger mass).

Material	Mass m_0 (kg)	Density (kg/m^3)	Classical Radius (meters)	r_{loc} in steady-state universe $H_0 = 2.2\times10^{-18}$ (meters)
Critical mass for average density of universe	1.7×10^{-20}	3×10^{-28}	350	350
Critical mass for Air	1.86×10^{-9}	1.225	1.04×10^{-3}	1.04×10^{-3}
Critical mass for a Styrofoam ball	8.20×10^{-9}	50	5×10^{-4}	5×10^{-4}
Critical mass for interstellar dust grain (amorphous SiO_2)	3.59×10^{-8}	2000	2.4×10^{-4}	2.4×10^{-4}
Critical mass for neutron star density	0.01	10^{17}	4×10^{-7}	4×10^{-7}

As an aside, we note the existence of a pseudopotential in flat space that reproduces the same stationary states as described here (outline of derivation presented in Appendix 3).

4.1. Consistency with Decoherence Theories of Spontaneous Quantum Collapse

The physical problem of classical localization in a quantum world has long been recognized as demonstrating an incompleteness of quantum theory. Most physicists would agree that the best current theory for classical localization comes from decoherence theory. Decoherence has been described as resulting from the inter-connectedness of all matter in the universe through the fundamental forces. No particle can ever be considered completely isolated, as illustrated by Zeh in [14] p. 27. "*Borel* (*Borel* 1914, *pp.* 27-35) *showed long ago that even the gravitational effect resulting from shifting a small piece of rock as distant as Sirius by a few centimeters would completely change the microscopic state of a gas in a vessel here on earth within seconds after the retarded field of force had arrived*."

Connectedness means that the Hamiltonian of any subsystem expressed in terms of subsystem coordinates (q, p), also depends on the actual state of the entire universe or environment. That is, $H_{\text{subsystem}} \to H_{\text{subsystem}}\left(q, p; q_{env}(t), p_{env}(t)\right)$, after Joos in [14] p. 35. It is as if the system under study is constantly perturbed by collisions with a large number of very small virtual particles such as gravitons [15], generated by the finite temperature of the quantum vacuum state.

Some theories of spontaneous wavefunction collapse [16] address connectedness by adding a space- and time-dependent random factor to the Hamiltonian. We argue that the model presented here is consistent with such models, albeit with a different interpretation. We begin with a generalized Hamiltonian including spontaneous collapse

$$H_{\text{General}}\left(q_i, p_i, t\right) \approx H_{\text{subsystem}}\left(q_{sub}, p_{sub}, t\right) + T_{env}\left(q_{env}, p_{env}, t\right) \tag{23}$$

where T_{env} represents random perturbations in space and time. Averaging over many position and time realizations, we may replace T_{env} with its expectation value $\left\langle T_{env}(q, p, t)\right\rangle$. Assuming these perturbations δE_T are small and linear, it is reasonable to assume their probability distribution [14] [17]

$$P\left(\delta E_T\right) \approx \exp\left[-t^2/\sigma_t\right] \times \prod_i \exp\left[-q_i^2/\sigma_{q_i}^2\right] \propto \pi^{-1/2} \exp\left[-\delta E_T^2/\sigma_{E_T}^2\right] \tag{24}$$

In our localization theory, we find that the particle rest mass has an intrinsic uncertainty which can be described with additive term describing the distribution of possible rest masses (20). In the non-relativistic case these distributions have the same form as Equation (24) (referencing Equation (20)):

$$P\left(\delta E_T\right) \propto P_{m_0}\left(mc^2\right) \tag{25}$$

and the distributions remain similar even at relativistic energies. *We conclude that in* (*some*) *decoherence theories of spontaneous collapse*, *the full Hamilto-*

nian including system and environment will induce particle localization essentially indistinguishable from that of localization theory presented here.

While spontaneous collapse theory and the present theory can give rise to almost indistinguishable results, they derive from very different physics. The present theory has no free parameters while decoherence theory is adjusted to fit reality by careful choice of the free parameter σ_{E_T} (Equation (24)). Decoherence theory does not yet have a model to predict σ_{E_T} from first principles.

The present theory assumes a steady-state universe and does not address the very real mystery of wavefunction collapse. Conversely, spontaneous collapse theories presume that wavefunction collapse is the chief ingredient for particle localization.

Although the interpretation is different, we emphasize that our theory in no way invalidates the results of decoherence theory; the two theories can be made to be consistent. With the acceptance of our localization theory, the vast literature and most if not all the deductions of decoherence theory remain viable.

4.2. Blurring of Gravity

In a more speculative vein, we consider the potential effects of localization theory on gravitational interactions. Some theories of quantum gravity begin by assuming that a mass whose wavefunction is dispersed in space can be treated by classical general relativity with a continuous mass spatial distribution proportional to the probability density of the wavefunction. Not all theories include this assumption, but it seems like a reasonable place to start.

For small particles, localization theory also assumes a dispersed center of mass wavefunction with the localization radius of Equation (21). Consulting **Table 1** for a solitary hydrogen atom isolated in vacuum, we predict its center of mass wavefunction is dispersed over a region ~10^6 meters in diameter. If this estimate is correct, then we expect the gravitational force between two hydrogen atoms to be essentially nil at microscopic distances. Hence the question, "what is the gravitational contribution to the binding energy of an isolated H_2 molecule in vacuum?" is answered by "none at all." The gravity from a hydrogen atom can be approximated as that of a point particle only at distances greater than the radius of localization.

Indeed, suppose a scientist wishes to repeat Cavendish's measurement of the gravitational constant G using a tiny torsion balance and very small lead spheres, of order 1 micron in diameter. In **Table 1** we see that the sphere's radius of localization on the order of 1 centimeter. Thus, in very sensitive measurements where the spheres are separated by distances less than 1 cm, the apparent value of G will be substantially smaller than its accepted value. Thus, localization theory makes a quantitative prediction of reduced gravitational attraction for very small separations which can, in principle, be compared with laboratory observations.

4.3. Final Remarks

This paper finds that classical localization of large masses is a direct result of the expansion of space and is inversely related to the Hubble constant. This means that in a flat universe ($H_0 = 0$) classical localization does not occur, in conflict with the predictions of decoherence theory. Localization theory and decoherence theory may both be true at once, but contemplation of this difference may lead to experiments that distinguish the two theories.

The original impetus for our development of localization theory was the intuition that Hubble expansion must have some detectable effect on the interaction between microscopic particles. We were further motivated by the intuitive conviction that quantum theory (QT) applies even to large "particles" like planets, and there should be a detectable effects of ordinary QT even over cosmological distances.

In a lighthearted closing, we note that localization theory may have something to say about dark matter. Observations show that dark matter tends to be confined on galactic scales, say 200,000 light years. If this confinement were explained by localization theory, it is possible to estimate the mass of a typical "dark matter particle," and we do this in Table 1. The mass of such a small particle is on the order of 10^{-58} kg.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix 1: Alternative Derivation/Extension of the Steady-State Theory to a Dynamic Universe

A1. Mass Uncertainty Based on Duration of Measurement

In very precise mass measurements, the energy-time uncertainty principle plays an important role. We consider a thought experiment where a scientist with a perfect scale wishes to weigh (find mass of) a single muon[8]. After placing the muon and releasing the scale, she must wait a finite time for the scale to settle, and the accuracy of her measurement improves with increasing wait time. Even a perfect scale can never display the exact particle mass in a finite amount of time, because of Heisenberg's uncertainty principle for time and energy, usually expressed as $\Delta E \Delta t \ge \hbar/2$. Since rest energy is proportional to mass, we may write

$$\sigma_{\min} \Delta t \ge \frac{\hbar}{2c^2} \approx 5.87 \times 10^{-52}\ \mathrm{kg \cdot s} \tag{26}$$

Equation (25) is relevant to our thought experiment because the muon is unstable and spontaneously decays into lighter particles with an average lifetime of 2.2 μs. As she pursues higher accuracy, Equation (25) and muon decay places a fundamental limit on the uncertainty in her measurement; in this case $\sigma_\mu m_\mu^{-1} \approx 1.4 \times 10^{-18}$. Even idealized mass measurements of single muons will show statistical fluctuations about the nominal muon mass with an uncertainty of σ_μ. A simplistic interpretation of the uncertainty is that muons come in a random distribution of different mass values centered on the mean value.

This argument applies at all mass scales, from elementary particles to planets. Because no particle has ever existed longer than the age of the universe (T = thirteen point eight billion years), Equation (25) predicts that no particle mass may have an uncertainty smaller than $\sigma_{\min} = 1.3 \times 10^{-69}$ kg. This is a lower limit for the mass indeterminacy since Equation (25) is an inequality.

In the theory of the main text, we computed the minimum mass uncertainty to be $\sigma_H = 2.94 \times 10^{-68}$ kg. In other words, the uncertainty σ_H is indeed larger than the lower limit set by the uncertainty principle, by a factor of 22. We consider this level of agreement to be a success, and the result from the uncertainty principle supports the result from localization theory.

A2. Model of a Dynamic Universe

A still better estimate of $\sigma_{\min}$ is derived for the dynamic universe in **Figure 3**. We model the dynamic universe as a homogeneous spherical region bounded on all sides by the comic horizon. This horizon is expanding at the speed of light and within its bounds the Hubble parameter is time dependent $H_0 \to H(t = T_0)$. The origin of the time axis is taken to be the time of the big bang and T_0 refers to the current epoch.

In the real universe, the value of the mass quantum might depend on the un-

[8]The fact that the muon is a Fermion has no relevance to this part of the discussion. The accepted muon mass is 1.88×10^{-18} kg.

iverse' peculiar expansion history. However, we model the passage of time as a Markov chain of events, where the observed properties at time t depend only on the properties at the time t' immediately preceding t. Since the separation between t and t' is infinitesimal, our model assumes that the current states of matter can be predicted using only current physics and current values of physical constants and their time derivatives. The Markov assumption is justified by its common usage in most domains of physics including pure quantum field theory and all classical physics.

Our perspective is that the dynamic properties of the universe can be modeled *as if* it was a linear extrapolation of current dynamics. Figure 3 makes a schematic comparison of our models of the steady-state and dynamic universe. In our dynamic universe, the Hubble parameter is equal to the inverse of the effective age of the universe; $H(T_0+\Delta t)=(T_0+\Delta t)^{-1}$.

The time derivative of the Hubble parameter is only one part in 10^{-18} per second. We assume that this may be neglected over short periods of time, and the wavefunctions describing matter in the dynamic universe can be well approximated by the wavefunctions of the steady-state universe (Equations (4) and (5)), except with a time-dependent Hubble parameter. This permits us to carry over most of the results of the steady state theory.

Using $H_0=T_0^{-1}$ we may rewrite Equation (26)

$$\sigma_{\min} \geq \frac{\hbar H_0}{2c^2} \tag{27}$$

And with the replacement $\sigma_{\min} \to \sigma_H$, we write down the equation for the localization radius, $r_{loc,dynamic}$ corresponding to Equation (21)

$$r_{loc,dynamic}(m_0,T_0) \leq \sqrt{\frac{4\pi^2\hbar}{m_0 H_0}}. \tag{28}$$

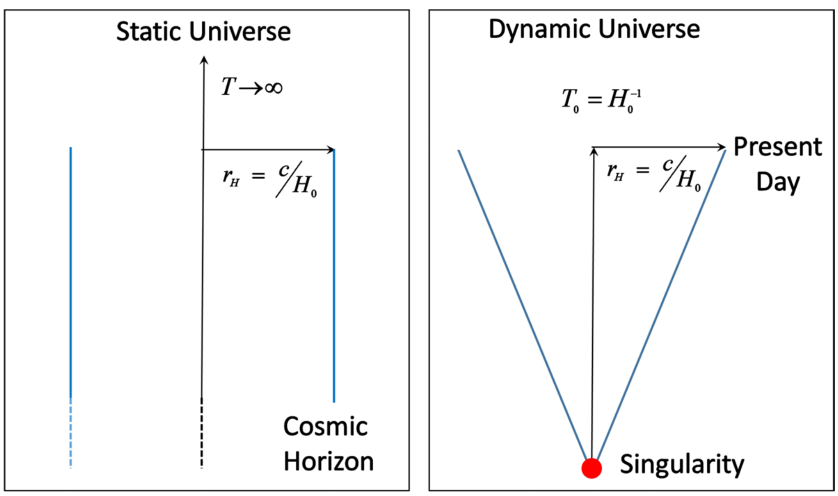

Figure 3. Schematic comparison of the two models of the universe discussed in this paper. On the left we have a steady-state universe where the cosmic horizon does not move with time. On the right, the universe comes into existence as a singularity and the cosmic horizon moves outward from that time at the speed of light.

The right hand side of Equation (27) is larger than the steady state localization radius by a factor of $\sqrt{4\pi}$. Since Equation (27) is an *inequality*, the results for the dynamic universe are entirely consistent with those of the steady-state universe. Therefore, Equation (27) provides further support for localization theory as derived in the main paper.

Appendix 2: Criticisms of the Derivation and Initial Responses

Single particle theory

This theory describes only isolated particles and the concept of multi-particle interactions is not addressed. A quantum field version of the theory may solve this problem. For now, it is at least plausible to approximate particle-particle gravitational interactions assuming the particles are localized to begin with (see 4.2).

Ambiguity in value of δm

We remind the reader that in Equation (17) the width of the mass distribution δm that satisfies the boundary condition is multi-valued, depending on the quantum number *n*. To estimate the minimum permitted mass, it is reasonable to choose the lowest quantum number $n = 1$. But other values are permissible for δm and we ignore those solutions here. We note that with increasing quantum number, the value of δm becomes more and more dependent on the exact choice of mass distribution function $a(m)$, and such ambiguities are minimized when $n = 1$.

Ambiguity between Hubble constant and age of universe

In our primary derivation, the universe is ageless, and localization derives from a nonzero Hubble constant. In our secondary derivation (Appendix 1), localization derives from the finite age of the universe. It is no coincidence that our dynamic universe' age is equal to H_0^{-1}, but even so there is an ambiguity about the origins of mass quantization for the two models. In the steady-state model, mass uncertainty is caused by spatial localization of particles within cosmic horizon. In the dynamic theory, mass uncertainty comes from time localization inferred from the singularity of the big bang. A more complete theory will combine both effects.

Choice of basis set for wavefunction expansion

We have chosen to express generic particle states with a Huygens-like basis set of spherically symmetric functions $\Psi(\boldsymbol{r}_{cm})$ with variable center of mass positions. We choose this basis for convenience of comparison with the $H_0 = 0$ plane wave states. However, direct integration shows that two functions $\Psi(\boldsymbol{r}_1)$ and $\Psi(\boldsymbol{r}_2)$ centered at different positions $\boldsymbol{r}_1 \neq \boldsymbol{r}_2$ are only approximately orthogonal.

A more natural basis would be a set of functions all centered at the same point and distinguished by three quantum numbers: $\Psi(n,l,m) = R_n(r) Y_l^m(\vartheta,\phi)$. The $Y_l^m(\vartheta,\phi)$ are spherical harmonics and the radial functions $R_n(r)$ are determined by boundary conditions. However correct this basis set may be, it is not so useful as a tool for understanding the stationary states. We leave the more exact derivation of a basis containing spherical harmonics to future work.

Use of Central Limit Theorem

The central limit theorem of statistics was derived for classical particles. Hence its application in Equation (19) relating to quantum particles may be dubious. We justify its use here based on an intuitive sense that it predicts the right behavioral trend, and in the end it provides a useful result that is consistent and explanatory for our universe. If nothing else, the comparison of theoretical predictions and experiments will either support or discredit our choice.

Wavefunction outside the cosmic horizon

Localization theory admits the possibility of a particle being detected at a point beyond the cosmic horizon r_H. Yet we focus only on particle states confined within the cosmic horizon and negligible amplitude outside. The form of the wavefunction beyond r_H derived from Equation (3) is given numerically by $A\mathrm{e}^{-3.67\times 10^{102} m_0 \sqrt{2\delta+\delta^2}}$ where δ is the radial distance beyond r_H. Even for the hydrogen atom, the e^{-1} decay distance is $4.7\times 10^{-112}/m_0^2 = 1.7\times 10^{-58}$ m, which can be compared to the localization radius of the total mass of the universe, which is much larger 2.2×10^{-23} m.

This provides some justification for neglecting the wavefunction outside, but the real particle distribution is entirely dependent on the initial conditions. These initial conditions were set at the moment of the big bang hence we know nothing about them. Thus, a fraction of the particles in the universe may lie outside (have support only outside) their own cosmic horizons. We intend to investigate this situation in a future paper, but we emphasize one key point. At the cosmic horizon, the wavefunction energy is infinite, and this acts as an impenetrable barrier for particles caught outside the horizon at $t = 0$. No physical interaction can push a particle from outside to inside, or *vice versa*. Thus, outside particles comprise a different state of matter as compared with inside particles. We conjecture that the (strong force, weak force) interactions between outside particles and inside particles are highly constrained. Only electronic and gravitational interactions would not suffer this restriction.

Appendix 3: Modeling $H_0 > 0$ with an Effective Potential

With an expression for $\Psi(r,t)$ (Equations (4) and (5)), it is straightforward to derive an effective potential $V(r,t)$ for a particle in flat space that gives the same solution as $\Psi(r,t)$ when $H_0 > 0$. In flat space, the free states of a spin-zero boson are solutions of the Klein Gordon equation, to which we add a fictitious potential

$$\left[\left(i\hbar\frac{\partial}{\partial t} - V(r,t)\right)^2 + \hbar^2 c^2\left(r^{-2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) - m_0^2 c^4 = 0\right]\Psi(r,t). \tag{29}$$

Plugging the desired solution $\Psi(r,t)$ into Equation (29) gives a second-order differential equation containing only derivatives of $V(r,t)$, and where most of the terms are algebraic. This shows another advantage of our novel deduction of stationary states in expanding space: with an expression for the wavefunction, the derivation of effective potential is straightforward.