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A New Analytical Solution for the Distance Modulus in Flat Cosmology

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Abstract

A new analytical solution for the luminosity distance in flat ΛCDM cosmology is derived in terms of elliptical integrals of first kind with real argument. The consequent derivation of the distance modulus allows evaluating the Hubble constant, $H_0 = 69.77 \pm 0.33$, $\Omega_M = 0.295 \pm 0.008$ and the cosmological constant, $\Lambda = (1.194 \pm 0.017) \times 10^{-52} \frac{1}{m^2}$.

Keywords

Galaxy Groups, Clusters, and Superclusters, Large Scale Structure of the Universe, Cosmology

1. Introduction

The release of two catalogs for the distance modulus of Supernova (SN) of type Ia, namely, the Union 2.1 compilation, see [1], and the joint light-curve analysis (JLA), see [2], allows matching the observed distance modulus with the theoretical distance modulus of various cosmologies. In this fitting procedure, the cosmological parameters are derived in a scientific and reproducible way.

We now focus our attention on the flat Friedmann-Lemaître-Robertson-Walker (flat-FLRW) cosmology. A first fitting formula has been derived by [3] and an approximate solution in terms of Padé approximant has been introduced by [4]. The presence of the elliptical integrals of the first kind in the integral for the luminosity distance in flat-FLRW cosmology has been noted by [5] [6] [7]. As a practical example the luminosity distance can be expanded into a series of orthonormal functions and the two cosmological parameters turn out to be $H_0 = 70.43 \pm 0.33$ and $\Omega_M = 0.297 \pm 0.002$, see [8]. This paper first introduces in Section 2 a framework useful to build a new solution for the luminosity dis-

tance in flat-FLRW cosmology, which will be derived in Section 3.

2. Preliminaries

This section reviews the adopted statistical framework, the ΛCDM cosmology, and an existing solution for the luminosity distance in flat-FLRW cosmology.

2.1. The Adopted Statistics

In the case of the distance modulus, the merit function χ^2 is

$$\chi^2 = \sum_{i=1}^{N} \left[\frac{(m-M)_i - (m-M)(z_i)_{th}}{\sigma_i} \right]^2, \tag{1}$$

where N is the number of SNs, $(m-M)_i$ is the observed distance modulus evaluated at redshift z_i, σ_i is the error in the observed distance modulus evaluated at z_i, and $(m-M)(z_i)_{th}$ is the theoretical distance modulus evaluated at z_i, see formula (15.5.5) in [9]. The reduced merit function χ^2_{red} is

$$\chi^2_{red} = \chi^2 / NF, \tag{2}$$

where $NF = N - k$ is the number of degrees of freedom, N is the number of SNs, and k is the number of parameters. Another useful statistical parameter is the associated Q-value, which has to be understood as the maximum probability of obtaining a better fitting, see formula (15.2.12) in [9]:

$$Q = 1 - \mathrm{GAMMQ}\left(\frac{N-k}{2}, \frac{\chi^2}{2} \right), \tag{3}$$

where GAMMQ is a subroutine for the incomplete gamma function.

The goodness of the approximation in evaluating a physical variable p is evaluated by the percentage error δ

$$\delta = \frac{\left| p - p_{approx} \right|}{p} \times 100, \tag{4}$$

where p_{approx} is an approximation of p.

2.2. The Standard Cosmology

We follow [10], where the *Hubble distance* D_{H} is defined as

$$D_{\mathrm{H}} \equiv \frac{c}{H_0}. \tag{5}$$

The first parameter is Ω_{M}

$$\Omega_{\mathrm{M}} = \frac{8\pi G \rho_0}{3H_0^2}, \tag{6}$$

where G is the Newtonian gravitational constant and ρ_0 is the mass density at the present time. The second parameter is Ω_{Λ}

$$\Omega_{\Lambda} \equiv \frac{\Lambda c^2}{3H_0^2}, \tag{7}$$

where Λ is the cosmological constant, see [11]. These two parameters are connected with the curvature Ω_K by

$$\Omega_{\mathrm{M}} + \Omega_{\Lambda} + \Omega_K = 1. \tag{8}$$

The comoving distance, D_{C}, is

$$D_{\mathrm{C}} = D_{\mathrm{H}} \int_0^z \frac{\mathrm{d}z'}{E(z')} \tag{9}$$

where $E(z)$ is the "Hubble function"

$$E(z) = \sqrt{\Omega_{\mathrm{M}} (1+z)^3 + \Omega_K (1+z)^2 + \Omega_{\Lambda}}. \tag{10}$$

The above integral does not have an analytical solution but a solution in terms of Padé approximant has been found, see [12].

2.3. A First Formula for a Flat-FLRW Universe

The first model starts from Equation (2.1) in [4] for the luminosity distance, d_{L},

$$d_{\mathrm{L}}(z; c, H_0, \Omega_{\mathrm{M}}) = \frac{c}{H_0}(1+z) \int_{\frac{1}{1+z}}^1 \frac{\mathrm{d}a}{\sqrt{\Omega_{\mathrm{M}} a + (1-\Omega_{\mathrm{M}}) a^4}}, \tag{11}$$

where H_0 is the Hubble constant expressed in km·s^{-1}·Mpc^{-1}, c is the speed of light expressed in km·s^{-1}, z is the redshift and a is the scale-factor. The indefinite integral, $\Phi(a)$, is

$$\Phi(a, \Omega_{\mathrm{M}}) = \int \frac{\mathrm{d}a}{\sqrt{\Omega_{\mathrm{M}} a + (1-\Omega_{\mathrm{M}}) a^4}}. \tag{12}$$

The solution is in terms of F, the Legendre integral or incomplete elliptic integral of the first kind, and is given in [13].

The luminosity distance is

$$d_{\mathrm{L}}(z; c, H_0, \Omega_{\mathrm{M}}) = \Re\left(\frac{c}{H_0}(1+z) \left(\Phi(1) - \Phi\left(\frac{1}{1+z} \right) \right) \right), \tag{13}$$

where $\Re$ means the real part. The distance modulus is

$$(m-M) = 25 + 5\log_{10}\left(d_{\mathrm{L}}(z; c, H_0, \Omega_{\mathrm{M}}) \right). \tag{14}$$

3. A New Formula for a Flat-FLRW Universe

The second model for the flat cosmology starts from Equation (1) for the luminosity distance in [14]

$$d_{\mathrm{L}}(z; c, H_0, \Omega_{\mathrm{M}}) = \frac{c(1+z)}{H_0} \int_0^z \frac{1}{\sqrt{\Omega_{\mathrm{M}} (1+t)^3 + 1 - \Omega_{\mathrm{M}}}} \mathrm{d}t. \tag{15}$$

The above formula can be obtained from formula (9) for the comoving distance inserting $\Omega_K = 0$ and the variable of integration, t, denotes the redshift.

A first change in the parameter Ω_{M} introduces

$$s = \sqrt[3]{\frac{1-\Omega_{\mathrm{M}}}{\Omega_{\mathrm{M}}}} \tag{16}$$

and the luminosity distance becomes

$$d_{\mathrm{L}}\left(z;c,H_0,s\right)=\frac{1}{H_0}c\left(1+z\right)\int_0^z\frac{1}{\sqrt{\frac{\left(1+t\right)^3}{s^3+1}+1-\left(s^3+1\right)^{-1}}}\mathrm{d}t. \tag{17}$$

The following change of variable, $t=\frac{s-u}{u}$, is performed for the luminosity distance which becomes

$$d_{\mathrm{L}}\left(z;c,H_0,s\right)=-\frac{c}{H_0s^2}\left(1+z\right)\left(s^3+1\right)\int_s^{\frac{s}{1+z}}\frac{u}{u^3+1}\sqrt{\frac{s^3\left(u^3+1\right)}{u^3\left(s^3+1\right)}}\mathrm{d}u. \tag{18}$$

Up to now we have followed [14] which continues introducing a new function $T\left(x\right)$; conversely we work directly on the resulting integral for the luminosity distance: which is

$$\begin{aligned}&d_{\mathrm{L}}\left(z;c,H_0,s\right)\\&=-1/3\frac{c\left(1+z\right)3^{3/4}\sqrt{s^3+1}}{\sqrt{s}H_0}\times\left(F\left(2\frac{\sqrt{s\left(s+1+z\right)}\sqrt[4]{3}}{s\sqrt{3}+s+z+1},1/4\sqrt{2}\sqrt{3}+1/4\sqrt{2}\right)\right.\\&\left.-F\left(2\frac{\sqrt[4]{3}\sqrt{s\left(s+1\right)}}{s+1+s\sqrt{3}},1/4\sqrt{2}\sqrt{3}+1/4\sqrt{2}\right)\right),\end{aligned} \tag{19}$$

where s is given by Equation (16) and $F\left(\phi,k\right)$ is Legendre's incomplete elliptic integral of the first kind,

$$F\left(\phi,k\right)=\int_0^{\sin\phi}\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \tag{20}$$

see [15]. The distance modulus is

$$\left(m-M\right)=25+5\log_{10}\left(d_{\mathrm{L}}\left(z;c,H_0,s\right)\right), \tag{21}$$

and therefore

$$\left(m-M\right)=25+5\frac{1}{\ln\left(10\right)}\ln\left(-\frac{1}{3}\frac{c\left(1+z\right)3^{3/4}\left(F_1-F_2\right)\sqrt{s^3+1}}{\sqrt{s}H_0}\right), \tag{22}$$

where

$$F_1=F\left(2\frac{\sqrt{s\left(s+1+z\right)}\sqrt[4]{3}}{s\sqrt{3}+s+z+1},1/4\sqrt{2}\sqrt{3}+1/4\sqrt{2}\right) \tag{23}$$

and

$$F_2=F\left(2\frac{\sqrt[4]{3}\sqrt{s\left(s+1\right)}}{s+1+s\sqrt{3}},1/4\sqrt{2}\sqrt{3}+1/4\sqrt{2}\right), \tag{24}$$

with s as defined by Equation (16).

Data Analysis

In recent years, the extraction of the cosmological parameters from the distance

modulus of SNs has become a common practice, see among others [8] [16] [17]. The best fit to the distance modulus of SNs is here obtained by implementing the Levenberg-Marquardt method (subroutine MRQMIN in [9]). This method requires the fitting function, in our case Equation (22), as well the first derivative $\frac{\partial(m-M)}{\partial H_0}$, which has a simple expression, and the first derivative $\frac{\partial(m-M)}{\partial \Omega_M}$, which has a complicated expression. A simplification can be introduced by imposing a fiducial value for the Hubble constant, namely $H_0 = 70\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$, see [2] [18]. We call this model "flat-FLRW-1", where the "1" stands for there being one parameter. Table 1 presents H_0 and Ω_M for the Union 2.1 compilation of SNs and Figure 1 displays the best fit. The reading of this table allows to evaluate the goodness of the approximation, see (4), in the derivation of the Hubble constant in going from the supposed true value ($H_0 = 70\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$) to the deduced value ($H_0 = 69.77\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$), which is $\delta = 99.67\%$. The

Table 1. Numerical values from the Union 2.1 compilation of χ^2, χ^2_{red} and Q, where k stands for the number of parameters.

Cosmology	SNs	k	parameters	χ^2	χ^2_{red}	Q
flat-FLRW	580	2	$H_0 = 69.77 \pm 0.33$; $\Omega_M = 0.295 \pm 0.008$	562.55	0.9732	0.66
flat-FLRW-1	580	1	$H_0 = 70$; $\Omega_M = 0.295 \pm 0.008$	563.52	0.9732	0.669
ΛCDM	580	3	$H_0 = 69.81$; $\Omega_M = 0.239$; $\Omega_\Lambda = 0.651$	562.61	0.975	0.658

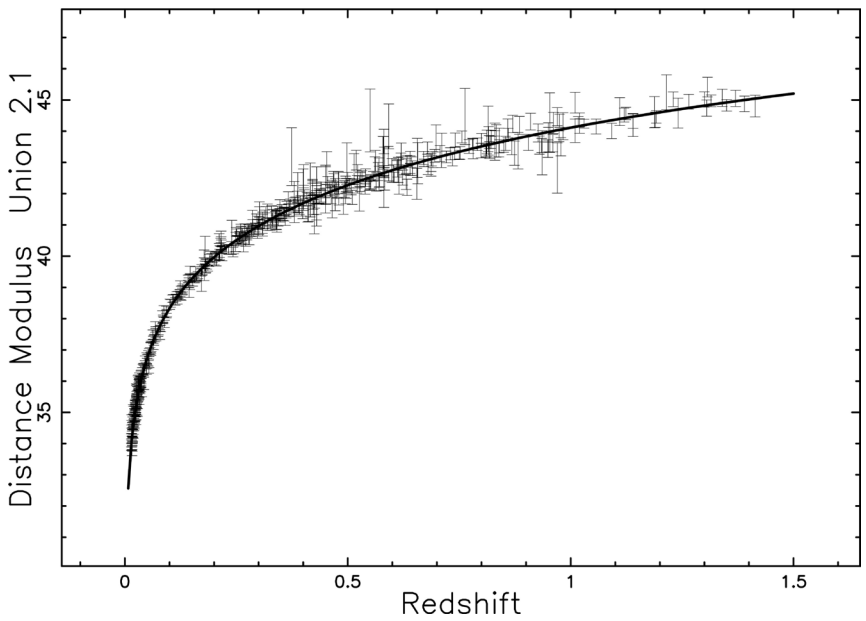

Figure 1. Hubble diagram for the Union 2.1 compilation. The solid line represents the best fit for the exact distance modulus in flat-FLRW cosmology as represented by Equation (22). Parameters as in first line of Table 1; Union 2.1 compilation.

JLA compilation is available at the Strasbourg Astronomical Data Centre (CDS), and consists of 740 type I-a SNs for which we have the heliocentric redshift, z, the apparent magnitude $m_B^\star$ in the B band, the error in $m_B^\star$, $\sigma_{m_B^\star}$, the parameter $X1$, the error in $X1$, σ_{X1}, the parameter C, the error in C, σ_C, and $\log_{10}(M_{stellar})$. The observed distance modulus is defined by Equation (4) in [2]:

$$m - M = -C\beta + X1\alpha - M_b + m_B^\star. \tag{25}$$

The adopted parameters are $\alpha = 0.141$, $\beta = 3.101$ and

$$M_b = \begin{cases} -19.05 & \text{if } M_{stellar} < 10^{10} M_\odot \\ -19.12 & \text{if } M_{stellar} \geq 10^{10} M_\odot \end{cases} \tag{26}$$

see line 1 in Table 10 of [2]. The uncertainty in the observed distance modulus, σ_{m-M}, is found by implementing the error propagation equation (often called the law of errors of Gauss) when the covariant terms are neglected, see Equation (3.14) in [19],

$$\sigma_{m-M} = \sqrt{\alpha^2 \sigma_{X1}^2 + \beta^2 \sigma_C^2 + \sigma_{m_B^\star}^2}. \tag{27}$$

The parameters as derived from the JLA compilation are presented in **Table 2** and the fit is presented in **Figure 2**.

Table 2. Numerical values from the JLA compilation of χ^2, χ^2_{red} and Q, where k stands for the number of parameters.

Cosmology	SNs	k	parameters	χ^2	χ^2_{red}	Q
flat-FLRW	740	2	$H_0 = 69.65 \pm 0.231$; $\Omega_M = 0.3 \pm 0.003$	627.91	0.85	0.998

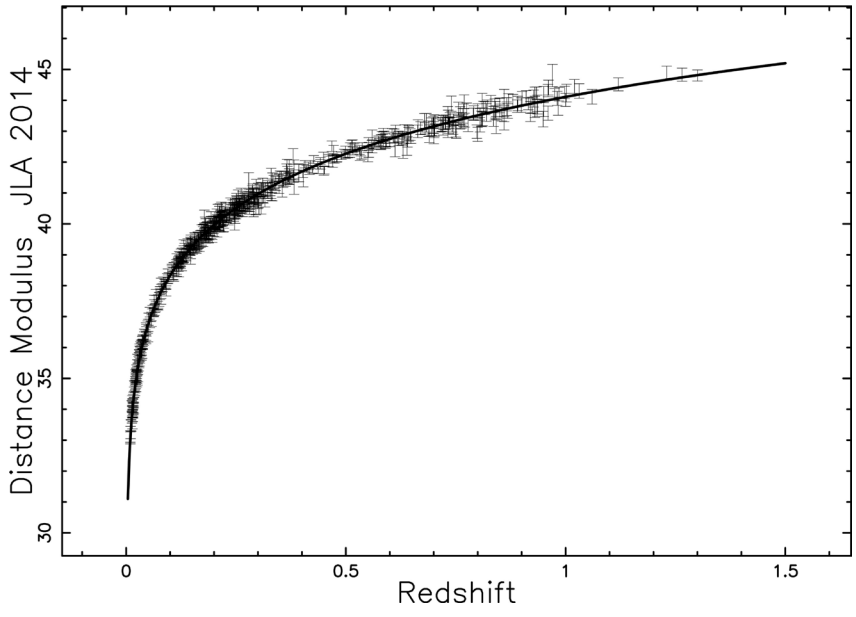

Figure 2. Hubble diagram for the JLA compilation. The solid line represents the best fit for the exact distance modulus in flat-FLRW cosmology as represented by Equation (22). Parameters as in first line of **Table 2**.

As an example the luminosity distance for the Union 2.1 compilation with data as in the first line of Table 1 is

$$d_{\mathrm{L}}(z)=8147.04(1.0+z)\times\left(-0.637664F\left(2.63214\frac{\sqrt{3.1188+1.33542z}}{4.64846+z},0.965925\right)+1.75322\right)\mathrm{Mpc} \tag{28}$$

when $0<z<1.5$

and the distance modulus is

$$m-M=25.0+2.17147\ln\left(8147.04(1.0+z)\times\left(-0.63766F\left(2.6321\frac{\sqrt{3.1188+1.3354z}}{4.6484+z},0.96592\right)+1.7532\right)\right). \tag{29}$$

when $0<z<1.5$

We now derive some approximate results without Legendre integral for the flat-FLRW case and Union 2.1 compilation with data as in Table 1, first line. A Taylor expansion of order 6 around $z=0$ of the luminosity distance as given by Equation (19) for the flat-FLRW case and Union 2.1 compilation gives

$$d_{\mathrm{L}}(z)=0.000423646+4296.57z+3344.13z^2-1186.94z^3+979.403z^4-42078.6z^5\ \mathrm{Mpc} \tag{30}$$

when $0<z<0.197$.

The upper limit in redshift, 0.197, is the value for which the percentage error, see Equation (4), is $\delta=1.16\%$. The asymptotic expansion of the luminosity distance with respect to the variable z to order 5 for the flat-FLRW case and Union 2.1 compilation gives

$$d_{\mathrm{L}}(z)\sim 14283.5z-15802\frac{1}{\sqrt{z^{-1}}}+14283.5-7901.01\sqrt{z^{-1}}+1975.25\left(z^{-1}\right)^{3/2}\ \mathrm{Mpc} \tag{31}$$

when $1.27<z<1.5$.

At the lower limit of $z=1.27$ the percentage error is $\delta=0.54\%$. The two above approximations at low and high redshift have a limited range of existence but does not contain the Legendre integral as solutions (28) and (29) which cover the overall range $0<z<1.5$.

A Taylor expansion of order 6 of the distance modulus as given by Equation (22) around $z=0.1$ for the flat-FLRW case and Union 2.1 compilation gives

$$(m-M)=36.0051+23.1777z-109.604(z-0.1)^2+724.464(z-0.1)^3-5429.06(z-0.1)^4+43429.8(z-0.1)^5 \tag{32}$$

when $0.1<z<0.197$.

The upper limit in redshift, 0.197, is the value at which the percentage error is $\delta=0.14\%$. Figure 3 reports both the numerical and the Taylor expansion of distance modulus in the above range.

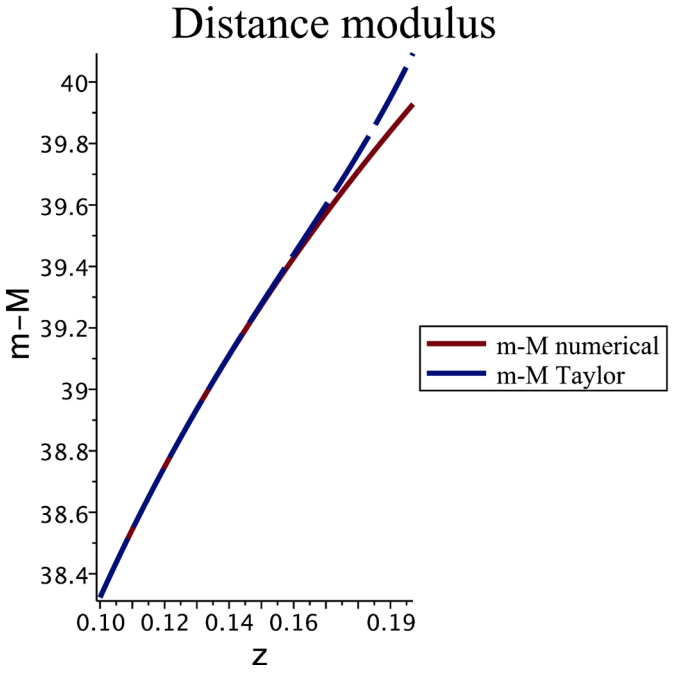

Figure 3. Distance modulus in flat-FLRW cosmology as represented by Equation (22) with parameters as in first line of **Table 1** (full red line) and Taylor solution (dash-dot-dash line) (blue line).

The asymptotic expansion of the distance modulus with respect to the variable z to order 5 for the flat-FLRW case and Union 2.1 compilation gives

$$\begin{aligned}(m-M) \sim & 45.7741+2.17147\ln(z)-2.40231\sqrt{z^{-1}}+0.842625z^{-1} \\ & +0.221081\left(z^{-1}\right)^{3/2}-0.570086z^{-2}-0.150471\left(z^{-1}\right)^{5/2} \\ & +0.357849z^{-3}+0.491842\left(z^{-1}\right)^{7/2}+0.179989z^{-4}\end{aligned} \tag{33}$$

when $1.27 < z < 1.5$.

The lower limit in redshift, 1.27, is the value at which the percentage error is $\delta = 0.54\%$. The ranges of existence in z for the analytical approximations here derived have the percentage error <2%, see Equation (4).

We now introduce the best minimax rational approximation, see [20] [21] [15], of degree (2, 1), for the distance modulus $m_{2,1}(z)$,

$$m_{2,1}(z) = \frac{a+bz+cz^2}{d+ez}. \tag{34}$$

In the case in which the distance modulus is represented by Equation (29) and given the interval $[0.001, 1.5]$, the coefficients of the best minimax rational approximation are presented in **Table 3**; the maximum error for the fit is $\approx 2.2\times10^{-5}$. **Figure 4** displays the data and the fit.

4. Conclusions

We have presented an analytical approximation for the luminosity distance in terms of elliptical integrals with real argument. The fit of the distance modulus

Table 3. Maximum error and coefficients of the distance modulus for the best minimax rational approximation for the flat-FLRW case and Union 2.1 compilation. Interval of existence $[0.001,1.5]$.

Name	value
maximum error	$2.28881836 \times 10^{-5}$
a	0.981622279
b	19.6473351
c	1.08210218
d	0.0309164915
e	0.462291896

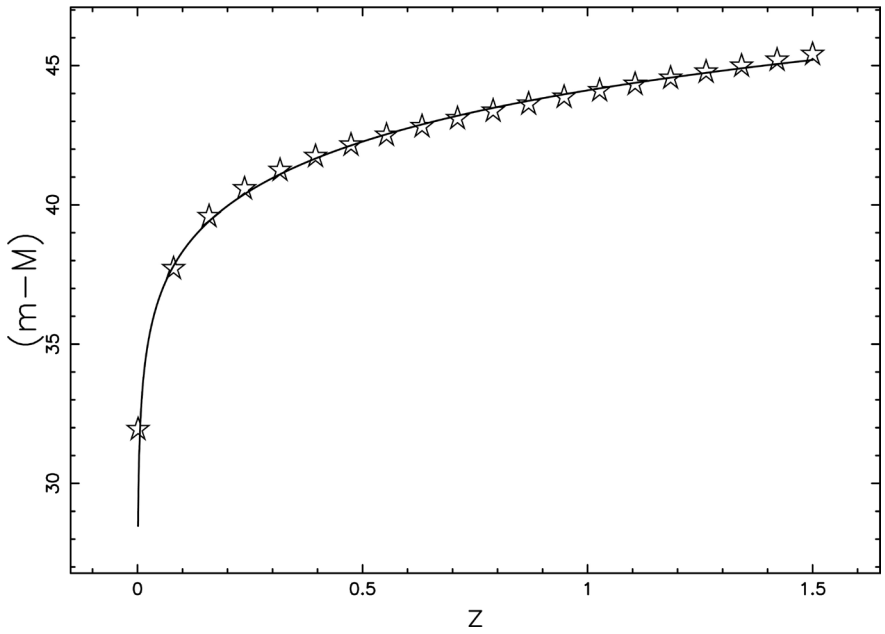

Figure 4. Distance modulus in flat-FLRW cosmology as represented by Equation (22) with parameters as in first line of **Table 1** (full line) and minimax rational approximation (empty stars); Union 2.1 compilation.

of SNs of type Ia allows finding the parameters H_0 and Ω_M for the two compilations in flat-FLRW cosmology

$$H_0 = (69.77 \pm 0.33)\,\mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}, \quad \Omega_M = 0.295 \pm 0.008 \tag{35}$$

flat-FLRW-Union 2.1,

$$H_0 = (69.65 \pm 0.23)\,\mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}, \quad \Omega_M = 0.3 \pm 0.003 \tag{36}$$

flat-FLRW-JLA,

A first comparison with [8] in the case of the Union 2.1 compilation gives a percentage error $p = 0.93\%$ for the derivation of H_0 and $p = 0.67\%$ for the derivation of Ω_M. A second comparison can be done with Equation (13) in [22]

$$H_0 = (67.27 \pm 0.60)\,\mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}, \quad \Omega_M = 0.3166 \pm 0.0084 \tag{37}$$

Planck 2018.

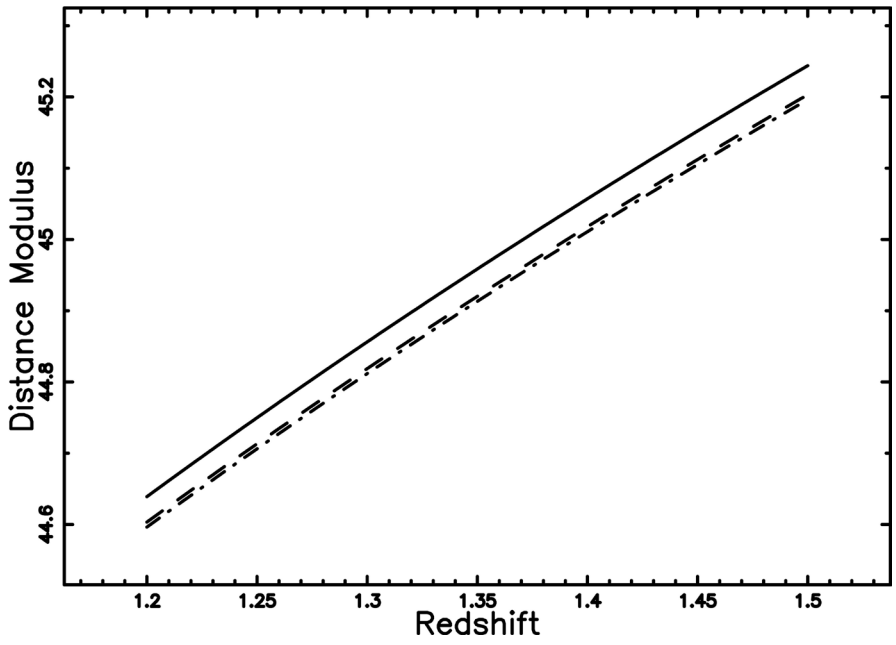

Figure 5. Distance modulus for ΛCDM cosmology (full line), flat-FLRW-1 (dot-dash-dot-dash line) and flat-FLRW cosmology (dashed line). Parameters as in **Table 1** and interval of existence $[1.2,1.5]$.

In the case of the Union 2.1 compilation, the percentage error $p=3.71\%$ for the derivation of H_0 and $p=6.82\%$ for Ω_{M}. A Taylor expansion at low redshift and an asymptotic expansion are presented both for the luminosity distance and the distance modulus. A simple version of the distance modulus is determined through the best minimax rational approximation. Adopting the cosmological parameters found here, the cosmological constant Λ turns out to be, for the Union 2.1 compilation,

$$\Lambda=(1.19457\pm 0.017)\times 10^{-52}\frac{1}{m^2} \tag{38}$$

flat-FLRW Union 2.1,

or introducing $c=1$ and the Planck time, t_p,

$$\Lambda=(3.12046\pm 0.0462942)\times 10^{-122}\frac{1}{t_p^2} \tag{39}$$

flat-FLRW Union 2.1.

The statistical parameters of the fits are given in Table 1 and Table 2 where the other two models are presented. The values of the χ^2 in the above table say that for the Union 2.1 compilation the flat cosmology produces a better fit than the ΛCDM does, but the situation is the reverse for the JLA compilation. As a concluding remark we point out that, thanks to the calibration on the distance modulus of SNs, the differences between the solutions here analyzed are minimum. Therefore a restricted range in redshift should be adopted in order to visualize the diverseness, see Figure 5.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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