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Pseudo-Hermitian Matrix Exactly Solvable Hamiltonian

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Abstract

The non *PT*-symmetric exactly solvable Hamiltonian describing a system of a fermion in the external magnetic field which couples to a harmonic oscillator through some pseudo-hermitian interaction is considered. We point out all properties of both of the original Mandal and the original Jaynes-Cummings Hamitonians. It is shown that these Hamiltonians are respectively pseudo-hermitian and hermitian [1] [2]. Like the direct approach to invariant vector spaces used in Refs. [3] [4], we reveal the exact solvability of both the Mandal and Jaynes-Cummings Hamiltonians after expressing them in the position operator and the impulsion operator.

Keywords

Pseudo-Hermiticity, Exact Solvability, Direct Method

1. Introduction

Several new theoretical aspects in quantum mechanics have been developed in last years. In the series of papers [5] [6], it is shown that the traditional self adjointness requirement (*i.e.* the hermiticity property) of a Hamilton operator is not necessary condition to guarantee real eigenvalues and that the weaker condition PT-symmetry of the Hamiltonian is sufficient for the purpose. Following the theory developed in Refs. [5] [6], let's remind that a Hamiltonian is invariant under the action of the combined parity operator P and the time reversal operator T if the relation $H^{PT} = H$ is proved (*i.e.* *PT*-symmetry is said to be broken). As a consequence, the spectrum associated the previous Hamiltonian is entirely real.

An alternative property called pseudo-hermiticity for a Hamiltonian to be associated to a real spectrum is shown in details in the Refs. [1] [2].

Referring the ideas of [1] [2], we recall here that a Hamiltonian is said to be η pseudo-hermitian if it satisfies the relation $\eta H\eta^{-1} = H^{+}$, where η denotes an invertible linear hermitian operator.

Another direction of quantum mechanics is the notions of quasi exact solvability and exact solvability [7] [8] [9] [10].

In the last few years, a new class of operators has been discovered. This class is intermediate between exactly solvable operators and non solvable operators. Its name is *the quasi-exactly solvable* (*QES*) operators, for which a finite part of the spectrum can be computed algebraically.

This paper is organized as follows:

In Section 2, we briefly describe the general model which is expressed in terms of the creation and the annihilation operators. We show that the Hamiltonian describing the model is pseudo-hermitian if $\phi = -1$, or it is hermitian if $\phi = +1$.

In Section 3, we show in details the properties of the Mandal Hamiltonian namely the non-hermiticity, the non *PT*-symmetry, the pseudo-hermiticity and the exact solvability.

In Section 4, as in the previous section, it was pointed out that the original Jaynes-Cummings Hamiltonian is hermitian and exactly solvable.

2. The Model

In this section, we consider a Hamiltonian describing a system of a fermion in the external magnetic field, $\boldsymbol{B}$ which couples the harmonic oscillator interaction (*i.e.* $\hbar\omega a^{+}a$) and the pseudo-hermitian interaction if $\phi = -1$, or the hermitian interaction if $\phi = +1$ (*i.e.* $\rho\left(\sigma_{+}a + \phi\sigma_{-}a^{+}\right)$) [1] [2]:

$$H = \mu\boldsymbol{\sigma}\cdot\boldsymbol{B} + \hbar\omega a^{+}a + \rho\left(\sigma_{+}a + \phi\sigma_{-}a^{+}\right), \tag{1}$$

where

$\boldsymbol{\sigma}$, $\sigma_{\pm}$ denote Pauli matrices,

ρ, μ are real parameters,

a^{+}, a refer the creation and annihilation operators respectively satisfying the usual bosonic commutation relation

$$\left[a, a^{+}\right] = 1, \ \left[a, a\right] = \left[a^{+}, a^{+}\right] = 0 \ \text{ and } \ \sigma_{\pm} \equiv \frac{1}{2}\left(\sigma_{x} \pm i\sigma_{y}\right).$$

Recall that the matrices $\sigma_{+}, \sigma_{-}, \sigma_{x}, \sigma_{y}$ and σ_{z} can be expressed in the following matrix forms:

$$\sigma_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \sigma_{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_{y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

For the sake simplicity, one can choose the external field in the *z*-direction (*i.e.* $\boldsymbol{B} = B_{0}\boldsymbol{z}$) in order to reduce the Hamiltonian given by the Equation (1) and it becomes [1] [2]:

$$H = \frac{\varepsilon}{2}\sigma_{z} + \omega a^{+}a + \rho\left(\sigma_{+}a + \phi\sigma_{-}a^{+}\right) \tag{3}$$

with $\varepsilon = 2\mu B_0$ and $\hbar = 1$.

3. Properties of the Original Mandal Hamiltonian

3.1. The Non-Hermiticity

In this section, we reveal that the Hamiltonian described by the Equation (3) is non- hermitian if $\phi = -1$. It is called Mandal Hamiltonian (*i.e.* H_M) and it takes the following form:

$$H_M = \frac{\varepsilon}{2}\sigma_z + \omega a^+ a + \rho\left(\sigma_+ a - \sigma_- a^+\right) \tag{4}$$

Taking account to the following identities:

$$\begin{aligned} &\left(a^+\right)^+ = a \\ &(a)^+ = a^+, \\ &\left(\sigma_+\right)^+ = \sigma_-, \\ &\left(\sigma_-\right)^+ = \sigma_+, \end{aligned} \tag{5}$$

let's show that the Mandal Hamiltonian given by the above Equation (4) is non hermitian:

$$H_M^+ = \left(\frac{\varepsilon}{2}\sigma_z\right)^+ + \left(\omega a^+ a\right)^+ + \left[\rho\left(\sigma_+ a - \sigma_- a^+\right)\right]^+,$$

$$H_M^+ = \frac{\varepsilon}{2}\sigma_z + \omega a^+ a - \rho\left(\sigma_+ a - \sigma_- a^+\right). \tag{6}$$

Comparing the expressions given by the Equations (4) and (6), we see that they are different (*i.e.* $H_M^+ \neq H_M$), as a consequence, we are allowed to conclude that the Mandal Hamiltonian H_M is non-hermitian.

3.2. The Non *PT*-Symmetry of H_M

In this section, we prove that the Mandal Hamiltonian is non *PT*-symmetric [5] [6]. Recall that the parity operator is represented by the symbol P and the time-reversal operator is described by the symbol T.

The effect of the parity operator P implies the following changes [1] [2]:

$$\begin{aligned} &P\varepsilon P^{-1} = \varepsilon,\ P\sigma_z P^{-1} = \sigma_z,\ P\sigma_+ P^{-1} = \sigma_+, \\ &P\sigma_- P^{-1} = \sigma_-,\ PaP^{-1} = -a,\ Pa^+ P^{-1} = -a^+. \end{aligned} \tag{7}$$

Notice also the changes of the following quantities under the effect of the time reversal operator T:

$$\begin{aligned} &T\varepsilon T^{-1} = \varepsilon,\ T\sigma_z T^{-1} = -\sigma_z,\ T\sigma_+ T^{-1} = -\sigma_-, \\ &T\sigma_- T^{-1} = -\sigma_+,\ TaT^{-1} = -a,\ Ta^+ T^{-1} = -a^+. \end{aligned} \tag{8}$$

Taking account to the relations (7) and (8), one can easily deduce the changes of the Mandal Hamiltonian under the effect of combined operators P et T as follows

$$(PT)H_M(PT)^{-1} = (PT)\left[\frac{\varepsilon}{2}\sigma_z + \omega a^+ a + \rho\left(\sigma_+ a^+ - \sigma_- a\right)\right](PT)^{-1},$$

$$(PT)H_M(PT)^{-1} = -\frac{\varepsilon}{2}\sigma_z + \omega a^+ a + \rho\left(\sigma_+ a^+ - \sigma_- a\right), \tag{9}$$

This above relation (9) can be written as follows

$$H_M^{PT} = -\frac{\varepsilon}{2}\sigma_z + \omega a^+ a + \rho\left(\sigma_+ a^+ - \sigma_- a\right) \tag{10}$$

Comparing the relations (4) and (10), we see that they are different (*i.e.* $H_M^{PT} \neq H_M$), it means that the Mandal Hamiltonian H_M is not invariant under the combined action of the parity operator *P* and the time-reversal operator *T*. In other words, the Mandal Hamiltonian H_M is not *PT*-symmetric.

3.3. Pseudo-Hermiticity of H_M

In this section, we first prove that the non *PT*-symmetric Mandal Hamiltonian is pseudo-hermitian with respect to third Pauli matrix σ_z [1] [2]:

$$\begin{aligned}\sigma_z H_M \sigma_z^{-1} &= \frac{\varepsilon}{2}\sigma_z\sigma_z\sigma_z^{-1} + \omega\sigma_z a^+ a\sigma_z^{-1} + \rho\left(\sigma_z\sigma_+\sigma_z^{-1}a - \sigma_z\sigma_-\sigma_z^{-1}a^+\right)\\ &= \frac{\varepsilon}{2}\sigma_z + \omega a^+ a - \rho\left(\sigma_+ a - \sigma_- a^+\right)\end{aligned} \tag{11}$$

with $\sigma_z\sigma_\pm\sigma_z^{-1} = -\sigma_\mp$ and $\sigma_z^{-1} = \sigma_z^t = \sigma_z$.

Comparing the Equations (6) and (11), it is seen that the following relation is satisfied:

$$\sigma_z H_M \sigma_z^{-1} = H_M^+ \tag{12}$$

Taking account to this above relation, we are allowed to conclude that the Mandal Hamiltonian is pseudo-hermitian with respect to σ_z.

Finally, we reveal a pseudo-hermiticity of H_M with respect to the parity operator *P*:

$$\begin{aligned}PH_M P^{-1} &= \frac{\varepsilon}{2}P\sigma_z P^{-1} + \omega P a^+ a P^{-1} + \rho\left(P\sigma_+ a P^{-1} - P\sigma_- a^+ P^{-1}\right)\\ &= \frac{\varepsilon}{2}\sigma_z + \omega a^+ a - \rho\left(\sigma_+ a - \sigma_- a^+\right)\\ &= H_M^+\end{aligned} \tag{13}$$

Here we have used the relations (7) in order to obtain this above equation (13). As a consequence, one can conclude that the Mandal Hamiltonian is pseudo-hermitian with respect to the parity operator *P*.

Note that even if H_M is non hermitian and non *PT*-symmetric, its eigenvalues are entirely real due to the pseudo-hermiticity property [1].

3.4. Differential Form and Exact Solvability of H_M

In this step, our purpose is to change the Mandal Hamiltonian given by the Equation (4) in appropriate differential operator (*i.e.* H_M is expressed in the position operator *x* and in the impulsion operator $p = -i\frac{\mathrm{d}}{\mathrm{d}x}$). Thus, referring to the ideas of exactly and quasi-exactly solvable operators studied in the Refs. [7] [8]

[9] [10], we reveal that H_M preserves a family of vector spaces of polynomials in the variable *x*.

With this aim, we use the usual representation of the creation and annihilation operators of the harmonic oscillator respectively a^+ and *a* [1] [2]:

$$a^+ = \frac{p + im\omega x}{\sqrt{2m\omega\hbar}}, \quad a = \frac{p - im\omega x}{\sqrt{2m\omega\hbar}} \tag{14}$$

where ω is the oscillation frequency, *m* denotes the mass, *x* refers to the position operator and the impulsion operator is $p = -i\frac{\mathrm{d}}{\mathrm{d}x}$, $p^2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Using appropriate units, we can assume $m = \hbar = 1$ and the operators a^+ and *a* take the following forms:

$$a^+ = \frac{p + i\omega x}{\sqrt{2\omega}}, \quad a = \frac{p - i\omega x}{\sqrt{2\omega}}. \tag{15}$$

Replacing the operators a^+ and *a* by their expressions given by this above Equation (15) in the Equation (4), the Mandal Hamiltonian H_M takes the following form:

$$H_M = \frac{\varepsilon}{2}\sigma_z + \frac{p^2 - \omega + \omega^2 x^2}{2} + \rho\frac{\left[\sigma_+\left(p - i\omega x\right) - \sigma_-\left(p + i\omega x\right)\right]}{\sqrt{2\omega}}. \tag{16}$$

In order to reveal the exact solvability of the above operator H_M, we first perform the standard gauge transformation [2]:

$$\tilde{H}_M = R^{-1} H_M R, \quad R = \exp\left(-\frac{\omega x^2}{2}\right). \tag{17}$$

After some algebraic manipulations, the new Hamiltonian $\tilde{H}_M$ (known as gauge Hamiltonian) is obtained

$$\begin{aligned} \tilde{H}_M &= \frac{\varepsilon}{2}\sigma_z - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} + \rho\frac{\left[\sigma_+ p - \sigma_-\left(p + 2i\omega x\right)\right]}{\sqrt{2\omega}} \\ &= \frac{\varepsilon}{2}\sigma_z + \frac{p^2}{2} + i\omega xp + \rho\frac{\left[\sigma_+ p - \sigma_-\left(p + 2i\omega x\right)\right]}{\sqrt{2\omega}} \end{aligned} \tag{18}$$

Replacing the Pauli matrices σ_z, σ_+ and σ_- by their respective expressions given by the relation (2), the final form of the gauge Hamiltonian is:

$$\tilde{H}_M = \frac{\varepsilon}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{p^2}{2} + i\omega xp & 0 \\ 0 & \frac{p^2}{2} + i\omega xp \end{pmatrix} - \rho\begin{pmatrix} 0 & \frac{p}{\sqrt{2\omega}} \\ \frac{p + 2i\omega x}{\sqrt{2\omega}} & 0 \end{pmatrix},$$

$$\tilde{H}_M = \begin{pmatrix} \frac{p^2}{2} + i\omega xp + \frac{\varepsilon}{2} & \rho\frac{p}{\sqrt{2\omega}} \\ -\rho\frac{p + 2i\omega x}{\sqrt{2\omega}} & \frac{p^2}{2} + i\omega xp - \frac{\varepsilon}{2} \end{pmatrix}. \tag{19}$$

Note that one can easily check if this above gauge Hamiltonian $\tilde{H}_M$ preserves the vector spaces of polynomials $V_n = \left(P_{n-1}\left(x\right), P_n\left(x\right)\right)^t$ with $n \in \mathrm{N}$. As the integer n doesn't have to be fixed (*i.e.* it is arbitrary), $\tilde{H}_M$ is exactly solvable. Indeed, its all eigenvalues can be computed algebraically. Even if the gauge Mandal Hamiltonian $\tilde{H}_M$ is non-hermitian and non *PT*-symmetric, its spectrum energy is entirely real due to the property of the pseudo-hermiticity [1] [2].

Thus, the vector spaces preserved by the operator H_M have the following form

$$W_n = \mathrm{e}^{-\frac{\omega x^2}{2}} \left(P_{n-1}(x), P_n(x)\right)^t \tag{20}$$

where $P_{n-1}(x)$ and $P_n(x)$ denote respectively the polynomials of degree $n-1$ and n.

As the gauge Mandal Hamiltonian $\tilde{H}_M$, it is obvious that the original Mandal Hamiltonian H_M is exactly solvable. Due to this property of exact solvability, the whole spectrum of H_M can be computed exactly (*i.e.* by the algebraic methods) [1] [2] [3].

4. Properties of the Jaynes-Cummings Hamiltonian

4.1. The Hermiticity

In this section, considering $\phi = +1$, the Hamiltonian given by the Equation (3) leads to the standard Jaynes-Cummings Hamiltonian of the following form

$$H_{JC} = \frac{\varepsilon}{2}\sigma_z + \omega a^+ a + \rho\left(\sigma_+ a + \sigma_- a^+\right) \tag{21}$$

Our aim is now to prove that the above Hamiltonian H_{JC} is hermitian.

Indeed, in order to reveal the hermiticity of the Jaynes-Cummings Hamiltonian given by the above relation (21), the following relation $H_{JC}^+ = H_{JC}$ must be satisfied.

Consider now the following relation

$$H_{JC}^+ = \left(\frac{\varepsilon}{2}\sigma_z\right)^+ + \left(\omega a^+ a\right)^+ + \left[\rho\left(\sigma_+ a + \sigma_- a^+\right)\right]^+, \tag{22}$$

Taking account to the identities of the relation (5), this above equation leads the following expression:

$$H_{JC}^+ = \frac{\varepsilon}{2}\sigma_z + \omega a^+ a + \left[\rho\left(\sigma_+ a + \sigma_- a^+\right)\right]. \tag{23}$$

Comparing the Equations (21) and (23), one can write that

$$H_{JC}^+ = H_{JC}. \tag{24}$$

Referring to this equation (24), it is obvious that the standard Jaynes-Cummings Hamiltonian is hermitian. As a consequence, its eigenvalues are real due to the property of hermiticity.

4.2. Differential Form and Exact Solvability of H_{JC}

Along the same lines as in the above section 3.4, our purpose is to change the Jaynes-Cummings Hamiltonian given by the Equation (21) in appropriate differential operator (*i.e.* H_{JC} is expressed in the position operator x and in the impulsion operator $p=-i\frac{\mathrm{d}}{\mathrm{d}x}$).

With this purpose, we use the usual expressions of the creation and annihilation operators of the harmonic oscillator respectively a^{+} and a given by the Equation (15).

Substituting (15) in the Equation (21), the Jaynes-Cummings Hamiltonian H_{JC} is written now as follows

$$H_{JC}=\frac{\varepsilon}{2}\sigma_z+\frac{p^2-\omega+\omega^2x^2}{2}+\rho\frac{\left[\sigma_+\left(p-i\omega x\right)+\sigma_-\left(p+i\omega x\right)\right]}{\sqrt{2\omega}} \tag{25}$$

Operating on the above operator H_{JC} the standard gauge transformation as

$$\tilde{H}_{JC}=R^{-1}H_{JC}R,\quad R=\exp\left(-\frac{\omega x^2}{2}\right), \tag{26}$$

after some algebraic manipulations, the new Hamiltonian $\tilde{H}_{JC}$ (known as gauge Hamiltonian) is obtained

$$\begin{aligned}\tilde{H}_M&=\frac{\varepsilon}{2}\sigma_z-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\omega x\frac{\mathrm{d}}{\mathrm{d}x}+\rho\frac{\left[\sigma_+p+\sigma_-\left(p+2i\omega x\right)\right]}{\sqrt{2\omega}}\\&=\frac{\varepsilon}{2}\sigma_z+\frac{p^2}{2}+i\omega xp+\rho\frac{\left[\sigma_+p+\sigma_-\left(p+2i\omega x\right)\right]}{\sqrt{2\omega}}\end{aligned} \tag{27}$$

Replacing the Pauli matrices σ_z,σ_+ and σ_- respectively by their matrix form given by the relation (2), the final form of the gauge Hamiltonian $\tilde{H}_{JC}$ is

$$\tilde{H}_M=\frac{\varepsilon}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}\frac{p^2}{2}+i\omega xp&0\\0&\frac{p^2}{2}+i\omega xp\end{pmatrix}+\rho\begin{pmatrix}0&\frac{p}{\sqrt{2\omega}}\\\frac{p+2i\omega x}{\sqrt{2\omega}}&0\end{pmatrix},$$

$$\tilde{H}_M=\begin{pmatrix}\frac{p^2}{2}+i\omega xp+\frac{\varepsilon}{2}&\rho\frac{p}{\sqrt{2\omega}}\\\rho\frac{p+2i\omega x}{\sqrt{2\omega}}&\frac{p^2}{2}+i\omega xp-\frac{\varepsilon}{2}\end{pmatrix}. \tag{28}$$

Note that one can easily check if this above gauge Hamiltonian $\tilde{H}_{JC}$ preserves the finite dimensional vector spaces of polynomials namely $V_n=\left(P_{n-1}(x),P_n(x)\right)^t$ with $n\in\mathrm{N}$. As the integer n is arbitrary, the gauge Jaynes-Cummings Hamiltonian $\tilde{H}_{JC}$ is exactly solvable.

As a consequence, its all eigenvalues can be computed algebraically. Indeed, the vector spaces preserved by the operator H_{JC} have the following form

$$W_n=\mathrm{e}^{-\frac{\omega x^2}{2}}\left(P_{n-1}(x),P_n(x)\right)^t \tag{29}$$

where $P_{n-1}(x)$ and $P_n(x)$ denote respectively the polynomials of degree $n-1$ and n.

As the gauge Jaynes-Cummings Hamiltonian $\tilde{H}_{JC}$, it is obvious that the standard Jaynes-Cummings Hamiltonian H_{JC} is exactly solvable. In other words, all eigenvalues associated to the Hamiltonian H_{JC} can be calculated algebraically (*i.e.* by the algebraic methods) [1-3].

5. Conclusion

In this paper, we have put out all properties of the original Mandal Hamiltonian. We have shown that the Mandal Hamiltonian H_M is non-hermitian and non-invariant under the combined action of the parity operator P and the time-reversal operator T. Even if the previous properties are not satisfied, it has been proved that the Mandal Hamiltonian H_M is pseudo-hermitian with respect to P and with respect to σ_3 also. With the direct method, we have revealed that H_M preserves the finite dimensional vector spaces of polynomials namely $V_n = \left(P_{n-1}(x), P_n(x)\right)^t$. Indeed, the Mandal Hamiltonian H_M is said to be exactly solvable [1] [2] [3] [4]. Along the same lines used in Section 3, we have pointed out that the standard Jaynes-Cummings Hamiltonian H_{JC} is hermitian and exactly solvable in Section 4.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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