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Frucht Graph is not Hyperenergetic

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Abstract: If $\lambda_1, \lambda_2, ..., \lambda_p$ are the eigen values of a p-vertex graph G, the energy of G is $E(G) = \sum_{i=1}^{p} \lambda_i$. If $E(G) > 2p-2$, then G is said to be hyperenergetic. We show that the Frucht graph, the graph used in the proof of well known Frucht's theorem, is not hyperenergetic. Thus showing that every abstract group is isomorphic to the automorphism group of some non-hyperenergetic graph. AMS Mathematics Subject Classification: 05C50, 05C35

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1. Introduction

The concept of hyperenergetic graphs was introduced by Gutman [1]. The existence of hyperenergetic graphs has been known for quite some time, their systematic design was first achieved by Walikar *et al.* [2]. Details can also be seen in [3–6].

The following result can be found in [4].

Theorem 1. A graph with p vertices and m edges such that $m < 2p-2$ is not hyperenergetic.

In this paper, we give the existence of one more class of hyperenergetic graphs, called the Frucht graphs.

2. Frucht Graph is not Hyperenergetic

Let Γ be a group with n elements and Λ be a set of Γ not containing the identity e. The Cayley digraph is defined to be the digraph with vertex set $V = \Gamma$ and arc set $A = \{(g, gh) : h \in \Lambda\}$. It is denoted by $D = D(\Gamma, \Lambda)$. If Λ = Γ -e, the resulting Cayley digraph is complete and is denoted by $K = K(\Gamma, \Lambda)$. If Λ is a set of generators for Γ, the Cayley digraph is called the basic Cayley digraph.

In the Cayley digraph $D = D(\Gamma, \Lambda)$, if (g, h) is an arc, $k = gh$ for some $h \in \Lambda$, that is $g^{-1}k \in \Lambda$ and $g^{-1}k$ is called the color of (g, k).

An automorphism of D is said to be color preserving if it preserves the colors of the arcs. It is well known that the group $C(D)$ of color preserving automorphisms of the Cayley digraph $D = D(\Gamma, \Lambda)$ is isomorphic to Γ.

The following result is the Frucht's Theorem [7–9].

Theorem 2. Every group is isomorphic to the automorphism group of some graph.

While proving Theorem 2, Frucht obtained the graph G_1 from $D(\Gamma, \Lambda)$ called as Frucht graph, whose automorphism group is isomorphic to $C(D)$.

The following is the construction of Frucht graph G_1.

Replace each arc $g_i g_j$ of D by a figure joining vertices g_i and g_j. The figure consists of the 3-path $g_i u_i v_i g_j$, and two paths- a path p_{2k} (containing $2k$ vertices) rooted at u_i, and a path p_{2k+1} (containing $2k+1$ vertices) rooted at v_i, where $g_i^{-1} g_j = g_k$ is the color of $g_i g_j$. (Note that there will be a similar figure corresponding to $g_j g_i$ for a different k).

Clearly, the Frucht graph $G_1(\Gamma)$ with $\Lambda = s$ has $n(s+1)(2s+1)$ vertices.

Theorem 3. The Frucht graph G_1 has $ns(2s+1)$ edges.

Proof. The number of edges m in $G_1(\Gamma)$ is given by

$$m = \sum_{i=1}^{n} \sum_{k=1}^{s} (4k-1) = n\left[\frac{4s(s+1)}{2} - s\right] = ns(2s+1)$$

Theorem 4. The Frucht graph G_1 is not hyperenergetic.

Proof. We observe that,

$$m = ns(2s+1) < 2\left[n(s+1)(2s+)\right] - 2 = 2p-2 .$$

Thus the result follows from Theorem 1.

Lovasz [10] gives an alternate construction of the graph $G_2(\Gamma)$ used to prove the Fruchts theorem. In this case the figure of color k is a path of length $k+2$, including the end vertices g_i and g_j, in which to the first k internal vertices are attached a path P_2 and to the last internal vertex (near g_j) is attached a path P_3.

Theorem 5. $G_2(\Gamma)$ has $n(s^2+4s+1)$ vertices, where $|\Lambda| = s$.

Proof. In $G_2(\Gamma)$ each arc $g_i g_j$ is replaced by a figure with $k+1+2k+3=3k+4$ internal vertices(that is excluding g_i and g_j) if $g_i^{-1}g_j = g_k$.

Therefore total number of extra vertices introduced to form $G_2(\Gamma)$ is equal to

$$\sum_{i=1}^{n}\sum_{k=1}^{s}(2k+3) = \sum_{i=1}^{n}(s^2+4s) = n(s^2+4s)$$

Hence $|V(G_2(\Gamma))| = n(s^2+4s)+n = n(s^2+4s+1)$.

Theorem 6. $G_2(\Gamma)$ has $n(s^2+5s)$ edges, where $|\Lambda| = s$.

Proof. The number of edges in $G_2(\Gamma)$ is given by

$$m = \sum_{i=1}^{n}\sum_{k=1}^{s}(k+2+k+2) = \sum_{i=1}^{n}(s^2+5s) = n(s^2+5s) .$$

Theorem 7. $G_2(\Gamma)$ is not hyperenergetic.

Proof. We see that

$$m = n(s^2+5s) < 2\left[n(s^2+4s+1)\right] - 2 = 2p-2 .$$

Thus the result follows from Theorem 1.

Combining the above observations, we conclude with the following result.

Theorem 8. Every group is isomorphic to the automorphism group of some non-hyperenergetic graph.

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