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Analytical Solution of Van Der Pol's Differential Equation Using Homotopy Perturbation Method

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Abstract

In this research work, Homotopy perturbation method (HPM) is applied to find the approximate solution of the Van der Pol Differential equation (VDPDE), which is a well-known nonlinear ODE. Firstly, the approximate solution of Van Der Pol equation is developed using Dirichlet boundary conditions. Then a comparison between the present results and previously published results is presented and a good agreement is observed. Finally, HPM method is applied to find the approximate solution of VDPDE with Robin and Neumann boundary conditions.

Keywords

Homotopy Perturbation Method (HPM), Van Der Pol Differential Equation (VDPDE), Nonlinear Differential Equations, Analytic Solution, Boundary Conditions

1. Introduction

The Van Der Pol differential equation [1] can be written as

$$y'' + u\left(y^2 - 1\right)y' + y = 0 \tag{1.1}$$

where, u is a scalar parameter. It is considered as an example of an oscillator with nonlinear damping, energy being dissipated at large amplitudes and generated as low amplitude. Such systems typically possess limit cycles: sustained oscillations around a state at which energy generation and dissipation balance [1] [2]. The main application by VDP models an electric circuit with a triode valve, the resistive properties of which change with current, the low current, negative resistance becoming positive as current increases. This model has been widely applied in science and engineering [3]. The parameter u is indicating the nonlinearity and the strength of the damping.

The concept of HPM was first established by He [4] [5], which has been used to solve a large number of non-linear problems. It is found that such approximations rapidly converges to the exact solution [4] [5]. Since it is not easy to find the solution of Equation (1.1) by usual methods such as the perturbation method, separation of variables etc., HPM can be used to find the analytical solution of nonlinear differential equations with different types of initial and boundary conditions [6] [7] [8] [9]. "In addition, the merging of perturbation method and the homotopy method is called homotopy perturbation method [4] [5], which has banished the deficiency of the traditional perturbation methods. On the other hand, this method can use the full benefit of perturbation techniques."

2. Formulation of HPM

The details of the following formulation is given in He which is presented below:

Consider the following non-linear differential equation [5]:

$$M(y)-q(x)=0,\ \ x\in\phi \tag{2.1}$$

with

$$N\left(y,\frac{\partial y}{\partial n}\right)=0,\ \ x\in\Phi, \tag{2.2}$$

Here, M has two parts (linear and nonlinear), Ln and Nln respectively. Then re-write Equation (2.1) as

$$Ln(y)+Nln(y)-q(x)=0 \tag{2.3}$$

Then He [4] [5] introduced a homotopy $g(r,t):\phi\times[0,1]\to\mathbb{R}$ which satisfy

$$H(g,t)=(1-t)\left[Ln(g)-Ln(v_0)\right]+t\left[M(g)-q(x)\right]=0,\ \ t\in[0,1],x\in\phi \tag{2.4a}$$

which is equivalent to

$$H(g,t)=Ln(g)-Ln(v_0)+tLn(v_0)+t\left[Nln(g)-s(x)\right]=0 \tag{2.4b}$$

It follows from Equation (2.4a) and Equation (2.4b) that

$$H(g,0)=Ln(g)-Lln(v_0)=0 \tag{2.5}$$

$$H(g,1)=R(g)-q(x)=0 \tag{2.6}$$

"Thus, the changing process of t from zero to unity is just that of $g(x,t)$ from $v_0(x)$ to $g(x)$. In topology, this is called deformation, and $Ln(g)-Lln(v_0)$, $M(g)-q(x)$ are called homotopic." [4] [5]

Here, t is very small and assume that

$$g=g_0+tg_1+t^2g_2+\cdots \tag{2.7}$$

Setting $t=1$, approximate solution of Equation (2.1) can be obtained as,

$$g=g_0+g_1+g_2+\cdots \tag{2.8}$$

The convergence of series Equation (2.8) has been proved by He [4] [5] in his paper.

3. Numerical Examples

In Example 1, the VDPDE with Dirichlet boundary conditions is solved by HPM and comparisons between the HPM and exact solutions are presented here after.

Example 1:

$$y'' + u\left(y^2 - 1\right)y' + y = 0,\ \ y(0) = 0,\ \ y(20) = 1$$

Considering $y'' + y$ as the linear part and $u\left(y^2 - 1\right)y'$ as the nonlinear part we get the following Homotopy,

$$Y'' + Y - \left(y_0'' + y_0\right) + t\left(y_0'' - y_0\right) + t\left[u\left(Y^2 - 1\right)Y'\right] = 0 \tag{3.1}$$

Putting,

$$Y = Y_0 + tY_1 + t^2Y_2,\ \ y_0 = \csc[20]\sin[x],$$

in Equation (3.1) and equating the coefficients of t from both sides, we get

$$y_0'' + y_0 = 0,\ \ y_0(0) = 0,\ \ y_0(20) = 1$$

$$y_1'' + y_1 + uy_0^2 y_0' - uy_0' = 0,\ \ y_1(0) = 0,\ \ y_1(20) = 0$$

$$y_2'' + y_2 + 2uy_0 y_1 y_0' - uy_1' + uy_0^2 y_1' = 0,\ \ y_2(0) = 0,\ \ y_2(20) = 0$$

Solving we get,

$$y_0 = \csc[20]\sin[x]$$

$$\begin{aligned} y_1 = {} & \frac{1}{16} u \csc[20]^3 \sin[x]\big(-40 + x\left(2 - 4\cos[40]\right) \\ & + 80\cos[40] - \sin[40] + \sin[2x]\big) \end{aligned}$$

$$\begin{aligned} y_2 = {} & \left(\frac{1}{3072}\right) u^2 \csc[20]^3 \big(-18\cos[5x]\cot[20] \\ & + 144\cos[20]^2\cos[40]\sin[x] - 960\cot[20]\sin[x] \\ & + 48x\cot[20]\sin[x] - 24\cot[20]^2\sin[x] \\ & - 12\cos[40]\cot[20]^2\sin[x] - 12\cos[80]\cot[20]^2\sin[x] \\ & - 9600\csc[20]^2\sin[x] + 960x\csc[20]^2\sin[x] \\ & - 24x^2\csc[20]^2\sin[x] - 18\cos[40]\csc[20]^2\sin[x] \\ & + 38400\cos[40]^2\csc[20]^2\sin[x] - 3840x\cos[40]^2\csc[20]^2\sin[x] \\ & + 96x^2\cos[40]^2\csc[20]^2\sin[x] + 9\cos[80]\csc[20]^2\sin[x] \\ & + 10\cos[120]\csc[20]^2\sin[x] + 12\big(7 + 4\cos[40] \\ & + 4\cos[80]\big)\cos[x]^2\csc[20]^2\sin[x] \\ & + 18\cos[2x]\csc[20]^2\sin[x] - 24\cos[80]\cos[2x]\csc[20]^2\sin[x] \\ & - 9\cos[4x]\csc[20]^2\sin[x] + 24\cos[40]\cos[4x]\csc[20]^2\sin[x] \\ & - 10\cos[6x]\csc[20]^2\sin[x] + 2160\cot[20]\csc[20]^2\sin[x] \\ & + 1920\cos[80]\cot[20]\csc[20]^2\sin[x] + 48\sin[40]^2\sin[x] \\ & - 480\csc[20]^2\sin[80]\sin[x] + 24x\csc[20]^2\sin[80]\sin[x] \end{aligned}$$

$$
\begin{aligned}
&-240\csc[20]^4\sin[80]\sin[x]+3\csc[20]^4\sin[80]^2\sin[x]\\
&-240\csc[20]^4\csc[40]\sin[80]^2\sin[x]\\
&-5\csc[20]^4\sin[40]\sin[120]\sin[x]\\
&+\cos[3x]\left(3\csc[20]^2\left(24x\cos[40]+5(-32-32\cos 80+\sin[40])\right)\right.\\
&\left.-72\cot[20]\sin[x]^2\right)+96x\sin[x]\sin[2x]\\
&+480\csc[20]^2\csc[x]\sin[4x]+3\csc[20]^2\csc[x]\sin[80]\sin[4x]\\
&-720\csc[20]^2\sin[x]\sin[4x]+36x\csc[20]^2\sin[x]\sin[4x]\\
&+1440\cos[40]\csc[20]^2\sin[x]\sin[4x]\\
&-72x\cos[40]\csc[20]^2\sin[x]\sin[4x]\\
&+\cos[x]\csc[20]^2\left(-2640+4320\cos[40]-48x\cos[40]\right.\\
&-480\cos[80]-48x\cos[80]-36(-20+x)\cos[40-4x]\\
&+48(-60+x)\cos[40-2x]+480\cos[80-2x]\\
&+1920\cos[2x]-96x\cos[2x]-720\cos[4x]\\
&+36x\cos[4x]+720\cos\left[4(10+x)\right]-36x\cos\left[4(10+x)\right]\\
&-2880\cos\left[2(20+x)\right]+48x\cos\left[2(20+x)\right]\\
&+480\cos\left[2(40+x)\right]-48\sin[40]+12\sin[40-4x]\\
&+21\sin[40-2x]-6\sin[80-2x]-21\sin[4x]+10\sin[6x]\\
&\left.-12\sin\left[4(10+x)\right]+21\sin\left[2(20+x)\right]-6\sin\left[2(40+x)\right]\right)\Big)
\end{aligned}
$$

Thus the two term solution by HPM is $Y = y_0 + y_1$ and the three term solution by HPM is $Y = y_0 + y_1 + y_2$.

From **Table 1**, **Figure 1(a)** and **Figure 1(b)**, it can be seen that there is an insignificant difference between the results of two and three terms HPM solutions and exact solution. It motivates us to move onto the next studies using different boundary conditions.

Example 2: Consider the VDP equation with first Robin type boundary condition

$$y'' + u\left(y^2 - 1\right)y' + y = 0,\ \ y(0) = 0,\ \ y'(20) = 1$$

The Homotopy is,

$$Y'' + Y - \left(y_0'' + y_0\right) + t\left(y_0'' - y_0\right) + t\left[u\left(Y^2 - 1\right)Y'\right] = 0$$

Putting,

$$Y = Y_0 + tY_1 + t^2 Y_2,\ \ y_0 = \sec[20]\sin[x],$$

in Equation (3.1) and equating the coefficients of *t* from both sides, we get

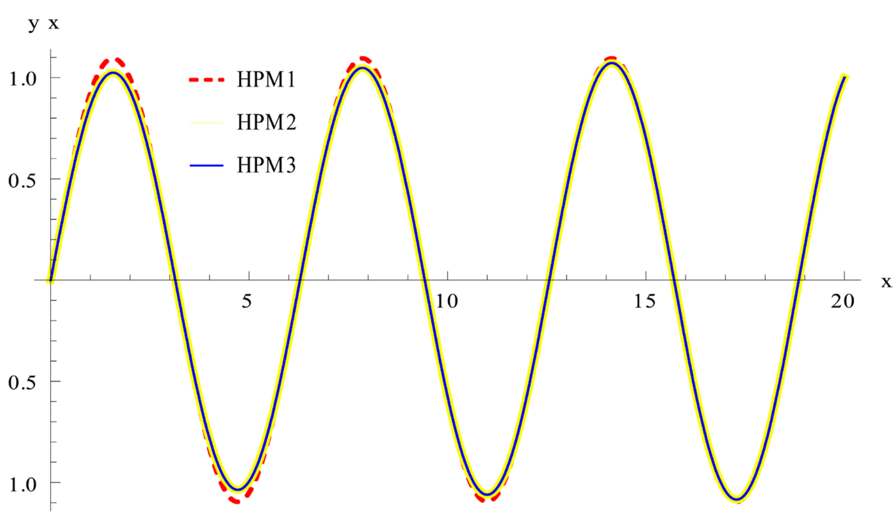
$$y_0'' + y_0 = 0,\ \ y_0(0) = 0,\ \ y_0'(20) = 1$$

$$y_1'' + y_1 + uy_0^2 y_0' - uy_0' = 0,\ \ y_1(0) = 0,\ \ y_1'(20) = 0$$

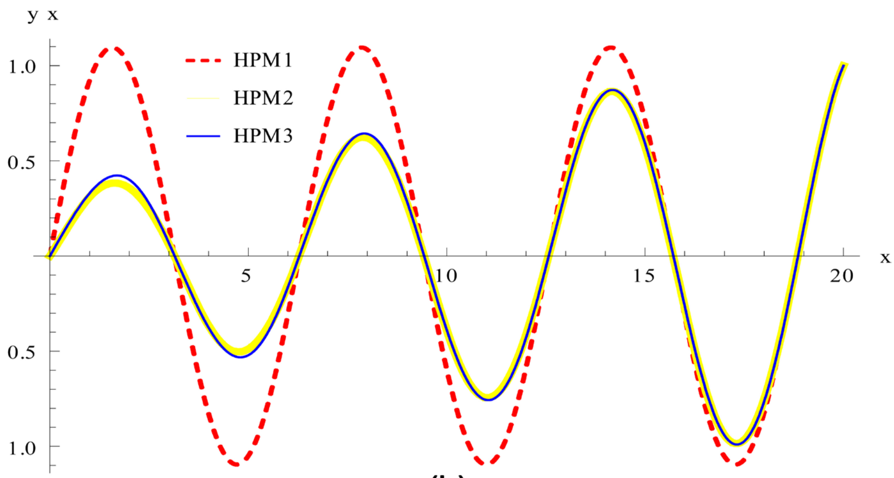
$$y_2'' + y_2 + 2uy_0 y_1 y_0' - uy_1' + uy_0^2 y_1' = 0,\ \ y_2(0) = 0,\ \ y_2'(20) = 0$$

Table 1. Relative errors for example 1 (Using $u = 0.01$).

x	HPM 2 terms	HPM 3 terms	Abbassi [2]	HPM 2 Error	HPM 3 Error
0	0	0	0	0	0
1	0.92171	0.860729	0.861666	0.06098172	0.000937481
2	0.996004	0.932566	0.931693	0.063437934	0.000873401
3	0.154577	0.145325	0.145255	0.009251442	7.01962E−05
4	−0.82897	−0.7828	−0.783643	0.046167623	0.000842517
5	−1.05036	−0.99452	−0.993778	0.055838839	0.000746558
6	−0.30606	−0.29088	−0.290629	0.015181896	0.000248529
7	0.719634	0.687061	0.687723	0.032572757	0.000661586
8	1.083699	1.037556	1.036991	0.046143471	0.000564957
9	0.451416	0.433663	0.433154	0.0177533	0.000509129
10	−0.5959	−0.57511	−0.575549	0.020785844	0.00043809
11	−1.09535	−1.06034	−1.059976	0.03500432	0.000364889
12	−0.58774	−0.57069	−0.569876	0.017048117	0.000814214
13	0.460232	0.448929	0.433154	0.011303398	0.01577506
14	1.085068	1.061946	1.061763	0.023121451	0.000183197
15	0.712297	0.699029	0.697907	0.013267577	0.001122069
16	−0.31536	−0.31084	−0.310907	0.004517348	6.77414E−05
17	−1.05307	−1.04186	−1.041802	0.011215634	5.4816E−05
18	−0.8226	−0.81587	−0.814473	0.006727765	0.001397574
19	0.164169	0.163462	0.163476	0.000707021	1.413E−05
20	1	1	1	0	0



(a)



(b)

Figure 1. (a) Approximate solutions using $u = 0.01$; (b) Approximate solutions using $u = 0.1$.

Solving we get,

$$y_0 = \sec[20]\sin[x]$$

$$y_1 = \frac{1}{32}u\sec[20]^3\sin[x]\big(-160 + x(4 + 8\cos[40]) - 80\cos[60]\sec[20]$$
$$-7\sec[20]\sin[60] + 4\cos[x]\sin[x] + \tan[20]\big)$$

$$y_2 = -\left(\frac{1}{6144}\right)u^2\sec[20]5\big(-100\cos[x]^6\sin[x]$$
$$-2\cos[x]^4(-118 + 72\cos[40] + 25\cos[2x])\sin[x]$$
$$+(36 - 48\cos[80])\sin[x]^3 + 6(3 + 8\cos[40])\sin[x]^5$$
$$-20\sin[x]^7 + 12\cos[x]^2\sin[x]\big(-17 + 8\cos[40]$$
$$-4\cos[80] - 4\cos[60]\sec[20]\sin[x]^2 + 15\sin[x]^4\big)$$
$$+36(-20 + x)\csc[x]\sin[2x]^3 - 18\cos[x]^5(-160$$
$$+x(4 + 8\cos[40]) - 80\cos[60]\sec[20] - 7\sec[20]\sin[60]$$
$$+\tan[20]) - \sin[x]\big(19431 + 48x^2(1 + 2\cos[80])$$
$$+19356\cos[60]\sec[20] + 18838\cos[100]\sec[20]$$
$$+73\cos[140]\sec[20] - 3840\sec[20]\sin[60]$$
$$+3360\sec[20]\sin[100] - 24x\sec[20]\big(80\cos[20]$$
$$+80\cos[60] + 80\cos[100] + 9\sin[20] - 8\sin[60]$$
$$+7\sin[100]\big) - 9120\sin[2x] + 81\sin[2x]^2 + 960\tan[20]$$
$$-1920\cos[60]\sec[20]\tan[20] - 960\cos[100]\sec[20]\tan[20]$$
$$-576\sec[20]\sin[60]\tan[20] + 220\sec[20]\sin[100]\tan[20]$$

$$+73\sec[20]\sin[140]\tan[20]+429\tan[20]^2)$$
$$+12\cos[x]^3\left(3\sec[20](-80\cos[60]+4x\cos[60]+\sin[20]\right.$$
$$-7\sin[60])\sin[x]^2-2(-2+\cos[40])(-160+x(4+8\cos[40])$$
$$-80\cos[60]\sec[20]-7\sec[20]\sin[60]+\tan[20]))$$
$$+6\cos[x](\sec[20](320\cos[20]+80\cos[60]$$
$$-160\cos[100]+4x(3\cos[20]+3\cos[60]+4\cos[100])$$
$$-21\sin[20]+37\sin[60]-14\sin[100])$$
$$-2\sec[20](-800\cos[60]-80\cos[100]+4x(19\cos[20]$$
$$+10\cos[60]+\cos[100])-55\sin[60]-7\sin[100])\sin[x]^2$$
$$+9\sin[x]^4\left(-160+x(4+8\cos[40])-80\cos[60]\sec[20]\right.$$
$$-7\sec[20]\sin[60]+\tan[20])))$$

Example 3: Consider the VDP equation with second Robin type boundary condition

$$y''+u\left(y^2-1\right)y'+y=0,\ \ y'(0)=0,\ \ y(20)=1$$

The Homotopy is,

$$Y''+Y-\left(y_0''+y_0\right)+t\left(y_0''-y_0\right)+t\left[u\left(Y^2-1\right)Y'\right]=0$$

Putting,

$$Y=Y_0+tY_1+t^2Y_2,\ \ y_0=\cos[x]\sec[20],$$

in Equation (3.1) and equating the coefficients of t from both sides, we get

$$y_0''+y_0=0,\ \ y_0'(0)=0,\ \ y_0(20)=1$$
$$y_1''+y_1+u\,y_0^2y_0'-uy_0'=0,\ \ y_1'(0)=0,\ \ y_1(20)=0$$
$$y_2''+y_2+2uy_0y_1y_0'-uy_1'+uy_0^2y_1'=0,\ \ y_2'(0)=0,\ \ y_2(20)=0$$

Solving we get,

$$y_0=\cos[x]\sec[20]$$

$$y_1=\frac{1}{64}u\sec[20]^3\left(-7\cos[x]^2\sin[x]+\sin[x](-1-16\cos[40]\right.$$
$$\left.+\sin[x]^2\right)+2\cos[x](-160+x(4+8\cos[40])$$
$$-80\cos[60]\sec[20]+5\sec[20]\sin[60]-3\tan[20]))$$

$$y_2=\left(\frac{1}{6144}\right)u^2\sec[20]^5\left(20\cos[x]^7+6\cos[x]^5\left(-18+16\cos[40]\right.\right.$$
$$+15\cos[2x])-4\cos[x]^3\left(33+96\cos[40]+12\cos[80]\right.$$
$$\left.+(-69+48\cos[40])\sin[x]^2+25\sin[x]^4\right)$$
$$+12\cos[x]^2\sin[x](3\sec[20](-160\cos[20]-80\cos[60]$$
$$+4x\cos[60]-3\sin[20]+5\sin[60])\sin[x]^2$$

$$
\begin{aligned}
&-2\left(4+\cos[40]\right)\left(-160+x\left(4+8\cos[40]\right)\right.\\
&-80\cos[60]\sec[20]+5\sec[20]\sin[60]-3\tan[20]))\\
&+54\cos[x]^4\sin[x]\left(-160+x\left(4+8\cos[40]\right)\right.\\
&-80\cos[60]\sec[20]+5\sec[20]\sin[60]-3\tan[20])\\
&-6\sin[x]\left(1280+36x+560\cos[60]\sec[20]\right.\\
&+20x\cos[60]\sec[20]-160\cos[100]\sec[20]\\
&-61\sec[20]\sin[60]+10\sec[20]\sin[100]-12x\sin[2x]^2\\
&+4\left(-2+\cos[40]\right)\sin[x]^2\left(-160+x\left(4+8\cos[40]\right)\right.\\
&-80\cos[60]\sec[20]+5\sec[20]\sin[60]-3\tan[20])\\
&+3\sin[x]^4\left(-160+x\left(4+8\cos[40]\right)-80\cos[60]\sec[20]\right.\\
&+5\sec[20]\sin[60]-3\tan[20])+49\tan[20])\\
&+\cos[x]\left(19455+48x^2\left(1+2\cos[80]\right)+19404\cos[60]\sec[20]\right.\\
&+19214\cos[100]\sec[20]-7\cos[140]\sec[20]\\
&+3840\sec[20]\sin[60]-24x\sec[20]\left(80\cos[20]+80\cos[60]\right.\\
&+80\cos[100]-11\sin[20]+8\sin[60]-5\sin[100])\\
&-2400\sec[20]\sin[100]+12\left(9+56\cos[40]-4\cos[80]\right)\sin[x]^2\\
&-6\left(31+48\cos[40]\right)\sin[x]^4+100\sin[x]^6+6720\tan[20]\\
&+5760\cos[60]\sec[20]\tan[20]-960\cos[100]\sec[20]\tan[20]\\
&-576\sec[20]\sin[60]\tan[20]+92\sec[20]\sin[100]\tan[20]\\
&-7\sec[20]\sin[140]\tan[20]+477\tan[20]^2))
\end{aligned}
$$

Example 4: Consider the VDP equation with Neumann type boundary condition

$$y''+u\left(y^2-1\right)y'+y=0,\ \ y'(0)=0,\ \ y'(20)=1$$

The Homotopy is,

$$Y''+Y-\left(y_0''+y_0\right)+t\left(y_0''-y_0\right)+t\left[u\left(Y^2-1\right)Y'\right]=0$$

Putting,

$$Y=Y_0+tY_1+t^2Y_2,\ \ y_0=-\cos[x]\csc[20],$$

in Equation (3.1) and equating the coefficients of t from both sides, we get

$$y_0''+y_0=0,\ \ y_0'(0)=0,\ \ y_0'(20)=1$$

$$y_1''+y_1+uy_0^2y_0'-uy_0'=0,\ \ y_1'(0)=0,\ \ y_1'(20)=0$$

$$y_2''+y_2+2uy_0y_1y_0'-uy_1'+uy_0^2y_1'=0,\ \ y_2'(0)=0,\ \ y_2'(20)=0$$

Solving we get,

$$y_0 = -\cos[x]\csc[20]$$

$$y_1 = \frac{1}{32}u\csc[20]^3\left(2\cos[x]\left(40-80\cos[40]+x\left(-2+4\cos[40]\right)-3\sin[40]\right) + 3\cos[x]^2\sin[x]-\sin[x]\left(-1+8\cos[40]+\sin[x]^2\right)\right)$$

$$y_2 = \left(\frac{1}{3072}\right)u^2\csc[20]^5\left(30\cos[x]^5-63\cos[x]^2\left(40-80\cos[40] + x\left(-2+4\cos[40]\right)-3\sin[40]\right)\sin[x]-\cos[x]^3\left(1+84\cos[40] + 25\sin[x]^2\right)+3\sin[x]\left(520-1680\cos[40]-640\cos[80] + 2x\left(9-14\cos[40]+8\cos[80]\right)-87\sin[40]-24\sin[80] + 3\left(40-80\cos[40]+x\left(-2+4\cos[40]\right)-3\sin[40]\right)\sin[x]^2\right) + \cos[x]\left(-\csc[20]\left(1440\cos[20]+3840\cos[60] - 240\cos[100]+9540\sin[20]-9399\sin[60]+9430\sin[100] + 24x^2\left(\sin[20]-\sin[60]+\sin[100]\right)+12x\left(-3\cos[20] + 3\cos[100]-80\left(\sin[20]-\sin[60]+\sin[100]\right)\right)\right) + 36\left(-2+7\cos[40]\right)\sin[x]^2+25\sin[x]^4\right)\right)$$

From Figures 2-4 it can be seen that VDPDE with different boundary

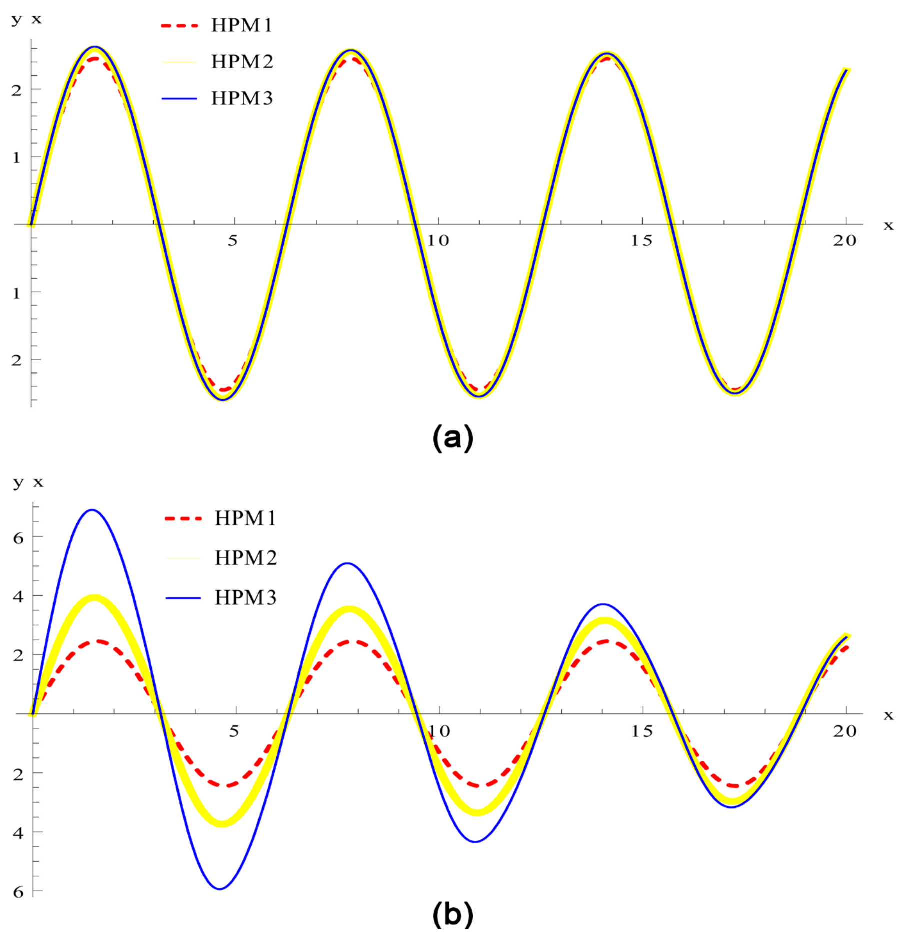


Figure 2. (a) Approximate solutions using $u = 0.01$; (b) Approximate solutions using $u = 0.1$.

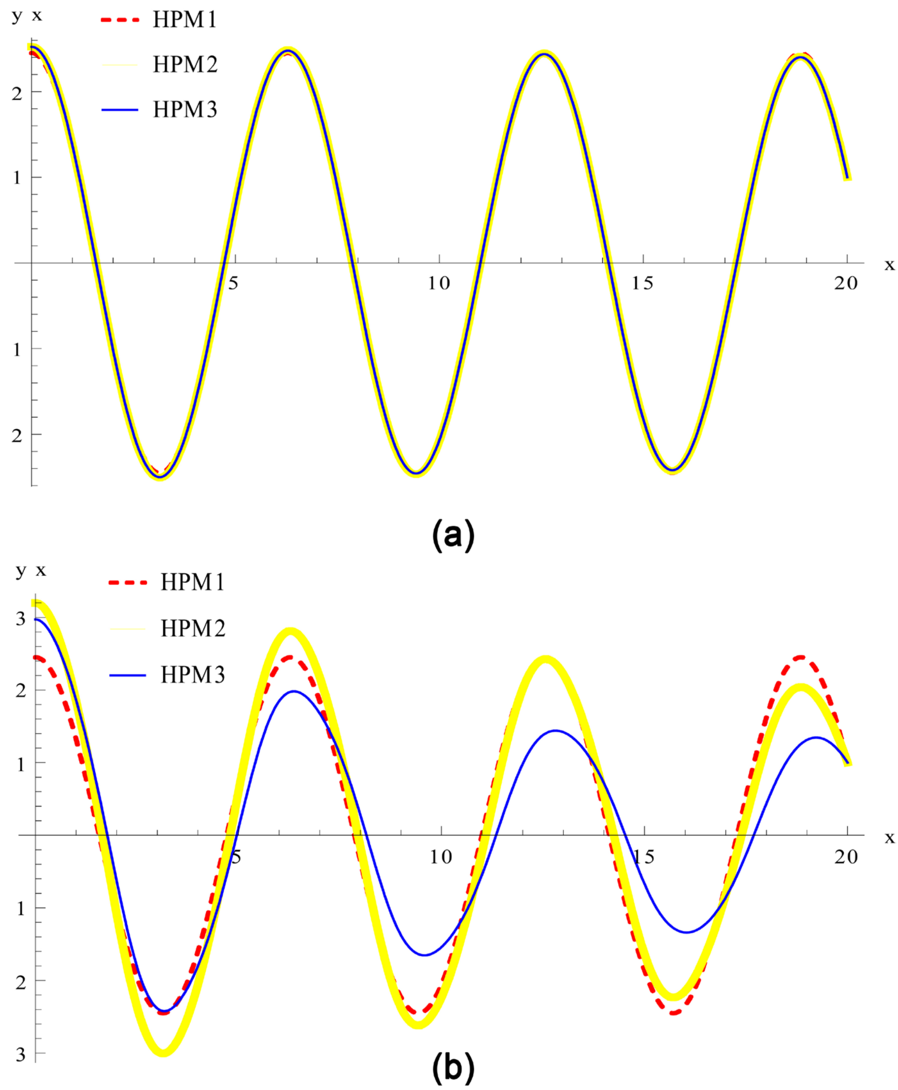
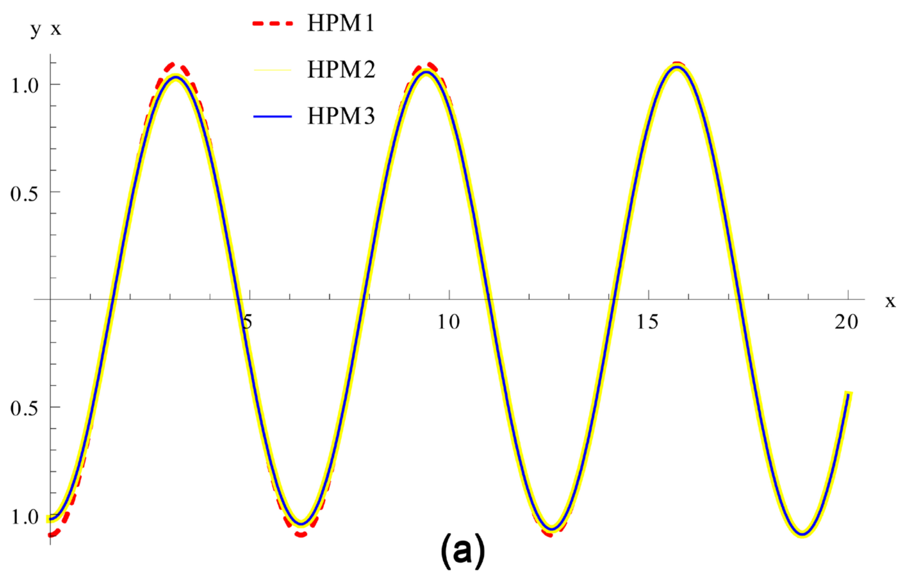


Figure 3. (a) Approximate solutions using $u = 0.01$; (b) Approximate solutions using $u = 0.1$.



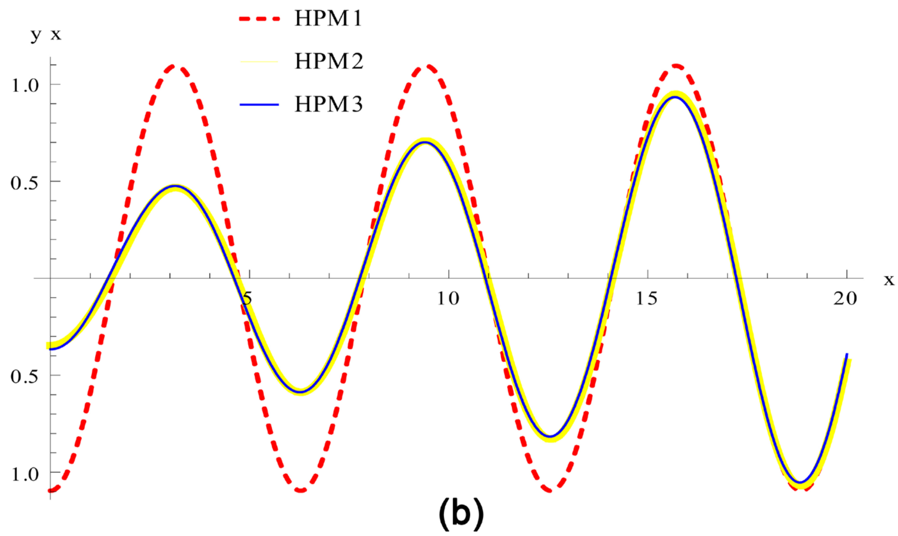


Figure 4. (a) Approximate solutions using $u = 0.01$; (b) Approximate solutions using $u = 0.1$.

conditions can be solved by HPM very easily. Two terms and three terms solution almost coincide. Increasing the number of terms, more accurate results can be found. The solution can be obtained by using different values of u.

4. Conclusion

In this research, HPM is applied for the solution of the Van Der Pol differential equation with different boundary conditions. One, two and three parameters HP solutions are developed and presented graphically. It is found that higher parameter shows good approximations of the analytical solution. It may conclude that HPM is a very effective technique to find the analytical solutions for highly non-linear ordinary differential equation with ICs/BCs.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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