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Evolution, Quantization, Relativity: An "*Ab Initio*" Model

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Abstract

"$\pi\alpha'\nu\tau\alpha\ \ \chi\omega\rho\varepsilon\tilde{\imath}$ = everything flows", Eraclitus, (Ephesus, 535-475 B.C.). If really in Nature everything changes and progresses, then at least two questions arise: 1) how can be these changes entropic but nonetheless somehow predictable without risk of oxymoronic behavior; 2) how can Science conform itself to follow this requirement of the Nature. To attempt an answer to these questions, the present paper introduces an *ab initio* theoretical model aimed to show that physical information is actually nothing else but straightforward quantum and relativistic implication of the concept of evolution.

Keywords

Evolution, Quantum Theory, Relativistic Theory

1. Introduction

Many physicists have emphasized the unreasonable effectiveness of mathematics in describing the physical world; among them the most authoritative one is Wigner [1]. An anecdote clarifies Wigner's perplexity. Two students were discussing the ability of describing the statistical distribution of hungers in the world through the Gauss function, which involves the number π. Strictly speaking, it is hard to realize what has to do the geometrical ratio between circumference and radius of a circle with the distribution of hungers; even in lack of a rational explanation, though, nobody could doubt about the ability of scientists to contribute to the advancement of science introducing π in the frame of sophisticated mathematical algorithms.

Even Bertrand Russel was concerned about the link between mathematics and physics [2]. In his book "Study of Mathematics" he says: "Mathematics, rightly

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viewed, possesses not only truth, but supreme beauty, a beauty cold and austere like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry."

Nevertheless, the outcomes of the natural sciences are subjected to experimental tests: what is false or true is definable regardless of hungers and geometrical distresses. On the one hand, abstract numbers express reliable physical laws describing properties and predicting behavior of Nature. On the other hand, however, this epistemological shortcut in fact leaves unexplained the link between science and reality, calculation and experiment, mental ideas and actual story of the Universe. Quoting Einstein "the most incomprehensible thing of the Universe is that it is comprehensible".

Paradoxically, it is easy to understand the correlation between mathematical algorithm and natural event assuming first deterministic evolution of systems according to the old classical physics: once having selected properly the initial conditions, the successive evolution is in principle uniquely determinable. In practice any deterministic model requires a suitable number of descriptive parameters exactly known of a whole system, whose time evolution is codified and described via appropriate functions of these parameters; the mathematical definitions valid at a given time $t = t_0$, remain also valid, if correctly chosen, at $t_0 + \delta t$. Extrapolating this reasoning, the outcomes of such a model hold at any times $t_0 + n\delta t$ even for $n \to \infty$: everything exactly known at $t = t_0$ remains exactly knowable forever. This should be true in principle also for a classical Universe, regarded as a whole physical system.

Actually however the problem is much more complicated.

The task of guessing the evolution of a physical system from a given initial condition must settle up with the probabilistic frame of the quantum theory: uncertainty relationships imply the impossibility of knowing simultaneously couples of conjugate dynamical variables. This constrain at the time t_0 prevents the possibility of their exact knowledge at any later time as well. Worse still, an initial energy uncertainty $\delta\varepsilon_0$ compels a subsequent range $\delta\varepsilon$ of possible values that depends itself upon the choice of δt. As a matter of fact, however, the fundamental laws of quantum physics are successful in conceiving correctly and designing operatively transistors and lasers.

The predictive ability of science becomes further at stake considering also the relativistic theory, according which δt and $\delta\varepsilon$ have meaning only relatively to the particular reference system where they are initially defined: e.g. the twin paradox exemplifies that the time is not an absolute parameter, as the reciprocal motion of their reference systems R and R' implies anyway admitting different time lapses δt and $\delta t'$ for a given event to occur and even for their aging. The necessity of specifying both reference systems to describe physical events explains why the physical laws must be formulated in a covariant way.

Moreover the link between quantum and relativistic theory is still a hard challenge even today.

To approach gradually the epistemological problem raised by these short considerations, suppose preliminarily that a given event K is allowed to occur in a given R at the arbitrary time t_1, waiving for the moment whether or not actually this time is exactly determinable; in this R are also defined the initial time t_o and the pertinent boundary conditions. Let K be for example the motion of a classical system of N particles, described by a total number J of descriptive parameters f_j: e.g. position coordinates $x_{r,i} = x_{r,i}(t_1)$, momentum components $p_{r,i} = p_{r,i}(t_1)$, energies $\varepsilon_i = \varepsilon_i(t_1)$ and so on of each i-th particle with $r = 1,2,3$. Of course f_j can include also mutual interactions, presence of external fields and anything else. Shortly, $f_{t_1,j}$ symbolize in general the j-th dynamical variables significant to define the state of each particle at the time t_1, in principle all measurable. Moreover let be known also the experimental value of the observable $V_1 = V(t_1)$ of a given property V characterizing the event K at the time t_1 and reproduce this value as a linear combination $V_1 = \sum_{j=1}^{J} A_j f_{t_1,j}$ of its descriptive parameters via appropriate coefficients A_j. A simple example clarifies this point. Consider a one dimensional system of two interacting particles having initial coordinates x_{o1} and x_{o2} at the time t_o; concerning first the initial boundary condition, write

$$V_o = A_1 x_{o,1} + A_2 x_{o,2} + A_3 \varepsilon_{o,1} + A_4 \varepsilon_{o,2} + A_5 \varepsilon_o^* + A_6 t_o$$

being all dynamical variables known by definition, regard the coefficients A_j as parameters that fit the initial value V_o of total energy of the system; also, are experimentally measured the space and time coordinates and the energies ε_{o1} and ε_{o2}, upon which depends the interaction energy ε_o^* assumed known as well. Repeat this reasoning, but considering now the total energy V_1 of the system at the arbitrary time t_1; it is in principle possible that the same equation links V_1 to the new space and time coordinates $x_{1,1}$ and $x_{1,2}$, energies $\varepsilon_{1,1}$ and $\varepsilon_{1,2}$, and interaction energy ε_1^*. Assuming experimentally known all these quantities, the new linear combination involving the same dynamical variables reasonably determining V_1 experimentally known as well reads

$$V_1 = A_1 x_{1,1} + A_2 x_{1,2} + A_3 \varepsilon_{1,1} + A_4 \varepsilon_{1,2} + A_5 \varepsilon_1^* + A_6 t_1.$$

In principle it is possible that the coefficients A_j, with $1 \geq j \geq 6$ in this example, fit not only the initial boundary condition but also this further equation. In practice, however, neither the former equation nor the latter are calculable because two equations do not define uniquely the six unknowns A_j; the system of equations is actually undetermined. But it is possible to measure all dynamical variables also at the subsequent arbitrary times t_2 or t_3 and so on, which yields

$$V_k = A_1 x_{1,k} + A_2 x_{2,k} + A_3 \varepsilon_{1,k} + A_4 \varepsilon_{1,k} + A_5 \varepsilon_k^* + A_6 t_k.$$

It is clear that further sets of six experimental data obtained measuring the

same quantities at five additional times with respect to the initial condition, yield a system of six equations with six unknowns. Now the system admits a unique solution for all A_j fulfilling by definition also the boundary condition. In principle this empirical procedure is possible no matter how complex is the system and how many its freedom degrees might be; a suitable number J of experimental measurements allow to obtain coefficients A_j that fit by definition all values of the observable V_k of interest at any time between t_o and t_J. The various $f_{t_k,j}$ are therefore not only the respective x_k but also any other dynamical variable that concur with all ε_k and ε_k^* to the resulting value of the observable V_k; it is clear why one of the dynamical variables must be just the pertinent time t_k. Note that, owing to the empirical character of the linear combination, even the higher powers of some descriptive parameters, e.g. $p_{r,i}^2$ are in principle admissible with their own A_j among the terms contributing to V_k.

Anyway take for granted that, by definition, all coefficients A_j fit correctly the known values of the experimental parameters $f_{t_k,j}$ of all particles concurring to the required value V_k.

On the one hand, is comprehensible the interest to describe the system at subsequent times after that of the initial condition for completeness of information. On the other hand, however, since in general the descriptive parameters are functions of time, e.g. the dynamical variables of the various particles, the evolution of the system during a given time range becomes in fact essential requirement for the mathematical approach: repeating the same numerical procedure at $J-1$ subsequent times t_k after the initial t_o, one can define a set of J equations and thus a square matrix of coefficients A_j whose lines fit exactly by definition the experimental values V_k of the observable V in the given time range. Write therefore

$$V_k = \sum_{j=1}^{J} A_j f_{t_k,j},\ 1 \le k \le J,\ V_k = V_k\left(x_j, t_k\right) \tag{1.1}$$

the system of equations removes the indeterminacy inherent a unique observation time and contextually describes how a given observable of the system changes at various times $t_1 \le t_k \le t_J$, although without rational or heuristic valence. Nonetheless the following evolution matrix represents the minimal condition able to characterize mathematically one property V of one event K of the system, although waiving any chance of physical explanation:

$$\boldsymbol{KA} = \left|\boldsymbol{V}\right|,\quad \boldsymbol{K} = \begin{pmatrix} f_{t_o,1} & f_{t_o,2} & \cdots & f_{t_o,J} \\ \vdots & \vdots & \ddots & \vdots \\ f_{t_k,1} & f_{t_k,2} & \cdots & f_{t_k,J} \\ \vdots & \vdots & \ddots & \vdots \\ f_{t_J,1} & f_{t_J,2} & \cdots & f_{t_J,J} \end{pmatrix},\quad \boldsymbol{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_j \\ \vdots \\ A_J \end{pmatrix},\quad \boldsymbol{V} = \begin{pmatrix} V_{t_o} \\ \vdots \\ V_{t_k} \\ \vdots \\ V_{t_J} \end{pmatrix}, \tag{1.2}$$

Every column of the matrix $\boldsymbol{K}$ represents the values of each descriptive parameter governing V at various times, every line concerns the values of all possible descriptive parameters contributing to the value V_k at the particular time

t_k regardless of explaining how it was at the past t_{k-1} or will be at the future t_{k+1}. The matrix elements defined by a set of successive measurements fit therefore "a posteriori" the evolution of the observable V, *i.e.* simply reproducing mathematically what is experimentally known in the considered time range. This empirical procedure, in principle non-predictive, is to be repeated at all times and extended to each observable of any event K of interest to characterize the whole system.

Moreover $\boldsymbol{K}$ implies neither past nor future: exchanging two lines, the change of sign of $\boldsymbol{K}$ is canceled by that of $\boldsymbol{A}$ concurrently necessary, *i.e.* V remains unchanged.

On the one hand this procedure, seemingly sterile, deserves attention as it shows that the link between numerical representation of the reality and physical events is in fact plausible: mathematics has its own rules to elaborate numbers; if these rules are implemented to reproduce the results of measurements, then the efforts of scientists are addressed to convert this empirical analysis of data, correct by definition, into rational information to be understood. So Wigner's doubts are bypassed regarding in fact the empiricism as an intermediate step between mere observation and profound knowledge of the reality, which however remains implicitly hidden in the raw data.

On the other hand all previous considerations evidence three key requirements necessary for any theoretical attempt to bridge abstract numbers and informative interpretation of results: 1) it must be holistic, 2) it must have space time structure, 3) it must inherently have evolutionary character. These three points prospect the non-trivial heuristic worth of $\boldsymbol{K}$: despite its pragmatic character, the coefficients of each line of the matrix and thus the matrix itself fulfill by definition these requirements and have thus physical valence. Also, $\boldsymbol{K}$ demonstrates the inherent rationality of Nature, without which no best fit technique could provide sensible outcomes. By consequence, no conceptual doubt exists about the effectiveness of a rational approach in describing mathematically the reality.

In principle is difficult to discern, on the basis of a linear combination of parameters only, whether for example two arbitrary time ranges δt and $\delta t'$ differ because they refer to different reference systems in reciprocal motion or because of the presence of a gravity field or even because the quantum uncertainty implies corresponding energy ranges $\delta\varepsilon$ and $\delta\varepsilon'$. Is evident thud the necessity of overcoming the mere empiricism hitherto preliminarily proposed, while acknowledging that the predictive ability of any theory is nothing else but its ability to reproduce the values of the aforesaid coefficients via rational path as general as possible, *i.e.* starting from first principles. In particular, it appears also necessary to identify rationally one by one the parameters f_j in fact concurring to describe exhaustively any physical event K.

The idea is at this point to bypass the best fit approach, valid by definition, by introducing a general function

$$\psi = \psi\left(x_{r,j}, p_{r,j}, \varepsilon_j, \cdots, \Phi_{r,j}, t\right), \quad p_{r,j} = p_{r,j}\left(x_{r,j}, t\right),$$
$$\varepsilon_j = \varepsilon_j\left(x_{r,j}, t\right), \quad \Phi_{r,j} = \Phi_{r,j}\left(x_{r,j}, t\right) \tag{1.3}$$

the index r stands for the set of three space coordinates and related vector components of all dynamical variables characterizing the system, e.g. possible internal and external vector fields $\Phi_{r,j}$ suitable to affect the evolution of all its constituting particles, the dots indicate any further j-th descriptive parameter additional to $p_{r,j}$ and ε_j possibly necessary. The last three positions allow writing implicitly and simply $\psi = \psi\left(x_{r,j}, t_k\right)$ via the various $x_{r,j}$ of all j-th descriptive parameters contributing to the k-th line of $\boldsymbol{K}$ at the time t_k. For example $\varepsilon_j\left(x_{r,j}, t\right)$ is itself a shortcut of $\varepsilon_j = \varepsilon_j\left(\Phi_{r,j}, x_{r,j}, t_k\right)$; indeed x_j at various t_k are somehow determined themselves by the strengths of the fields possibly acting on the system. So the Equation (1.3) can be shortened without loss of generality writing ψ as

$$\psi = \psi\left(X_{r,j}, t_k\right), \quad X_{r,j} = x_j, y_j, z_j, \tag{1.4}$$

having nested into X all possible descriptive parameters implicitly governing the physical state of the system.

It is clear that the strategy of implementing the form (1.4) as a starting point, requires to extract successively from $X_{r,j}$ information about the possible external fields concurring to the internal interactions in defining $\Phi_{r,j}$ previously quoted. But how could the primordial function ψ summarize the variety of phenomena symbolized by every possible observable V_k for all possible physical events K?

Try to simplify the problem: although in principle the following considerations hold even for $r > 3$, as postulated in some physical theories [3], assume for simplicity and without conceptual limitation a two dimensional space time, with the time coordinate and one space coordinate only. In this assumed one dimensional space $r = 1$ can be omitted, whereas the space coordinates and respective vector components are represented by the unique index j that now refers to the various particles of the system. Accordingly, it is eventually possible to write more shortly $\psi = \psi\left(x_j, t_k\right)$ intending now j extended to the freedom degrees of all particles of a given physical system at the time t_k. So any physical effect determining the behavior of the system is described via one dimensional approach with two space time coordinates only for each freedom degree; this bypasses the difficulty of guessing one by one the descriptive parameters that effectively govern case by case the event K. Compare now the early empirical expression (1.1) with the series expansion of ψ around arbitrary initial coordinates x_{oj} and t_o, which reads

$$V_k = V_o + \sum_{j=1}^{J}\sum_{i=1}^{I}\frac{1}{i!}\left(\left(x_j - x_{oj}\right)\frac{\partial}{\partial x_j} + \left(t_k - t_o\right)\frac{\partial}{\partial t}\right)^i \psi\Big|_{x_j = x_{oj}, t_k = t_o}, \quad \psi = \psi\left(x_j, t\right) \tag{1.5}$$

the summation over i accounts for the arbitrary number I of terms of the series, that on j reproduces the same number of terms of the linear combination (1.1),

the index k still represents the time at which the descriptive parameters j are expressed when defining the time change of an appropriate function ψ of all the necessary parameters. The notation indicates that the derivatives of ψ are calculated at arbitrary $x_j = x_{oj}$ and $t = t_o$ defining V_o in a given R, e.g. the laboratory. So each term j of (1.1) takes the form

$$A_j f_{t_k,j} = A_j f_{t_o,j} + \sum_{j=1}^{J}\sum_{i=1}^{I}\frac{1}{i!}\sum_{i'=0}^{i}\binom{i}{i'}\left(x_j - x_{oj}\right)^{i'}\left(t_k - t_o\right)^{i-i'}\partial_{oj}^{i'}\psi_o$$
$$\partial_{oj}^{i'}\psi_o = \frac{\partial^{i'}}{\partial x_j^{i'}}\frac{\partial^{i-i'}}{\partial t_k^{i-i'}}\psi\Big|_{x_j = x_{oj}, t_k = t_o} \tag{1.6}$$

the additive term is assumed known, being the initial boundary condition of the problem. Each j-th term is still related to the respective parameter f_j of the best fit procedure at the time t_k, although with a small difference. Previously $f_{t_k,j}$ were selected quantities implied by the physical event K (all measurable dynamical variables, among which x_j and t_k) tentatively introduced one by one; the best fit procedure aimed to calculate the respective coefficients A_j reproducing the known values of V_k (the specific physical property of interest) at the time t_k. Here instead the series expansion yields numerical coefficients $A_j = \partial_{oj}^{i'}\psi_o$ given by derivatives of a unique unknown function ψ calculated once for all at prefixed space and time coordinates initially set. The descriptive parameters are $(i,i')f_{t_k,j} = \delta x_j^{i'}\delta t_k^{i-i'}$, *i.e.* combinatorial factors times various products of space time ranges $\delta x_j = x_j - x_{oj}$ and $\delta t_k = t_k - t_o$: the dynamical variables previously tentatively introduced via the respective descriptive parameters $f_{t_k,j}$ correspond now to the space coordinates of all particles that still represent space and time experimental inputs. If these latter are known, then (1.6) and (1.1) are equivalent as concerns the best fit approach, defined again by a linear system of equations with best fit unknowns $\partial_{oj}^{i}\psi_o$. Yet, as by definition the coordinates depend upon all fields possibly acting on the system, summarized by $\Phi_{r,j}$ at given t_k and nested like in (1.4) and (1.3), further calculations are necessary to go back from these coordinates experimentally measured to the strength of the fields hidden in ψ. Nonetheless there is more information in (1.6) than in (1.1): the correlation of the actual experimental data to the initial conditions is not simply reasonable, it is required by the concept of space time ranges themselves.

The next step to overcome the legitimate Wigner doubts is just the time correlation (1.5), which does exist indeed and involves space time ranges as they appear in (1.6), not the local x_j, t_k and x_{oj}, t_o.

The worth of this information appears just from these equations comparing the particular cases where $i = 1$ and $i = 2$ in (1.6). Since in the former case $i' = 0,1$, the summation on i' yields for each j term $\partial_x \delta x_j + \partial_t \delta t_k$, where ∂_x and ∂_t are mere numerical coefficients corresponding to the respective $\partial_{oj}^{i'}\psi_o$. Whatever the numerical values of these coefficients might be, the space and time ranges appear separately: all δx_j on the one side and δt_k on the other side

can be put independently equal to zero to describe local or simultaneous events. The case $i=2$ is conceptually different and more interesting, as the *j*-th term of (1.6) reads $\partial'_x \delta x_j^2 + \partial'_t \delta t_k^2 + 2\partial'_x \partial'_t \delta x_j \delta t_k$, being ∂'_x and ∂'_t new numerical coefficients; the space and time ranges appear together in the mixed term $\delta x_j \delta t_k$ with mixed coefficient $\partial'_x \partial'_t$. In general all higher order terms of the sum over $i>1$ imply mixed space time ranges.

Hence the first order and second order terms of the series do not imply merely two different degrees of numerical approximations in calculating V_k of the Equation (1.5). It is clear that $i=1$ is the classical case: a glance to this equation indicates that space and time terms are in fact separate dynamical variables like in (1.1). In the linear combination (1.1) the time is an independent input parameter, arbitrarily set, as a function of which the *x*-coordinate is next calculable consistently with any event occurring in the system, e.g. the interaction between particles. But in general the mixed terms modify strongly this point of view; for example it is no longer possible to put $\delta t_k = 0$ independently of δx_j: simultaneity and locality are in general conflicting concepts.

Moreover the Equation (1.6) introduces contextually the concept of evolution regarding in the same way also the initial configuration of the system through products of ranges $\delta x_{oj} \delta t_o$. So (1.6) shows that the local space and time coordinates separately measured and purposely introduced to carry out best fit calculations are actually mere mathematical parameters useful for empirical calculations only; the space time ranges of coordinates are instead physical parameters collecting together sets of local space coordinates x_j included within δx_j that define the evolution of allowed states of physical systems during a finite time lapse δt_k. Without this correlation, the system would be that of the matrix (1.2), *i.e.* describable as if it would consist of a list of mathematical terms unrelated and disconnected each other at various times.

This is the first hint to reproduce the coefficients A_j of (1.1) from first principles, thus overcoming both empiricism and Wigner's doubts.

In effect it will be found in the following that $\delta x_j \delta t_k$, not the local $x_j t_k$, is a sensible definition of space time compatible with quantum requirements. This shows that (1.6) lays prospectively the basis of both relativity and quantum physics: the necessity of a space time frame defined via sets of local coordinates $\delta x_j \delta t_k$ is in principle also consistent with the quantum lack of determinism based on local coordinates both exactly knowable.

Anyway, apart from mathematical details, the known value of any V_k in (1.2) is still reproducible in principle solving once more a set of linear equations of the unknown $\partial^i_{oj} \psi_o$. The expected rationality inherent the best fit calculation appears now through the mathematical properties of ψ. With a correct choice of this function, the coefficients $\partial^{i'}_{oj} \psi_o \leftrightarrow A_j$ describe conceptually and not only mathematically the evolution of physical systems; in practice this function still maps the systems like the mere empirical approach (1.2) and makes plausible the numerical representation of the reality. The key point is the underlying link with the concept of time evolution of physical systems with respect to their initial

conditions.

Thus the basic idea is that a general function, ψ, must exist able to describe specific events of interest implementing the holistic concept of system evolution: if it is true that the Nature is a complex system under continuous modification, then the physical laws should also conform themselves to this principle. Accordingly, space and time should appear as inseparable properties in this evolutionary scenario that also implies the holistic view previously outlined as actual mathematical requirement.

On the one hand if the function ψ would be known, then there would be no necessity of determining in advance via best fit approach the power series (1.6) of the dynamical variables, which in fact would be calculable themselves through ψ and its derivatives; this chance exemplifies in principle the starting question of this section, *i.e.* to show why the rational knowledge of phenomena allows mapping the reality into numbers regardless of speculations about the geometrical origin of π. On the other hand this conclusion introduces the aim of the present paper, *i.e.* to understand how an appropriate function representing the physical phenomena through the concept of holistic evolution in fact prospects a conceptual path alternative to empirical best fit calculations; in this way ψ also removes the necessity of knowing in advance case by case the specific event to be described. Therefore the previous question about the mathematical structure of the reality overlaps to the following ones: "how all information codified in physical formulas is in fact deducible from ψ?" and also "are the current results of such theoretical basis susceptible of predictive outcomes prospecting the possible future Universe"?

Clearly the second question concerns the development of science and has heuristic valence in describing anything effectively allowed to happen in a changing Universe.

The purpose of the present paper is to highlight some straightforward hints towards this aim, *i.e.* how in principle could a single function ψ describe all variety of phenomena occurring in the Universe.

For simplicity and brevity of exposition the model is deliberately one dimensional: this choice does not represent a conceptual limit, it merely aims to simplify the theoretical approach with mathematical formalism as simple as possible. Also, the model purposely considers scalar quantities: for example v is the component of the velocity vector $\boldsymbol{v}$; analogous consideration holds for the component p of the momentum $\boldsymbol{p}$. These positions allow writing only $\psi = \psi(x,t)$ without subscripts. The time evolution of this function in a given R is therefore given by $\delta\psi = \psi(x+\delta x, t+\delta t) - \psi(x,t)$.

To add a further step forwards, consider more closely the particular space time interval introduced by (1.6)

$$\delta\ell_{st} = (x_j - x_{oj})(t_k - t_o) = \delta x_j \delta t_k \tag{1.7}$$

as $\delta\ell_{st}$ consists of two ranges, the first problem is how to define position and size of both δx_j and δt_k in an appropriate reference system R. For example

the coordinates x_{oj} and t_o can be defined in order to fix the distances of one boundary of δx_j and δt_k on the respective axes, e.g. x_{oj} and t_o, from their common origin O of R, imagined as a two dimensional space time plane with the time on the vertical axis and the length on the horizontal axis; so x_j and t_k fix the sizes of the ranges. However a better chance exists in this respect: it is possible to introduce the following average values calculated via the boundary coordinates themselves of the ranges only

$$\overline{x_j} = \frac{x_{oj} + x_j}{2}, \quad \overline{t_k} = \frac{t_o + t_k}{2}. \tag{1.8}$$

To describe self-consistently size and position of δx_j and δt_k in the space time plane, these mean values are defined on the respective axes of R as follows

$$v_c'' \overline{t_k} = \delta x_j, \quad v_c' \delta t_k = \overline{x_j} \tag{1.9}$$

the first definition relates δx_j to the average time $\overline{t_k}$ needed for a hypothetical particle to travel through the whole range size, whatever it might be, the second definition relates δt_k to the displacement rate of its average coordinate $\overline{x_j}$ related to the position of both range boundaries only. Clearly these definitions need introducing two velocities v_c' and v_c'' compliant with the strategy of having defined mean values characteristic of both ranges only; if indeed just these definitions characterize size and position of both ranges in a self-consistent way, then any reference to O, and thus to R, is lost. In other words, replacing (1.9) into (1.7) neither x_{oj} nor t_o appear anymore explicitly in $\delta \ell_{st} = \left(v_c'/v_c''\right)\overline{t_k}\,\overline{x_j}$: these mean values of coordinates are in effect identically compatible with different x_{oj}' and t_o', *i.e.* with any other O'. Multiplying now side by side (1.9), one finds $v_c' v_c'' \left(t_k^2 - t_o^2\right) = x_j^2 - x_{oj}^2$ *i.e.*

$$\delta \ell_{jk}^2 = v_c^2 t_k^2 - x_j^2 = v_c^2 t_o^2 - x_{oj}^2, \quad v_c^2 = v_c' v_c''. \tag{1.10}$$

The actual value of v_c does not require in principle any specific hypothesis; is however interesting its particular value, necessarily constant without contradicting (1.10), consistent with $\delta\left(t_k^2\right)$ and $\delta\left(x_j^2\right)$ regardless of the reference system R where are defined δx_j and δt_k; the expression at the right hand side, formally identical to that at the left hand side, can be referred indeed to another reference system R_o. It is significant that a unique constant v_c^2 fits different time and space coordinates and that this equation implies different time and space ranges in different inertial reference systems R and R_o even in reciprocal motion.

However, the fact that space time terms $\delta x_j^{i} \delta t_k^{i'}$ more complex than that of (1.7) also appear in (1.6), suggests that a more complex space time metric is to be expected too. Since now all of these hints seem a reasonable step towards the special and general relativity. This also suggests that a model prospectively aimed to account someway for these suggestions should consider since the beginning not only δt and δx but also, at least, $\delta(\delta t) = \delta^2 t = \delta t - \delta t'$ and $\delta(\delta x) = \delta^2 x = \delta x - \delta x'$. Thus the problem is how to handle methodically both changes $\delta\psi$ and $\delta^2\psi$, rather than ψ itself, to describe systematically the

physical properties of any system concerned by ψ. Despite ψ is not known, are essential and enough to this purpose the general definitions

$$\delta\psi = \psi\left(x+\delta x, t+\delta t\right) - \psi\left(x,t\right), \quad \psi = \psi\left(x,t\right) \tag{1.11}$$

and, increasing again $x+\delta x$ by δx and $t+\delta t$ by δt,

$$\delta\left(\delta\psi\right) = \delta^2\psi = \psi\left(x+2\delta x, t+2\delta t\right) - 2\psi\left(x+\delta x, t+\delta t\right) + \psi\left(x,t\right) \tag{1.12}$$

the former defines $\delta\psi/\delta\ell$, the latter $\delta^2\psi/\delta\ell^2$. Note that being by definition

$$\psi\left(\ell\right) = \psi\left(\ell_0\right) + \frac{\partial\psi}{\partial\ell}\delta\ell + \cdots, \quad \delta\ell = \ell - \ell_0,$$

where ℓ is any descriptive parameter of a physical system in the sense previously introduced, it is possible to put at the first order of approximation

$$\frac{\psi\left(\ell\right) - \psi\left(\ell_0\right)}{\ell - \ell_0} = \frac{\delta\psi}{\delta\ell} \approx \frac{\partial\psi}{\partial\ell} + \cdots$$

neglecting the higher order terms.

Here and in the following x and δx symbolize the r-th space coordinate of each j-th particle of the system and its change as a function of δt upon which depend possible changes of all dynamical variables and their x-components, e.g. $\delta\varepsilon_j$ and δp_j; the same holds for $\delta\left(\delta p_j\right)$ and $\delta\left(\delta\varepsilon_j\right)$, and so on. Although is considered for brevity and simplicity of notation one dimensional space coordinate only, from a conceptual point of view the number of actual coordinates is not necessarily limited to the usual three currently accepted.

The remainder of the paper concerns these points through an "*ab initio*" theoretical model whose exposition aims to be as self-contained as possible. Such model aims to deduce both well known results, as a validation, and new achievements, as innovative implications: in both cases, however, the assessment benchmark is its conceptual root in the Equations (1.11) and (1.12) only.

Despite for sake of brevity and clarity of exposition physical properties like energy and momentum have been taken for granted and explicitly mentioned as well acknowledged concepts in this introductory section, actually all of them will be inferred self-consistently themselves uniquely through (1.11) and (1.12); this holds also for quantities like charge and mass that apparently have nothing to do with the concept of evolution defined by these equations. Although seemingly trivial and innocuous, these two equations are unique source of information and unique input enough to infer all considerations exposed below in a consequential way, while overcoming Wigner's doubts and renouncing to any hints from physics theories currently existing. For completeness, when necessary, are also shortly sketched some results previously published to emphasize their connection with the present conceptual frame.

2. The Model

To infer information of physical interest from the initial positions (1.11) and (1.12), the simplest idea is to relate appropriately δx and δt, and possibly

even $\delta^2 x$ and $\delta^2 t$, to $\delta\psi$ and $\delta^2\psi$ in an arbitrary *R*. In principle this correlation can be expressed implementing $\delta\psi$ to obtain two identities $\delta\psi = (\delta\psi/\delta x)\delta x = (\delta\psi/\delta t)\delta t$ that merge into

$$\frac{\delta x}{\delta t} = \frac{\delta\psi/\delta t}{\delta\psi/\delta x}. \tag{2.1}$$

The ratio at the left hand side introduces a new concept implied by $\delta\psi$, the velocity *v*; this dynamical variable, not evident nor necessary in (1.11) and (1.12), is defined by the identity

$$\frac{\delta\psi}{\delta x} = \frac{1}{v}\frac{\delta\psi}{\delta t}, \quad v = \frac{\delta x}{\delta t}. \tag{2.2}$$

The significance of this result, which follows the Equation (1.11) only, appears rewriting both sides according to the Equation (1.12) *i.e.* implementing likewise the identity $\delta^2\psi = \delta^2\psi$. Dividing both sides by δx^2 still via $\delta x = v\delta t$ just introduced, an analogous reasoning yields the further identity

$$\frac{\delta^2\psi}{\delta x^2} = \frac{1}{v^2}\frac{\delta^2\psi}{\delta t^2}. \tag{2.3}$$

The explicit physical meaning of these identities appears when $\delta \to \partial$, *i.e.* when the range sizes described by δ tend to zero. On the one hand this is possible because no restrictive hypothesis has been introduced about the ranges, on the other hand $\delta x \to 0$ and $\delta t \to 0$ do not necessarily imply equal limits $\partial\psi/\partial x$ and $\partial\psi/\partial t$ of the Equations (1.11). As written, the left hand side of (2.3) reads $\left[\psi(x+2\delta x,t) - 2\psi(x+\delta x,t) + \psi(x,t)\right]/\delta x^2$, whereas the right hand side reads $v^{-2}\left[\psi(x,t+2\delta t) - 2\psi(x,t+\delta t) + \psi(x,t)\right]/\delta t^2$. The limits of these expressions for $\delta \to \partial$ are indeed $\partial^2\psi/\partial x^2$ and $v^{-2}\partial^2\psi/\partial t^2$; as such, they are defined in general by the local analytical dependence of ψ upon either dynamical variable.

All this makes sense, as in fact the symbols δ indicate arbitrary changes not only of ψ but also of *x* and *t*; just for this reason, therefore, nothing can be "a priori" inferred from the ratios between $\delta\psi$ and δx or δt since both these latter are arbitrary, unknown, unrelated and thus implementable separately and independently each other. Instead, despite (2.3) is trivial identity,

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\theta}{\partial t^2}, \quad \theta = \theta(x,t) \tag{2.4}$$

has physical meaning while the aforesaid limits imply contextually $\psi \to \theta$; the notation remarks that θ yields in particular the local analytical form of ψ resulting from the specific correlations of $\delta\psi$ with δx and δt (2.2) and (2.3). So the local behavior in the infinitesimal space range dx and time range dt fulfills at any x,t just the Equation (2.4). All quantities concerned by δ are arbitrary and finite by definition; thus they have been handled, and will be again handled also in the following, according to standard algebraic rules likewise any finite dynamical variable. Instead the limits $\delta \to \partial$ imposed to them define a further local condition/constrain that in fact eliminates their total arbi-

trariness and thus implies the mutual interdependence of both sides of (2.3) around a common limit: the initial analytic form of ψ, whatever it might be, turns locally into that, θ, fulfilling both local limits. In this specific case one has found the D'Alembert equation describing the dynamics of a homogeneous elastic string vibrating with fixed extremities and with constant propagation rate v of the perturbation around the equilibrium position of the string. Obviously the local dependence of θ upon x and t is found by solving the resulting differential equation.

This first example has emphasized how to infer information about one specific physical system through the local extrapolation θ of ψ as a function of both $\delta t \to 0$ and $\delta x \to 0$. Although the Equations (2.3) and (2.4) have identical analytical form, they remark the transition from non-local to local description of the concerned physical system: the former is in fact non-calculable, being mere identity, the latter takes physical meaning because is calculable and comparable with the experience. Otherwise stated, it is reductive to regard (2.3) as intermediate algebraic step towards (2.4); it actually describes a non-real and non-local world that does not have identifiable physical properties of the real and local world accessible to the experiment. Non-locality and non-reality are concurrent features of a further world, the quantum world, that can be not only guessed but also implemented to understand the microscopic properties of matter.

It is easy to generalize this result to the case where the string is non-homogeneous simply considering another possible chance of defining the link between $\delta\psi$ and both δx and δt via the trial positions $\delta x/k_1$ and $\delta t/k_2$, *i.e.* introducing two different proportionality factors k_1 and k_2 concerning separately the previous δx and δt. In this case $\delta\psi$ is defined via these generalized increments, both still unknown and arbitrary of course, where however the functions k_1 and k_2 prospect a new result even more general than (2.3). Now let us repeat the previous steps. To modify the correlation of $\delta\psi$ upon δx and δt via the respective factors k_1 and k_2, multiply first both sides of (2.2) by k_1, still keeping the definition $\delta x = v\delta t$ although with a different expectation value of the resulting local v. So the identity

$$k_1 \frac{\delta\psi}{\delta x} = \frac{k_1}{v}\frac{\delta\psi}{\delta t}, \quad k_1 = k_1(x,t) \tag{2.5}$$

yields the further identity according to (1.12)

$$\delta\left(k_1 \frac{\delta\psi}{\delta x}\right) = \delta\left(\frac{k_1}{v}\frac{\delta\psi}{\delta t}\right) \tag{2.6}$$

and thus, dividing both sides by δx,

$$\frac{\delta}{\delta x}\left(k_1 \frac{\delta\psi}{\delta x}\right) = \frac{\delta}{v\delta t}\left(\frac{k_1}{v}\frac{\delta\psi}{\delta t}\right), \quad \delta x = v\delta t \tag{2.7}$$

formally the Equation (2.7) results from two steps, taking first the changes (2.6)

of the quantities at both sides of (2.5), which are subsequently related to δx and δt to obtain (2.7). Of course the limit $\delta \to \partial$ is not implemented at this intermediate step, as this would mean differentiating the quantities at both sides before having introduced the second function k_2; instead it is convenient to keep still finite changes of δx and δt, which again can be further worked out regarding them like any finite physical variable, to introduce k_2 too. Write thus without loss of generality

$$\frac{\delta}{\delta x}\left(k_1\frac{\delta\psi}{\delta x}\right)=\frac{1}{v_0 v}\frac{\delta}{\delta t}\left(k_2\frac{\delta\psi}{\delta t}\right),\quad \frac{k_1}{v}=\frac{k_2}{v_0},\quad k_2=k_2\left(x,t\right), \tag{2.8}$$

being v_0 a constant velocity by definition; therefore

$$\frac{\delta}{\delta x}\left(k_1\frac{\delta\psi}{\delta x}\right)=\frac{1}{v'^2}\frac{\delta}{\delta t}\left(k_2\frac{\delta\psi}{\delta t}\right),\quad v'^2=v_0 v=v_0^2\frac{k_1}{k_2}. \tag{2.9}$$

Now it is possible to infer from the Equation (2.9) the pertinent differential equation once more via the position $\delta \to \partial$ that implies thus a new local function $\psi \to \vartheta$, *i.e.*

$$\frac{\partial}{\partial x}\left(k_1\frac{\partial\vartheta}{\partial x}\right)=\frac{1}{v'^2}\frac{\partial}{\partial t}\left(k_2\frac{\partial\vartheta}{\partial t}\right),\quad \vartheta=\vartheta\left(x,t\right),\quad v'=v'\left(x,t\right). \tag{2.10}$$

The particular result with $k_2 = const$, which thus can be included in $k_1\left(x,t\right)$ at left hand side, yields the well known equation of the wave propagating through a non-homogeneous string with one fixed extremity. Obviously the functions θ and ϑ fulfilling the respective local limits implied by (2.4) and (2.10) are different; is indeed different the local behavior of either function correspondingly to the respective differential equations. The notation emphasizes that $\theta \neq \vartheta$: these functions describe different physical systems because of the different correlation of $\delta\psi$ with δx and δt.

The outcomes (2.4) and (2.10) highlight the strategy of the present paper: the arbitrary function ψ initially introduced according to (1.4) to describe in principle the physical properties of any system is implementable in various ways, depending on how is expressed the possible correlation between its change $\delta\psi$ with respect to that of its dynamical variables δx and δt. In other words the crucial point is not the analytical form of ψ, but how it changes as a function of δx and δt: whatever ψ might be, in fact this procedure identifies itself the possible kind of problem and outlines its mathematical solution as well via the resulting differential equation.

These results are not accidental outcomes inherent the explanatory examples just carried out; in effect no "ad hoc" hypotheses have been made on the concerned systems, e.g. homogeneous or non-homogeneous string, having simply introduced two different ways of describing the local change, *i.e.* the evolution, of ψ.

Let us exemplify further possible ways to handle $\delta\psi$ and $\delta^2\psi$ to confirm further the general worth of this strategy. To this purpose multiply both sides of (2.9) by v'^2 so that

$$v_0 v\frac{\delta}{\delta x}\left(\frac{k_1\delta\psi}{\delta x}\right)=\frac{\delta}{\delta t}\left(\frac{k_2\delta\psi}{\delta t}\right), \tag{2.11}$$

which suggests the following definitions according to (2.8)

$$v_0 v\frac{\delta p}{\delta x}=\frac{\delta\left(pv_0\right)}{\delta t}=\frac{\delta\varepsilon}{\delta t},\quad p=k_1\frac{\delta\psi}{\delta x},\quad \varepsilon=k_2\frac{\delta\psi}{\delta t}. \tag{2.12}$$

As the unique Equation (2.9) cannot specify both k_1 and k_2, which are still undefined, nothing excludes in principle the chances $k_2\neq k_1$ or $k_2=k_1$. Yet, even so, it appears that the positions (2.12) are not merely formal. To understand the physical meaning of the "new" quantities p and ε, note that the first equation implies

$$\delta\left(pv_0\right)=\delta\varepsilon,\quad i.e.\quad pv_0=\varepsilon+const,\quad k_1\neq k_2. \tag{2.13}$$

By definition $\delta\varepsilon=\varepsilon_2-\varepsilon_1$ and $\delta\left(pv_0\right)=p_2v_0-p_1v=0$, whereas $\varepsilon_1\le\varepsilon\le\varepsilon_2$ and $p_1v_0\le pv_0\le p_2v_0$; of course all quantities labeled "1" and "2" are arbitrary. As it possible to multiply side by side these equations, write $pv_0\delta\left(pv_0\right)=\varepsilon\delta\varepsilon+const\,\delta\varepsilon$. Intuitively $\varepsilon\delta\varepsilon$ should read $\delta\left(\varepsilon^2\right)/2$ with notation that avoids confusion between $\delta\varepsilon^2=\left(\varepsilon_2-\varepsilon_1\right)^2$ and $\delta\left(\varepsilon^2\right)=\varepsilon_2^2-\varepsilon_1^2$: in effect if ε is specifically regarded as mean value within its own allowed range $\delta\varepsilon$ of variability, *i.e.* $\varepsilon=\left(\varepsilon_2+\varepsilon_1\right)/2$, one finds $\varepsilon\delta\varepsilon=\left(\varepsilon_2+\varepsilon_1\right)\left(\varepsilon_2-\varepsilon_1\right)/2=\delta\left(\varepsilon^2\right)/2$ whatever ε_2 and ε_1 might be. The same reasoning for $pv_0\delta\left(pv_0\right)$ yields $\delta\left(pv_0\right)^2/2$. The idea of local variables ε and pv_0 allows an interesting implication noting that in general $\delta\varepsilon\equiv\delta\left(\varepsilon\pm const\right)$; so merging (2.13) one finds

$$\frac{1}{2}\delta\left(pv_0\right)^2=\frac{1}{2}\delta\left(\varepsilon\right)^2+\delta\left(const\,\varepsilon\right) \tag{2.14}$$

i.e. $\left(pv_0\right)^2=\varepsilon^2+2const\,\varepsilon$. Also, since $\varepsilon^2+2const\,\varepsilon=\left(\varepsilon+const\right)^2-const^2$, then

$$\varepsilon'^2=\left(pv_0\right)^2+const^2,\quad \varepsilon'=\varepsilon+const. \tag{2.15}$$

To examine either chance, calculate with the help of (2.12), (2.2) and (2.9)

$$\varepsilon+pv=k_2\frac{\delta\psi}{\delta t}+k_1\frac{\delta\psi}{\delta x}\frac{\delta x}{\delta t}=\left(k_1+k_2\right)\frac{\delta\psi}{\delta t}=\left(1+\frac{k_1}{k_2}\right)\varepsilon=\left(1+\frac{v}{v_0}\right)\varepsilon. \tag{2.16}$$

Regarding separately the addends at the initial and final left and right hand sides, this chain of equations is consistent: ε and pv at the left hand side correspond respectively to ε and $\left(v/v_0\right)\varepsilon$; in effect $p=\varepsilon/v_0$ is nothing else but (2.13) with $const=0$. This justifies regarding p and ε of (2.16) as momentum and energy in agreement with (2.23). Yet another chance also consistent with $k_1\neq k_2$ is

$$k_1=-k_2, \tag{2.17}$$

i.e. $\varepsilon+pv=0$ so that $v=-v_0$: in effect v is actually velocity component defined by δx during the time range δt.

The well known Equation (2.15) will be inferred again later; these short notes

aim to justify preliminarily the positions (2.12) according which, regarding from now on $v_0 = c$ with usual notation, p and ε are nothing else but momentum and energy of a relativistic free particle. Simply regarding p and ε as local random values in their allowed ranges δp and $\delta\varepsilon$, *i.e.* anticipating here the concept of quantum uncertainty, it also appears in (2.14) why v_0 must be upper bound: if not, then p necessarily finite in its finite range δp could be consistent with an infinite energy ε' allowed by diverging $\delta\varepsilon$ once multiplied by a value of $v_0 \to \infty$.

So the finite value of c follows as a corollary.

Also, it is not surprising that the energy is defined an arbitrary constant apart; it will be shown shortly, however, that the constant has in this context a peculiar physical meaning. If $k_1 = k_2$, then $v = v_0$ and thus $\varepsilon + pv = 2\varepsilon$ *i.e.* $\varepsilon = pc$. The implications of this chance will be examined in the following.

2.1. Diffusion Equations

With $v = v_0 = c$, according to (2.8), the Equation (2.11) reads

$$\frac{\delta}{\delta x}\left(\frac{c^2 k_2 \delta\psi}{\delta x}\right) = \frac{\delta\psi'}{\delta t}, \quad \psi' = \frac{k_2 \delta\psi}{\delta t}, \quad k_1(x,t) = k_2(x,t); \tag{2.18}$$

since it is certainly possible to introduce an arbitrary function g such that $g\delta\psi = \tau\delta\psi'$, being τ a time dimensional constant, this equation reads

$$\frac{\delta}{\delta x}\left(D\frac{\delta\psi'}{\delta x}\right) = \frac{\delta\psi'}{\delta t}, \quad D = v_0^2 \tau \frac{k_2}{g}, \quad \delta\psi' = \frac{g}{\tau}\delta\psi, \quad g = g(x,t). \tag{2.19}$$

Whatever the function k_2/g might be, D has physical dimensions of diffusion coefficient; in effect with the position $\delta \to \partial$, which implies the local behavior of ψ' described by $\psi' \to \beta$, the last equation reads

$$\frac{\partial}{\partial x}\left(D\frac{\partial\beta}{\partial x}\right) = \frac{\partial\beta}{\partial t}, \quad D = D(x,t), \quad \beta = \beta(x,t). \tag{2.20}$$

This is just the general form of diffusion equation in a homogeneous and isotropic medium in the absence of internal sources or sinks. But diffusion of what? Although β is by definition dimensionless function, two relevant examples are reported below. To this purpose are anticipated here for clarity the concepts of mass and energy kT; both concepts will be inferred later self consistently in the frame of the present theoretical model.

It is possible to multiply β at both sides by a constant mass per unit volume m_0/V_0; so the equation

$$\frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) = \frac{\partial C}{\partial t}, \quad C = C(x,t) = \frac{\beta m_0}{V_0}, \tag{2.21}$$

where C is an appropriate function describing the local value of mass density, concerns the matter transport function under non-equilibrium concentration gradient. It is known that other important phenomena fulfill (2.20); in fact the extension to these cases, e.g. the Fourier heat diffusion, is also possible in an

analogous way. Implementing a different dimensional factor to the local function β, *i.e.* multiplying both sides by an appropriate constant energy ϵ_0, one finds the famous equation

$$\frac{\partial}{\partial x}\left(\mathrm{K}\frac{\partial \epsilon}{\partial x}\right)=\frac{\partial \epsilon}{\partial t},\quad \epsilon(x,t)=\beta\epsilon_0=\beta kT_0,\quad \mathrm{K}=\mathrm{K}(x,t),\quad T=\beta T_0=T(x,t) \tag{2.22}$$

where now with usual notation K replaces D to express the heat diffusion coefficient simply identifying $\epsilon(x,t)\equiv kT$.

Note that the present strategy to infer information about physical systems reveals unexpected links between seemingly different laws: it is significant the fact that elementary manipulations of the equation of vibrating string lead to the diffusion equations.

2.2. Energy and Momentum

The Equation (2.8) reads according to (2.9)

$$\frac{\delta p}{\delta x}=\frac{1}{c^2}\frac{\delta \varepsilon}{\delta t},\quad p=k_1\frac{\delta \psi}{\delta x},\quad \varepsilon=k_1\frac{\delta \psi}{\delta t},\quad k_1=k_2,\quad v=c \tag{2.23}$$

because of course the positions (2.12) still hold also in this particular case. This equation can be implemented in two ways.

The first way is

$$\frac{\delta p}{\delta x}\frac{\delta t}{\delta \varepsilon}=\frac{\delta p}{\delta \varepsilon}\frac{1}{v}=\frac{1}{c^2} \tag{2.24}$$

and thus the second equality yields

$$\delta p=\frac{v}{c^2}\delta\varepsilon. \tag{2.25}$$

Here v still appears because the ratio $\delta x/\delta t$ is explicitly present in (2.24). The ranges explicitly written as $\delta p=p_2-p_1$ and $\delta\varepsilon=\varepsilon_2-\varepsilon_1$ by definition, where of course the quantities labeled with subscripts 1 and 2 are arbitrary, yield

$$p_2-\frac{v}{c^2}\varepsilon_2=p_1-\frac{v}{c^2}\varepsilon_1;$$

this result reads therefore

$$p=\frac{v}{c^2}\varepsilon,\quad p_1\le p\le p_2,\quad \varepsilon_1\le\varepsilon\le\varepsilon_2, \tag{2.26}$$

where p and ε are random values by definition included within the respective ranges.

The second way is highlighted rewriting (2.25) as

$$\frac{\delta(pc)}{\delta x}=\frac{\delta(\varepsilon/c)}{\delta t},$$

which yields

$$\delta p_o\delta x=\delta\varepsilon_o\delta t=const=n\hbar,\quad p_o=\frac{\varepsilon}{c},\quad \varepsilon_o=pc; \tag{2.27}$$

the constant $n\hbar$, required to fulfill products of different variables, will be justi-

fied soon below; the notation emphasizes that p_o and ε_o are not constants. Thus, in agreement with (2.2), one also finds

$$\delta\varepsilon_o = v\delta p_o, \quad v = \frac{\delta x}{\delta t}. \tag{2.28}$$

A few remarks help to simplify the notations in the following:

- the subscripts of δp_o and $\delta\varepsilon_o$ will be omitted as both ranges are arbitrary, so they actually symbolize any sizes of the respective δp and $\delta\varepsilon$;
- the velocities v and c are profoundly different, as the former is defined as ratio of two range sizes whereas the latter is a universal constant of the Nature;
- the definitions of two "new" quantities, momentum p and energy ε, have been guessed in (2.23) by dimensional reasons according to the constant $\hbar$, once having defined dimensionless the coefficients k_1 and k_2.

The lack of specific assumptions on p and ε, e.g. about the sizes of their allowed ranges, implies their physical definition on mere dimensional basis. At the moment n has been formally introduced in (2.27) as mere proportionality factor of a constant, $\hbar$; dimensional reasons are enough to justify this position. In the following, see next Equations (3.1) and (3.2), it will be shown that n is actually an arbitrary integer, whereas the pertinent reasoning will also explain why the physical laws need quantization. The Equation (2.27) is particularly interesting as it correlates the products $\delta x\delta p$ and $\delta\varepsilon\delta t$ of four ranges of different dynamical variables, regardless of the necessity of the position $\delta \to \partial$ and regardless of the range sizes; despite all changes of dynamical variables are arbitrary, the fact of having introduced a relationship between δx and δt implies the general and non-local character of this connection.

Consider now the Equations (2.28) and (2.26): the former concerns ranges, the latter local values. Let us show that relevant physical information is obtainable merging these equations. Multiplying side by side

$$\delta\varepsilon = v\delta p, \quad \varepsilon = \frac{c^2}{v}p \tag{2.29}$$

one finds

$$\varepsilon\delta\varepsilon = c^2 p\delta p; \tag{2.30}$$

thus (2.30) is compatible with

$$\varepsilon^2 + const' = c^2\left(p^2 + const''\right). \tag{2.31}$$

So follow three relevant equations

$$p = \varepsilon\frac{v}{c^2}, \quad \varepsilon^2 = (pc)^2 + const^2, \quad const^2 = const''v_0^2 - const' \tag{2.32}$$

Introduce now the boundary condition $p = 0$ to which corresponds $\varepsilon = \varepsilon_0 \neq 0$, because in general the third equation is different from zero; strictly speaking, in effect, there is no reason to expect that ε_0 is necessarily null too. So this boundary condition yields $\varepsilon_0 = \pm const$; moreover it implies defining a "new" quantity m not yet explicitly mentioned hitherto although implicitly

inherent the physical dimensions of p and ε, *i.e.*

$$\lim_{v\to 0}\frac{p}{v}=\frac{\varepsilon_0}{c^2}=m \tag{2.33}$$

so m is the rest mass. Calling c the constant velocity v_0, with usual notation, the last result reads thus

$$p=\varepsilon\frac{v}{c^2},\quad \varepsilon^2=(pc)^2+\left(mc^2\right)^2. \tag{2.34}$$

Clearly the particular case $\varepsilon=pc$ corresponds to $v=c$, which however requires $m=0$ in the second (2.34). It is immediate to verify that the two Equations (2.34) are consistent for $m\neq 0$, as they imply the Lorentz factor $\sqrt{1-v^2/c^2}$, whereas it also follows

$$\varepsilon^2=\frac{\left(mc^2\right)^2}{1-(v/c)^2},\quad p^2=\frac{(mv)^2}{1-(v/c)^2}. \tag{2.35}$$

The second equation is compatible with $\pm p$; this is not surprising because actually the component of p along an arbitrary direction can have both signs. Much more interesting is the analogous conclusion for $\pm\varepsilon$, which implies states of negative and positive energy separated by a gap 2ε.

Note that in addition to the concepts of mass, momentum and energy, follow from (1.11) and (1.12) the constancy of light speed and Lorentz transformations of energy and momentum.

A problem however arises now about why the first (2.34) is consistent with $pc=\varepsilon$ for $v=c$ whereas both (2.35) and the second (2.34) itself do not. A rational answer to this question will be given in the next Section 4.3. Note at the moment that the factor c^2/v of (2.34) yields $c^2/v\nu=length$, being ν an arbitrary reciprocal time; so, calling "wavelength" the new length λ defined in this way and multiplying both sides of the first (2.34) by ν^{-1}, one finds $\varepsilon/\nu=const=p\lambda$. Thus

$$\frac{\varepsilon}{\nu}=const,\quad p=\frac{const}{\lambda},\quad \lambda=\frac{c^2}{v\nu}, \tag{2.36}$$

where obviously $const=h$; so $\nu=c/\lambda$ is defined even for $v=c$. These positions, here reasonably guessed, are easily verified starting again from (2.34) rewritten as $p=\varepsilon\delta x/c^2\delta t$. With the help of (2.27), trivial manipulation turn equivalently this result into both forms

$$\frac{(pc)\delta(pc)}{\varepsilon}=energy=\frac{\hbar}{\delta t},\quad \frac{(\varepsilon/c)\delta(\varepsilon/c)}{p}=momentum=\frac{\hbar}{\delta x}.$$

In both cases, dimensional considerations confirm the validity of the three positions (2.36), regarding in particular $\delta x\leftrightarrow\lambda$: *i.e.* the range size δx corresponds to one or more momentum wavelengths, the range size δt corresponds to one or more frequency quanta. This suggests that actually $\lambda=n\lambda_o$, with $n=integer$, which formally is compatible with the constant appearing in (2.36) as $const=n\hbar$ as in effect it has been guessed in (2.27). Therefore it is

possible to write, in agreement with (2.2),

$$\varepsilon_{\omega}=\frac{\hbar}{\delta t},\quad \frac{1}{\delta t}=2\pi\nu,\quad p_{\omega}=\frac{\hbar}{\delta x},\quad \frac{1}{\delta x}=\frac{2\pi}{\lambda}. \tag{2.37}$$

It is interesting the fact that the Equations (2.36), pillars of quantum mechanics, are obtained contextually to the relativistic expressions of momentum, energy and rest mass.

2.3. Lagrange and Hamilton Equations

Write (2.28) as

$$\frac{\delta p}{\delta t}=\frac{\delta\varepsilon}{\delta x};$$

rewriting left hand side via (2.27) with $n=1$ for simplicity, this equation reads then according to (2.23)

$$\frac{\delta}{\delta t}\left(\frac{\hbar}{\delta x}\right)=\frac{\delta}{\delta t}\left(\frac{\delta\varepsilon}{\delta x/\delta t}\right)=\frac{\delta\varepsilon}{\delta x}$$

and thus

$$\frac{\delta}{\delta t}\left(\frac{\delta\varepsilon}{\delta\dot{x}}\right)=\frac{\delta\varepsilon}{\delta x},\quad \delta\dot{x}=\frac{\delta x}{\delta t}.$$

Also now the general concept of energy takes physical meaning via the limit $\delta\to\partial$, which implies $\varepsilon\to\phi$ as well; hence the result is

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial\dot{x}}\right)=\frac{\partial\phi}{\partial x},\quad \phi=\phi\left(x,\dot{x}\right). \tag{2.38}$$

So ϕ is the particular local energy resulting from ε whose local behavior is described just by this equation. It is easy to realize that the resulting ϕ turns out to be Lagrangian energy. The most intuitive interpretation of ϕ compatible with both sides of (2.38) is indeed

$$\frac{\partial\phi}{\partial\dot{x}}=p,\quad \dot{p}=\frac{\partial\phi}{\partial x}=F,\quad \dot{p}=\frac{\partial p}{\partial t}; \tag{2.39}$$

as all of this is coherent with ϕ equal to energy, these equations define ϕ reasonably consistent with the Lagrangian $\phi=\mathrm{T}-\mathrm{U}$ of a physical system. In effect defining

$$\mathrm{S}=\int\phi\mathrm{d}t,\quad F=-\frac{\partial\mathrm{U}}{\partial x},$$

one finds

$$\frac{\partial\mathrm{S}}{\partial t}=\phi,\quad \frac{\partial\mathrm{S}}{\partial x}=\int\frac{\partial\phi}{\partial x}\mathrm{d}t=\int\frac{\partial p}{\partial t}\mathrm{d}t=p \tag{2.40}$$

in agreement with the well known definition of action S. Moreover, the second (2.39) yields

$$\dot{p}=\frac{\partial\phi}{\partial x}=-\frac{\partial\mathrm{U}}{\partial x}=\frac{\partial\left(\mathrm{T}-\mathrm{U}\right)}{\partial x},\quad \dot{x}\frac{\partial\phi}{\partial\dot{x}}=\dot{x}p=2\mathrm{T} \tag{2.41}$$

owing to Euler's theorem of homogeneous functions. Hence

$$\dot{x}\frac{\partial\phi}{\partial\dot{x}}-\phi=\mathrm{H}=\mathrm{T}+\mathrm{U}. \tag{2.42}$$

It is immediate to conclude that (2.42) yields the Hamilton function.

As the Equations (2.3) and (2.9) have sensible implications, (2.4) and (2.10), whereas (2.39) and (2.42) allow describing correctly the dynamics of any particle, the present approach appears significant: a relationship between space and time ranges δx and δt has been established even without knowing anything about the initial ψ, simply admitting possible relationships between arbitrary $\delta(\delta x)$ and $\delta(\delta t)$. Once more, however, it is worth emphasizing that everything follows via (2.27) from (1.11) and (1.12) only.

Instead of attempting to explain some particular physical event on the basis of the intuition about its presumed theoretical foundation, we started from arbitrary changes of an introductory function, ψ, which is not "a priori" specified but rather is "a posteriori" identified case by case depending on its possible local change described by the analytical form of the pertinent differential equation.

2.4. The Group Velocity

In (2.2) v is defined by the time range δt necessary for a particle to travel ideally the range size δx. Note now that (2.1) reads formally

$$\frac{\delta x}{\delta t}=\frac{\delta\omega}{\delta\mathrm{k}},\quad \delta\omega=\frac{\delta\psi}{\delta t},\quad \delta\mathrm{k}=\frac{\delta\psi}{\delta x},$$

where ω and k are two "new" quantities called frequency and wave vector respectively; in this case the concept of velocity at the left hand side is different from that of (2.2). These definitions introduce a further concept of velocity, because at the local limit $\delta\to\partial$ one finds

$$\frac{\partial x}{\partial t}=\frac{\partial\omega}{\partial\mathrm{k}}. \tag{2.43}$$

It is immediate to show that also the positions (2.36), in particular the third one, allow calculating consistently the group velocity of a matter wave packet through the following simple chain of equations. Implementing $\delta\varepsilon$ and δp with the help of (2.28) one finds

$$v=\frac{\delta\varepsilon}{\delta p}=\frac{\delta\nu}{\delta\lambda^{-1}}=\frac{\delta 2\pi\nu}{\delta 2\pi\lambda^{-1}}=\frac{\delta\Omega}{\delta\kappa},\quad \kappa=2\pi\lambda^{-1},\quad \Omega=2\pi\nu \tag{2.44}$$

whatever *const* might be. This suggest a possible quantum definition of velocity additional to the direct ratio between space range δx and time range δt. Once more the position $\delta\to\partial$ implies the local definitions $\Omega\to\omega$ and $\kappa\to\mathrm{k}$, whereas v turns to the first equality into local group velocity v_g, *i.e.*

$$v_g=\frac{\partial\omega}{\partial\mathrm{k}}. \tag{2.45}$$

Eventually, note that the third Equation (2.36) alone is enough itself to con-

firm this result. Write

$$\lambda = \frac{\mathrm{n}c}{\nu}, \quad \mathrm{n}c = \lambda\nu = \frac{\omega}{\mathrm{k}}, \quad \mathrm{n} = \frac{c}{v}; \tag{2.46}$$

as $c\delta\mathrm{n} = \lambda\delta\nu + \nu\delta\lambda$, trivial manipulations yield

$$c\lambda\frac{\delta\mathrm{n}}{\delta\lambda} = \lambda^2\frac{\delta\nu}{\delta\lambda} + \mathrm{n}c = \mathrm{n}c - \frac{\delta\nu}{\delta\lambda^{-1}},$$

whence

$$\frac{\delta\nu}{\delta\lambda^{-1}} = \frac{c^2}{v} = \mathrm{n}c - c\lambda\frac{\delta\mathrm{n}}{\delta\lambda} = c\left(\mathrm{n} - \lambda\frac{\delta\mathrm{n}}{\delta\lambda}\right).$$

Therefore

$$v = \frac{c}{\mathrm{n} - \lambda\dfrac{\delta\mathrm{n}}{\delta\lambda}}$$

yields for $\delta \to \partial$ the local dispersion equation

$$v_g = \frac{c}{\mathrm{n} - \lambda\dfrac{\partial\mathrm{n}}{\partial\lambda}}, \tag{2.47}$$

i.e. the well known group velocity of a matter packet wave.

In summary, relevant equations of physics are simply inferred and described through various chances of changing an arbitrary function ψ of time and space, regardless of its early specific physical meaning and without need of introducing initial hypotheses. This concerns crucially the functions k_1 and k_2 introduced in general in (2.8) and (2.9), whose specific analytical form determines the correlation of $\delta\psi$ with δx and δt: as it has been just highlighted, if $k_1 = k_2$ in (2.9) then one obtains the diffusion Equation (2.10), if instead $k_1 \neq k_2$ then one obtains further results concerned later thanks to the additional freedom degree allowed to k_2.

2.5. The Relativistic Velocity

The starting point is the first Equation (2.34), which must be rearranged in order to find a sum rule between two arbitrary velocities $v_1 + v_2$ and their corresponding v_1' and v_2', e.g. in two different inertial reference systems R and R'. As $v^{-1} = \varepsilon/pc^2$, calculate first

$$\frac{\varepsilon_1}{p_1c^2} + \frac{\varepsilon_2}{p_2c^2} = \frac{1}{v_1} + \frac{1}{v_2} = \frac{1}{v_o}, \quad v_o = \frac{v_1v_2}{v_1 + v_2};$$

in this way one has introduced $v_1 + v_2$ through the invariant momentum. It is necessary now to define in general v_o in a form suitable to relate $v_1 + v_2$ and $v_1' + v_2'$, e.g. in another reference system. A reasonable position is the following linear combination that does not involve neither v_1v_2 nor $v_1'v_2'$, *i.e.*

$$v_o = \frac{c^2}{v_1' + v_2'} - \frac{c^2}{v_1 + v_2},$$

so that the sought result is

$$v_1' + v_2' = \frac{v_1 + v_2}{v_1 v_2 / c^2 + 1}. \tag{2.48}$$

Accordingly any v summed to or subtracted from c still yields c.

3. Preliminary Implications of the Model

The results so far obtained are enough to get four relevant consequences, exposed below.

3.1. Statistical Formulation of Quantum Uncertainty

Write (2.27) as

$$\delta x \delta p = n \times const = \delta\varepsilon\delta t \tag{3.1}$$

being n an arbitrary integer. The reason of this definition is to make (2.27) independent of a specific reference system. Suppose that (3.1) holds for ranges defined in R whereas $\delta x' \delta p' = n' \times const$ holds for that defined in any R', with n' arbitrary integer as well; the prime symbols account for the respective Lorentz transformations of range sizes. Actually the reference systems are indistinguishable because neither n nor n' are specific numbers, they instead symbolize by definition whole sets of allowed integer numbers: so any specific n of the first set that turns into a new specific n' of the primed set does not imply in fact distinguishable sets of the respective reference systems. This point is better understood introducing appropriate measure units $x_{Pl}, p_{Pl}, \varepsilon_{Pl}, t_{Pl}$ to express the respective range sizes; for example it is possible to express the size of δx as $n_x^* \times x_{Pl}$; *i.e.* n_x^* is a dimensionless length expressing the actual range size in x_{Pl} units.

Is evident the hint to the well known Planck units, whose choice implies $x_{Pl} p_{Pl} = \hbar = \varepsilon_{Pl} t_{Pl}$ by definition. Without having introduced the gravity constant yet, this explicit reference appears here premature; it is enough to emphasize that the Planck units fulfill this equation by definition. The crucial fact is that introducing the dimensionless lengths $n_x^*, n_p^*, n_\varepsilon^*, n_t^*$, the couples x_{Pl}, p_{Pl} and ε_{Pl}, t_{Pl} fulfill the condition $x_{Pl} p_{Pl} = \varepsilon_{Pl} t_{Pl}$. In this way, dividing side by side with the Equations (3.1), one finds

$$\frac{\delta x}{x_{Pl}} \frac{\delta p}{p_{Pl}} = n = \frac{\delta\varepsilon}{\varepsilon_{Pl}} \frac{\delta t}{t_{Pl}}, \quad const = \hbar = x_{Pl} p_{Pl} = \varepsilon_{Pl} t_{Pl}.$$

It implies that with this choice of measure units, the statistical formulation of quantum uncertainty reads simply

$$n_x^* n_p^* = n = n_\varepsilon^* n_t^* \tag{3.2}$$

the stars indicate arbitrary real numbers, n is instead an arbitrary real integer number. This reasoning shows that in fact the Equations (2.27) hold regardless of any reference system; otherwise stated, the problem of specifying the reference system where are defined the four uncertainty ranges is physically meaningless, provided that the local dynamical variables are systematically replaced

by respective uncertainty range totally unknown in any physical problem. This holds also for the derivatives, which are defined in the present model as mere ratios of uncertainty ranges arbitrary, unknown and conceptually unknowable: for example is meaningless to inquire whether δx refers to Cartesian or curvilinear or cylindrical reference frame. What is crucial in this reasoning is that the four starred numbers be not specifiable and unspecified in any physical problem formulated via the Equations (2.10); in short, the quantization of n is necessary to make (3.1) independent of any specific R. For clarity and self-contained exposition, this is shortly sketched in the next subsection. The results quoted here for completeness are reported more in detail elsewhere [4] [5].

3.2. The Old Quantum Mechanics

It is usually assumed that the quantum problems are tackled via the operator formalism of wave mechanics, introducing operators and wave equations. For comparison purposes, this section sketches very shortly results concerning one case where the wave equation can be exactly solved: the non relativistic hydrogenlike atom. The aim is to show that identical information is obtainable via a a "corpuscular approach", which does not require solving any wave equation; it is enough to replace $x \to \delta x$ and $p \to \delta p$, instead of $-\hbar i \partial\Psi/\partial x$, and proceed via elementary algebraic manipulations. These results help understanding how the relativity fits the conceptual frame so far outlined.

The starting point is the classical component of $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}$ along an arbitrary direction defined by the unit vector $\boldsymbol{w}$ is $M_w = \boldsymbol{r} \times \boldsymbol{p} \cdot \boldsymbol{w}$. Consider thus

$$M_w = (\Delta \boldsymbol{r} \times \Delta \boldsymbol{p}) \cdot \boldsymbol{w} = (\boldsymbol{w} \times \Delta \boldsymbol{r}) \cdot \Delta \boldsymbol{p} = \Delta \boldsymbol{W} \cdot \Delta \boldsymbol{p}, \quad \Delta \boldsymbol{W} = \boldsymbol{w} \times \Delta \boldsymbol{r},$$

which introduces a range of possible values for M_w included in ΔM_w. If $\Delta \boldsymbol{p}$ and $\Delta \boldsymbol{W}$ are orthogonal, then $M_w = 0$; else, rewriting $\Delta \boldsymbol{W} \cdot \Delta \boldsymbol{p}$ as $(\Delta \boldsymbol{p} \cdot \Delta \boldsymbol{W}/\Delta W)\Delta W$ with $\Delta W = |\Delta \boldsymbol{W}|$, the component $\pm \Delta p_W = \Delta \boldsymbol{p} \cdot \Delta \boldsymbol{W}/\Delta W$ of $\Delta \boldsymbol{p}$ along $\Delta \boldsymbol{W}$ yields $M_w = \pm \Delta W \Delta p_W$. Thus, according to Equations (3.1), $M_w = \pm l\hbar$, being l the usual notation for the integer quantum number of angular momentum. So M_w is effectively a multi-valued quantized function because of the uncertainties initially postulated for $\boldsymbol{r}$ and $\boldsymbol{p}$. One component of $\boldsymbol{M}$ only is actually knowable; the same considerations for the y and x components would trivially mean changing $\boldsymbol{w}$.

Just this conclusion on the physical uniqueness of M_w suggests that the average values $\langle M_x^2 \rangle$, $\langle M_y^2 \rangle$ and $\langle M_z^2 \rangle$ should be equal; so the quantity of physical interest to describe the properties of quantum angular momentum is l, as a function of which M^2 is now inferred as well. The components averaged over the possible states summing $(l\hbar)^2$ from $-L$ to $+L$, where L is an arbitrary maximum value of l, yield $\langle M_i^2 \rangle = \sum_{l_i=-L}^{l_i=L} (\hbar l)^2 / (2L+1)$ *i.e.*

$$M^2 = \sum_{i=1}^{3} \langle M_i^2 \rangle = L(L+1)\hbar^2, \quad M_w = l\hbar. \tag{3.3}$$

Consider the quantum system formed by a particle in a central force field, e.g.

an electron around a nuclear charge; the concept of force will be justified in the conceptual frame of (1.12) and (1.11). Assuming the origin O of R on the nucleus, let $\varepsilon = p^2/2m - Ze^2/r$ be the classical electron energy, where m is the electron mass. As $p^2 = p_r^2 + M^2/r^2$, putting again $p_r \to \Delta p_r$ and $r \to \Delta r$, one finds

$$\varepsilon = \frac{\Delta p_r^2}{2m} + \frac{M^2}{2m\Delta r^2} - \frac{Ze^2}{\Delta r}. \tag{3.4}$$

Two numbers of states, *i.e.* two quantum numbers, are expected because of the radial and angular uncertainties. In effect the Equations (2.1) and the quantum M^2 yield $\varepsilon = n^2\hbar^2/2m\Delta r^2 + l(l+1)\hbar^2/2m\Delta r^2 - Ze^2/\Delta r$, which reads

$$\varepsilon = \varepsilon_o + \frac{l(l+1)\hbar^2}{2m\Delta r^2} - E_o, \quad E_o = \frac{Z^2e^4m}{2n^2\hbar^2}, \quad \varepsilon_o = \frac{\left(n\hbar/\Delta r - Ze^2m/n\hbar\right)^2}{2m}. \tag{3.5}$$

Minimize ε putting $\varepsilon_o = 0$, which yields

$$\Delta r = \frac{n^2\hbar^2}{Ze^2m} = \left(\frac{\alpha Z}{n}\right)^{-1} n\lambda_C, \quad \lambda_C = \frac{\hbar}{mc}, \quad \alpha = \frac{e^2}{\hbar c} \tag{3.6}$$

and thus $\varepsilon_{\min} = \left[l(l+1)/n^2 - 1\right]E_o/n^2$; so $l \le n-1$ in order to get $\varepsilon < 0$, *i.e.* a bound state. The reason of both ways to express Δr will be explained in the section 6. Here are of interest the electron energy levels and rotational energy of the atom as a whole around O

$$\varepsilon_{\min} = \varepsilon_{el} + \varepsilon_{rot}, \quad \varepsilon_{el} = -\frac{Z^2}{n^2}E_0 = -\frac{Ze^2}{2\Delta r}, \quad \varepsilon_{rot} = E_0\frac{Z^2l(l+1)}{n^4}E_0 = \frac{e^4m}{2\hbar^2}. \tag{3.7}$$

The physical meaning of Δr is related to the early Bohr radius, *i.e.* ε_{el} is due to charges of opposite sign delocalized within a diametric distance $2\Delta r$ apart. So n and l are properties of the phase space, *i.e.* numbers of allowed quantum states.

Consider now the identity $\Delta r/n\hbar \equiv \omega\Delta r/n\hbar\omega$. So it is consequently true that

$$\frac{2\pi\Delta r}{nh} = \frac{v}{\epsilon} = \frac{1}{p}, \quad v = \omega\Delta r, \quad \epsilon = n\hbar\omega,$$

where the last equation of the chain introduces the momentum p by dimensional reasons and reads

$$2\pi\Delta r = n\lambda, \quad p = \frac{h}{\lambda}. \tag{3.8}$$

It shows the link between De Broglie momentum, Planck energy and condition $n\lambda = 2\pi\Delta r$, according which an integer number of steady electron wavelengths λ is defined along a circumference of radius Δr along which the electron wave propagates at rate v. For such electron waves one finds

$$\varepsilon_{el} = -\alpha\frac{Z}{n}\frac{pc}{2} = -\left(\frac{\alpha Z}{n}\right)^2\frac{mc^2}{2}. \tag{3.9}$$

The first chain of equalities will be explained in the next section 6, in particular as concerns the evident link of pc and mc^2 with E_0. Note here that introducing α to express the quantum energy levels implies defining the De Broglie

momentum as a corollary, in agreement with (2.36) and (2.37): appears interesting that the energy levels ε_{el} of the system are linked to the kinetic energy pc of the electron moving along the circumference of radius Δr via the coefficient $\alpha Z/2n$. On the one hand, this result emphasizes the electromagnetic character of the interaction between electron and nucleus; comprehensibly Δr is proportional to α^{-1}, as the coupling constant determines the force exerted in an interaction, *i.e.* the greater α the smaller Δr. On the other hand, it also appears that the key role of the quantum uncertainty in determining the allowed energy levels (3.7) also evidences the kind of interaction itself.

These results confirm that the operator formalism and the uncertainty equations are equivalent in describing the quantum systems. As concerns the spin, the paper [6] [7] has shown that it can be inferred without additional hypotheses from the quantization itself. Simply rewriting identically
$M^2 = (l+1/2)^2\hbar^2 - (\hbar/2)^2$, one finds

$$M^2 + (\hbar/2)^2 + \left(l+\frac{1}{2}\right)\hbar^2 = M^2 = \left(l+\frac{1}{2}\right)\left[\left(l+\frac{1}{2}\right)+1\right]\hbar^2 \tag{3.10}$$

after having added $(l+1/2)\hbar^2$ at both sides. Trivial manipulations of the initial $\boldsymbol{M}$ exposed in the quoted paper show that

$$\mathcal{M}^2 = J(J+1)\hbar^2, \quad J = l_{or} + l_s, \quad l_s = \frac{1}{2} \tag{3.11}$$

and that in general these consideration introduce the spin component $l'/2\hbar$; being of course l' an arbitrary integer, the quantum uncertainty implies itself the existence of bosons and fermions. No information is necessary about Δr and Δp_r, which in effect are unknown and unknowable because of the quantum uncertainty.

Besides its inherent worth, the hydrogenlike model has been explicitly quoted here because it also provides useful information about the characteristic lengths in the atom, the first of which is of course the Bohr radius inferred in (3.7). The first powers of α scale further significant lengths starting from this radius, whose essential form reads $r_B = \hbar^2/e^2 m$ as a function of the fundamental constants. One infers the following lengths

$$r_B = \frac{\hbar^2}{e^2 m}, \quad r_B\alpha = \lambda_C = \frac{\hbar}{mc}, \quad r_B\alpha^2 = r_e = \frac{e^2}{mc^2}, \quad r_B\alpha^3 = r_N = \frac{e^8}{\hbar^3 mc^5} \tag{3.12}$$

whose values are

$$r_B \approx 5.3\times10^{-9}\ \text{cm}, \quad \lambda_C \approx 3.6\times10^{-11}\ \text{cm},$$
$$r_e \approx 2.8\times10^{-13}\ \text{cm}, \quad r_N \approx 2.4\times10^{-15}\ \text{cm}.$$

the Bohr radius scales r_B down to λ_C, electron Compton length, and then to r_e, classical electron radius. Further lengths, shorter and shorter, will be introduced later to extend these definitions and sketch short range nuclear forces. Indeed the fact of having found these well known specific lengths suggests that even the fourth position should reasonably have its own physical meaning at the

smaller α^3 scale too; if so, r_N can be related to nothing else but the scale of lengths within the atomic nuclei, whose sizes in effect are known to fall between $\lesssim 10^{-13}$ cm (proton of hydrogen) to $\sim 10^{-12}$ cm (heavier nuclei).

3.3. Velocity Dependence of Mass

Owing to (2.28), $\varepsilon_2 - \varepsilon_1 = vp_2 - vp_1$ reads $\varepsilon_2 - vp_2 = const = \varepsilon_1 - vp_1$; so (2.35) yields

$$\varepsilon - vp = \frac{mc^2 - mv^2}{\sqrt{1 - v^2/c^2}} = const.$$

This result is more expressively rewritten in the form $mc^2\sqrt{1 - v^2/c^2} = const$ fulfilled by

$$m_0 c^2 = const, \quad m_0 = m\sqrt{1 - v^2/c^2}, \quad m = m(v). \tag{3.13}$$

The physical meaning of this result, the dependence of m on v via the constant m_0, will be clarified soon below. In the following are introduced three interesting ways to implement further the Equations (2.25) and (2.34), to show in particular how results of special relativity are obtainable regarding the local dynamical variables ε and p as random and unknown values defined in the respective quantum uncertainty ranges $\delta\varepsilon$ and δp.

3.4. Quantum Correction to Special Relativity

The strategy is still that followed to find (2.15) and to infer (2.31) and (2.32) from (2.29). Consider the Equation (2.25) and (2.26) rewritten in the particular case $v = c$ as

$$\delta\varepsilon^* = c\delta p, \quad \varepsilon^* = pc, \quad \varepsilon_1^* \le \varepsilon^* \le \varepsilon_2^*, \quad p_1 \le p \le p_2; \tag{3.14}$$

the former equation defines the maximum energy range $\delta\varepsilon^*$ allowed to the local ε^* consistently with the given momentum range δp allowed to any local p. Here energy and momentum ranges are linked each other, whereas in fact they were independent in the Equation (2.28) owing to the arbitrariness of v; so, the upper limit allowed for v implies an upper limit to the size of $\delta\varepsilon^*$ compliant with any possible δp. Anyway this latter is arbitrary; thus both energy and momentum ranges are in fact arbitrary as well, but now correlated. The second position emphasizes the local dynamical variables ε^* and p allowed in the respective ranges. The fact that (3.14) is not mere formal way of rewriting (2.25) but contains additional physical information, is easily proven: multiplying side by side both (3.14) one finds $\delta(\varepsilon^*)^2 = c^2\delta(p)^2$ *i.e.*, as in (2.14), $\varepsilon^{*2} + const' = (pc)^2 + const''$. So the second (2.34) is instantly inferred via the correlation between δp and $\delta\varepsilon^*$ through c.

However just the fact that the (3.14) appears suitable to be directly linked to (2.34) rises a quantum problem. Replace (2.36) in the Equation (2.34) via the positions $\varepsilon^* = h\nu^*$ and thus $p = h/\lambda^*$, being $\lambda^* = c/\nu^*$ in fact implied itself by the third (2.36) for $v = c$ too. Then $(h\nu^*)^2 = (hc/\lambda^*)^2 + (mc^2)^2$ requires

$m=0$. On the one hand nothing hinders in principle to express (2.34) via the corresponding quantum energy and momentum, in agreement with the dual wave/corpuscle character of matter. On the other hand (2.34), as written, seems inadequate to allow both $m\neq 0$ and (2.36). It is reasonable to expect that further terms to be included in (2.34) could overcome this difficulty: the attempt to generalize the standard result of the early special relativity is not only legitimate but also necessary.

The subsection 3.2 has been explicitly enclosed in the present exposition to emphasize that the quantum eigenvalues leave out any information about the range sizes; the Equations (3.3) to (3.12) elucidate this assertion. In other words the previous results obtained implementing $\delta p = p_2 - p_1$ could have been identically obtained considering any other $\delta p' = p_2' - p_1'$, as the range boundary coordinates are inessential as concerns the quantized eigenvalues of angular momentum and energy. The same holds of course even implementing a linear combination of momentum ranges, e.g. $\delta p'' = p_2'' - p_1'' = a\delta p + b\delta p'$ via the constant arbitrary coefficients *a* and *b*. This means that the local value p defined by $p_1 \le p \le p_2$ could be identically replaced by any p' defined by $p_1' \le p' \le p_2'$; the same holds of course for any p'' defined by $ap_1 + bp_1' \le p'' \le ap_2 + bp_2'$: the only essential requirement is that any range sizes δx and δp fulfill (3.1), whatever the boundary values might be. Now let us introduce in the relativistic domain this peculiarity of the quantum world. This means that the local values of pc and ε^* defined the respective ranges (3.14) can be replaced by linear combinations of momentum and energy.

The chance of demonstrating the actual effectiveness of this reasoning has heuristic worth in demonstrating the close connection between quantum and relativistic theories.

In practice, to generalize the standard relativistic result (2.34), implement again the first (3.14) with the same steps from (2.29) to (2.31) and then to (2.32), but rewriting the third and fourth positions as

$$\varepsilon_1^* \le \varepsilon_0 + \sigma_\varepsilon \varepsilon^* + \sigma_{o\varepsilon}\varepsilon^{*2} \le \varepsilon_2^*, \quad p_1 \le p_0 + \sigma_p p + \sigma_{op} p^2 \le p_2$$

σ_ε and σ_p are dimensionless arbitrary constants, *a* and *b* are arbitrary constants having physical dimensions $mass^{-1}$ and expressing conveniently the σ_o coefficients. The equations to be implemented are thus

$$\delta\varepsilon^* = c\delta p, \quad \sigma_\varepsilon \varepsilon^* - \frac{b}{c^2}\varepsilon^{*2} = \sigma_p pc - ap^2 + \epsilon_a, \tag{3.15}$$

where in fact $a \neq b \neq 0$ extend the previous procedure simply introducing additional ε^{*2} and p^2 terms with respect to (3.14) while however keeping a physical meaning still compliant with that of ranges $\delta\varepsilon^*$ and δp, as it appears via dimensional considerations. In other words, the second (3.15) still has the usual form $\delta\mathcal{E}^* = \delta(\mathcal{P}c) + const$. Repeat therefore exactly the same procedure just outlined to merge (3.14), *i.e.* multiply side by side the second and first (3.15) with the *a* and *b* terms exchanged of place; omitting for simplicity of notation the asterisk, one finds

$$\sigma_\varepsilon \varepsilon\delta\varepsilon + ap^2\delta\varepsilon = bp\varepsilon\delta p + \sigma_p pc^2\delta p + \delta\left(\epsilon_a pc\right), \tag{3.16}$$

which yields

$$\frac{1}{2}\delta\left(\sigma_\varepsilon\varepsilon^2\right) = \frac{1}{2}\delta\left(\sigma_p\left(pc\right)^2\right) - ap^2\delta\varepsilon + bp\varepsilon\delta p + \delta\left(\epsilon_a pc\right)$$

and then

$$\frac{1}{2}\delta\left(\sigma_\varepsilon\varepsilon^2\right) = \frac{1}{2}\delta\left(\sigma_p\left(pc\right)^2\right) - \delta\left(ap^2\varepsilon\right) + \delta\left(\epsilon_a pc\right), \quad b = -2a. \tag{3.17}$$

Hence, reasoning as before, this result implies:

$$\sigma_\varepsilon\varepsilon^2 + const' = \sigma_p\left(pc\right)^2 + const'' - 2ap^2\varepsilon + const_1 + \epsilon_a pc + const_2. \tag{3.18}$$

As hold for (3.18) the same considerations carried out for (2.34), because also the new terms $ap^2\varepsilon$ and $\epsilon_a pc$ vanish for $p \to 0$. Merging the constants, one finds

$$\begin{aligned} &\sigma_\varepsilon\varepsilon^2 = \sigma_p\left(pc\right)^2 - 2ap^2\varepsilon + \epsilon_a pc + \left(mc^2\right)^2 + \epsilon_0^2 \\ &const'' - const' = \left(mc^2\right)^2, \quad const_2 + const_1 = \epsilon_0^2. \end{aligned} \tag{3.19}$$

The notation $\left(mc^2\right)^2$ has been kept resulting from the primed constants like in (2.32), in order that this equation reduces to (2.34) in the particular case $a = 0$ and $\epsilon_a = 0$. Of course the constants σ_ε and σ_p can be included in the respective energies; *i.e.* with the positions

$$\begin{aligned} &p' = \sqrt{\sigma_p}\,p, \quad \varepsilon' = \sqrt{\sigma_\varepsilon}\,\varepsilon, \quad a' = \frac{a}{\sigma_p\sqrt{\sigma_\varepsilon}}, \\ &\epsilon_a' = \frac{\epsilon_a}{\sqrt{\sigma_p}}, \quad m'c^2 = \sqrt{\left(mc^2\right)^2 + \epsilon_0^2} \end{aligned}$$

(3,19) reads

$$\varepsilon'^2 = (p'c)^2 + \left(m'c^2\right)^2 - 2a'p'^2\varepsilon' + \epsilon_a' p'c. \tag{3.20}$$

As expected, thanks to the higher order terms ε^2 and p^2 in (3.16) one finds again an equation like the second (2.34) plus two additional terms $-2a'p'^2\varepsilon'$ and $\epsilon_a' p'c$ not present in the standard special relativity. The quantum correction terms are negligible in (3.20) if $-2a'p'^2\varepsilon' + \epsilon_a' p'c \ll \varepsilon'^2$, *i.e.* if

$$-2\frac{p'^2/m_a}{\varepsilon'} + \frac{\epsilon_a' p'c}{\varepsilon'^2} \ll 1, \quad m_a = \frac{1}{a'}, \tag{3.21}$$

then ϵ_a and m_a fix the scale where the quantum correction plays a significant role. Moreover, if in particular $\epsilon_a' \ll 2a'p'\varepsilon'/c$, then is effective only the term $-2a'p'^2\varepsilon'$ in (3.20). These points deserve attention.

First of all, replace $\varepsilon' = h\nu'$ and $p' = h/\lambda'$; being again $\left(h\nu'\right)^2 = \left(hc/\lambda'\right)^2$, (3.20) reads

$$\left(m'c^2\right)^2 = 2a'\frac{h^2}{\lambda'^2}h\nu' - \epsilon_a'\frac{hc}{\lambda'} = h\nu'\left(2a'\frac{h^2}{\lambda'^2} - \epsilon_a'\right) = h\nu'\left(2m_ac^2\frac{\lambda_a^2}{\lambda'^2} - \epsilon_a'\right)$$

$$\lambda_a = \frac{h}{m_ac}, \quad m_a \geq \frac{\epsilon_a}{2}$$

hence (3.20) is compatible with the quantum condition (2.36) even for $m' \neq 0$, whereas $m' = 0$ is also possible if in particular vanishes the quantity in parenthesis. The last inequality holds for $\lambda' = \lambda_a$, because in general $\lambda' \geq \lambda_a$, whatever λ' might be.

Moreover rewrite the second (3.15) with the help of (2.36) as

$$\sigma_\varepsilon h\nu + \frac{2}{m_a c^2}(h\nu)^2 = \sigma_p h\nu - \frac{h^2}{m_a \lambda^2} + \epsilon_a, \quad \nu = \frac{c}{\lambda}. \tag{3.22}$$

To recognize the physical meaning of this equation under the condition that m_a is the constant mass defined by the Equation (3.21), useful positions are:

$$\frac{h^2}{m_a \lambda^2} = m_a c^2 \frac{\lambda_a^2}{\lambda^2}, \quad h\nu = r m_a c^2, \quad \lambda = n\lambda_a, \quad r = n'\sigma_\varepsilon. \tag{3.23}$$

The first one is an identity, whose left hand side is simply rewritten introducing the Compton length λ_a of m_a. The second one is a formal way to link $h\nu$ and $m_a c^2$ via the parameter r to be defined. The third one regards λ of (3.22) as an integer multiple of λ_a; in fact the conceptual difference between p defined by (2.23) and $p = h/\lambda$ is that δx is a mere space range that can take in principle any value, the wavelength λ requires introducing quantized lengths $n\lambda$, which explains why anyway the quantization must be introduced via h in $n\hbar/\delta x$ of (3.1). Although this idea is introduced here as a reasonable input, a previous paper [6] has shown that in effect a huge amount of interesting results is accordingly obtainable. The fourth one will be explained after having replaced the first three (3.23) into (3.22), which reads

$$rh\nu + \sigma_\varepsilon \frac{h\nu}{2} = \frac{m_a c^2}{2}\left(r\sigma_p - \frac{1}{n^2}\right) + \frac{\epsilon_a}{2}. \tag{3.24}$$

For sake of generality the notation emphasizes that n' defining r is not necessarily coincident with n defining the ratio λ/λ_a. Is attracting the fourth position (3.23) with n' arbitrary integer that expresses the left hand side as σ_ε times the harmonic oscillator energy; indeed (3.24) becomes

$$n'h\nu + \frac{h\nu}{2} = \frac{m_a c^2}{2}\left(n'\sigma_p - \frac{1}{\sigma_\varepsilon n^2}\right) + \frac{\epsilon_a}{2\sigma_\varepsilon}.$$

Now it is necessary to express the fact that m_a is a constant, which in fact means regarding the quantum numbers n and n' as proportionality factors linking $m_a c^2$ and $h\nu$. The limit $n' \to \infty$ yields $n'h\nu = \left(n'\sigma_p/2\right)m_a c^2$ and thus, by comparison with the second and fourth positions (3.23), $\sigma_p/2 = \sigma_\varepsilon$; so the last equation reads

$$n'h\nu + \frac{h\nu}{2} = \frac{m_a c^2}{2}\left(2n'\sigma_\varepsilon - \frac{1}{\sigma_\varepsilon n^2}\right) + \frac{\epsilon_a}{2\sigma_\varepsilon}, \quad \sigma_p = 2\sigma_\varepsilon. \tag{3.25}$$

It appears that if $m_a = 0$, then $\epsilon_a/2\sigma_\varepsilon$ is the energy of harmonic oscillator of frequency ν. Analogous conclusion holds if $\sigma_\varepsilon^2 = 1/2(n'n)^2$, in which case $\epsilon_a/2 = n'h\nu_0 + h\nu_0/2$ with $\nu_0 = \nu/\sqrt{2}n'n$; as both n and n' are arbitrary integers, $n'n$ must be regarded as a new arbitrary integer itself and thus anyone

among the numbers already implied by n^2 and n'^2. So ν is an arbitrary multiple of the fundamental frequency ν_0. The fourth position also allows expressing (3.24) as a function of quantum numbers only

$$n'^2\sigma_\varepsilon + \frac{n'\sigma_\varepsilon}{2} = \frac{1}{2}\left(2n'^2\sigma_\varepsilon - \frac{1}{\sigma_\varepsilon n^2}\right) + \frac{1}{\sigma_\varepsilon}\frac{\epsilon_a}{m_a c^2},$$

which yields

$$\frac{\epsilon_a}{m_a c^2} = \frac{1}{2}\left(n'\sigma_\varepsilon^2 + \frac{1}{n^2}\right). \tag{3.26}$$

Next, inserting the positions (3.23) in (3.24) trivial manipulations yield

$$\left(\frac{m'}{m_a}\right)^2 = n'\sigma_\varepsilon\left(\frac{2}{n^2} - \frac{\epsilon_a}{m_a c^2}\right) = n'\sigma_\varepsilon\left(\frac{3}{2n^2} - \frac{1}{2}n'\sigma_\varepsilon\right)$$

Clearly ν appearing in (3.24) implied by $a' \neq 0$ and $\epsilon_a \neq 0$ is different from ν^* previously found consistent with $m = 0$ only; (3.20) skips this restriction.

It is known that (3.20) is a valuable equation of quantum gravity able to solve three cosmological paradoxes [8]. It is hard to guess what has to do the cosmology in this conceptual frame; but in fact this is not the correct way to regard this equation. Rather it is correct to say that the additional terms due to $a \neq 0$ and $b \neq 0$ add a quantum correction to the standard relativistic formula, actually having quantum character itself being inferred from (2.30) and (2.31); then, once having acknowledged this result, further studies also acknowledge that this correction has valuable cosmological implications as well.

3.5. Operator Formalism

The subsection 3.2 has shown that the corpuscular approach to quantum mechanics provides sensible results in agreement with the wave formalism. This subsection shows that also the wave formalism enters in the conceptual frame hitherto exposed. Implement the quantum relativistic Equation (3.20), noting that

$$\varepsilon''^2 = \left(mc^2\right)^2 + \left(pc\right)^2 = \left(mc + ip\right)\left(mc - ip\right)c^2, \quad \varepsilon''^2 = \varepsilon^2 + 2ap^2\varepsilon - \varepsilon_a pc. \tag{3.27}$$

Admitting that even the single factors at the right hand side have physical meaning, it is possible to introduce imaginary momentum $\mathcal{P}$ and energy $\mathcal{E}$ in agreement with the early positions (2.9) and (2.12); the momentum and energy equations take indeed the forms

$$\mathcal{P} = \mp i\hbar\frac{\delta\psi}{\delta x}, \quad \mathcal{E} = \pm i\hbar\frac{\delta\psi}{\delta t}, \quad \mathcal{E}^2 = \varepsilon''^2, \tag{3.28}$$

being simply required

$$k_1 = \mp ik_0, \quad k_2 = \pm ik_0, \quad k_0 = \hbar.$$

The correct correspondence of signs in (3.28) is indeed such that $k_1 + k_2 = 0$

and thus $\mathcal{P}v+\mathcal{E}=0$, in agreement with (2.17). Whatever the specific form of ψ might be, replacing p and ε'' of (3.27) with the new definitions (3.28) one finds

$$\left(mc+\hbar\frac{\delta\psi}{\delta x}\right)\left(mc-\hbar\frac{\delta\psi}{\delta x}\right)=-\left(\frac{\hbar}{c}\frac{\delta\psi}{\delta t}\right)^2; \tag{3.29}$$

in this way the Equation (3.27) turns again into a real form. Introduce now the positions

$$\frac{\delta^2\psi}{\delta t^2}=\pm\left(\frac{\delta\psi}{\delta t}\right)^2,\quad \frac{\delta^2\psi}{\delta x'^2}=\pm\left(\frac{\delta\psi}{\delta x'}\right)^2, \tag{3.30}$$

being

$$\delta x'=c\delta t, \tag{3.31}$$

which are justified soon below. In principle the positions (3.30) are compatible each other because δt and $\delta x'$ are arbitrary finite ranges that can be determined in order to fulfill both equations. Note that the more general positions $\delta^2\psi/\delta t^2=q\left(\delta\psi/\delta t\right)^2$ and $\delta^2\psi/\delta x^2=q\left(\delta\psi/\delta x\right)^2$, with q arbitrary factor, would have been in principle reasonable and possible; however q could be included in m of (3.29), so its specific value is inessential; more important are instead the signs of q, as it will appear shortly. Taking the upper signs (3.30), (3.29) reads as follows

$$\left(\frac{mc}{\hbar}\right)^2=\frac{\delta^2\psi}{\delta x^2}-\frac{1}{c^2}\frac{\delta^2\psi}{\delta t^2}-\frac{1}{v^2}\frac{\delta^2\psi}{\delta t^2}+\frac{1}{v^2}\frac{\delta^2\psi}{\delta t^2}.$$

The addend $\delta^2\psi/v^2\delta t^2$ has been summed and subtracted at the right hand side in order to split this equation as follows

$$\frac{\delta^2\psi}{\delta x^2}-\frac{1}{v^2}\frac{\delta^2\psi}{\delta t^2}=0,\quad \left(\frac{mc}{\hbar}\right)^2=-\frac{1}{c^2}\frac{\delta^2\psi}{\delta t^2}+\frac{\delta^2\psi}{\delta x^2},\quad \delta x^2=v^2\delta t^2 \tag{3.32}$$

the first equation is still the precursor (2.3) of the D'Alembert Equation (2.4) and is clearly an identity $0=0$ owing to $\delta x=v\delta t$; the second equation only involves explicitly m through its reciprocal Compton length. To show why, and how to implement further these equations, note that the first couple of Equations (3.32) merged together yields

$$\hbar^2\frac{\delta^2\psi}{\delta t^2}=\hbar^2v^2\frac{\delta^2\psi}{\delta x^2},\quad \hbar^2\frac{\delta^2\psi}{\delta t^2}=\frac{\left(mcv\right)^2}{1-\left(v/c\right)^2},$$

so that one finds

$$\hbar^2\frac{\delta^2\psi}{\delta\left(x/c\right)^2}=\frac{\left(mc^2\right)^2}{1-\left(v/c\right)^2},\quad \hbar^2\frac{\delta^2\psi}{\delta(ct)^2}=\frac{\left(mv\right)^2}{1-\left(v/c\right)^2}; \tag{3.33}$$

then the position

$$\delta t'=\frac{\delta x}{c} \tag{3.34}$$

yields by consequence

$$\hbar^2\frac{\delta^2\psi}{\delta t'^2}=\frac{\left(mc^2\right)^2}{1-\left(v/c\right)^2},\quad \hbar^2\frac{\delta^2\psi}{\delta x'^2}=\frac{\left(mv\right)^2}{1-\left(v/c\right)^2}. \tag{3.35}$$

Reasonably therefore the positions (3.30) imply (3.32), which yield (3.35) in agreement with (2.35). Hence

$$\pm\hbar^2\left(\frac{\delta\psi}{\delta t}\right)^2=\frac{\left(mc^2\right)^2}{1-\left(v/c\right)^2},\quad \pm\hbar^2\left(\frac{\delta\psi}{\delta x'}\right)^2=\frac{\left(mv\right)^2}{1-\left(v/c\right)^2}. \tag{3.36}$$

It is useful to introduce now the local limit $\delta\to\partial$ of the Equations (3.30); once more, the resulting equations take then physical meaning via this limit condition, which introduces an appropriate function $\chi=\chi\left(x,t\right)$ defined by the local properties of ψ; also now indeed the consequent position $\psi\to\chi$ turns the Equations (3.30) into the respective differential equations that represent the actual behavior of the particle. So

$$\frac{\partial^2\chi}{\partial t^2}=\pm\left(\frac{\partial\chi}{\partial t}\right)^2,\quad \frac{\partial^2\chi}{\partial x'^2}=\pm\left(\frac{\partial\chi}{\partial x'}\right)^2,\quad \chi=\chi\left(x't\right), \tag{3.37}$$

obtained equating the left hand sides of (3.35) and (3.36), are both fulfilled by

$$\chi=\mp\log\left(\xi\left(t-t_o\right)\left(x'-x_o\right)+\eta\right)+\zeta,\quad \xi=\frac{1}{\delta t_0\delta x_0}, \tag{3.38}$$

being ξ, ζ and η three arbitrary constants. The second equation remarks through the constants x_0 and t_0 that the physical dimensions of ξ are $\left(space\times time\right)^{-1}$. This equation, which emphasizes the space time range $\left(t-t_o\right)\left(x'-x_o\right)$ already found in (1.7), will be also implemented in the shortened form

$$\chi=\mp\log\left(\xi tx'+\eta\right)+\zeta. \tag{3.39}$$

An interesting corollary of (3.38) follows from $\chi=\zeta\mp\left(\log\left(\delta t/\delta t_0\right)+\log\left(\delta x/\delta x_0\right)\right)$ valid for $\eta=0$. As $\delta x/\delta x_0=m\delta x/m\delta x_0=C_0/C$, strictly speaking $C=m/\delta x$ and $C_0=m/\delta x_0=const$ are linear mass densities in the present one dimensional model; of course in a realistic four dimensional space time C and C_0 must be intended as usual mass densities, as emphasized in (2.21) and in the next (4.13). Consider χ during a fixed time range, so that $\delta t/\delta t_0$ is regarded as a time constant; then

$$\chi=const\mp\log\left(C/C_0\right),\quad const=\zeta\mp\log\left(\delta t/\delta t_0\right). \tag{3.40}$$

Although χ is dimensionless, appropriate units clarify its physical meaning: multiplying for example both sides by the energy kT already introduced preliminarily in (2.22) but to be defined shortly later, one finds

$$\chi kT=\mp\left(\mu+\mu_0\right),\quad \mu=kT\log\left(C/C_0\right),\quad \mu_0=const kT. \tag{3.41}$$

Hence χ is proportional to the chemical potential μ an arbitrary space time constant μ_0 apart.

Eventually note that $\delta x'$ and $\delta t'$ defined in (3.31) and (3.34) fulfill $\delta x/\delta t' = \delta x'/\delta t$; this equation is also fulfilled putting $\delta x' = \delta x/\gamma$ and $\delta t' = \gamma\delta t$ with γ arbitrary factor, in which case it reduces to identity. In particular γ could be the Lorentz factor, in fact introduced in (2.34); so one infers that

$$\delta x\delta t = \delta x'\delta t'. \tag{3.42}$$

is a relativistic invariant in different inertial reference systems. Moreover, dividing both sides by ν, write the identity

$$\frac{\delta x'}{\gamma\nu'} = \frac{\delta x}{\nu}, \quad \nu' = \frac{\nu}{\gamma}, \quad \nu' = \frac{1}{\delta t'}, \quad \nu = \frac{\gamma}{\delta t'}. \tag{3.43}$$

With γ equal in particular to the Lorentz factor $\sqrt{1-v^2/c^2}$, as suggested by δt and $\delta t'$, it is possible to regard the frequency ν' as that related to ν in different inertial reference systems R' and *R*. Moreover it is also possible to regard ν as the frequency recorded by an observer moving in *R* at rate *v* with respect to the frequency ν_o emitted by the source. Noting that $\nu_o c > \nu_o v$, let ν be such that $\nu_o c = \nu_o v + \nu c$ so that $\nu = \nu_o\left(1-v/c\right)$. Thus replacing in the second (3.43) one finds

$$\nu' = \nu_o\frac{1-v/c}{\gamma}.$$

This equation is nothing else but the Doppler shift of frequencies reciprocally moving at rate *v* along their sight line.

As (3.39) shows that both signs of (3.30) are admissible, consider now separately either sign of the Equations (3.36).

1) The negative sign yields

$$\pm i\hbar\frac{\delta\psi}{\delta t} = \varepsilon, \quad \mp i\hbar\frac{\delta\psi}{\delta x'} = p, \quad \varepsilon = \frac{mc^2}{\sqrt{1-\left(v/c\right)^2}}, \quad p = \frac{mv}{\sqrt{1-\left(v/c\right)^2}}, \tag{3.44}$$

which of course confirm (3.28); so, for $\delta\to\partial$ and thus $\psi\to\varphi$, the local limits read

$$\pm i\hbar\frac{\partial\varphi_\varepsilon}{\partial t} = \varepsilon, \quad \mp i\hbar\frac{\partial\varphi_p}{\partial x'} = p, \quad \varphi_\varepsilon = \varphi_\varepsilon\left(x',t\right), \quad \varphi_p = \varphi_p\left(x',t\right). \tag{3.45}$$

In these equations the physical meaning of $\pm p$ is immediately evident: *p* is actually a component of the vector *p* along the *x* axis on which is defined δx. Instead $\pm\varepsilon$ is more interesting, as it indicates the existence of states of negative energy.

Note that holds for (3.44) and (3.45) the same remark carried out for (2.3) and (2.4): also now the left hand side of (3.44) are in fact not calculable explicitly because are indeterminate not only $\delta\psi$ but also δx and δt. However are in principle calculable their limits for $\delta\to\partial$. Now also the relativistic quantities (3.45) come from and are compliant with the non-real and non-local (3.44). In effect even the Equations (3.45) bring back to the early postulates of the old quantum mechanics, despite obtained from the relativistic (3.27): this is immediately evident via the following positions:

$$\varphi = \log(\Psi), \quad \mp i\hbar \frac{\partial \varphi}{\partial x'} = \mp \frac{i\hbar}{\Psi} \frac{\partial \Psi}{\partial x'} = p, \quad \pm \frac{i\hbar}{\Psi} \frac{\partial \Psi}{\partial t} = \varepsilon, \quad \Psi = \Psi(x't/x_0 t_0). \quad (3.46)$$

In this case Ψ has the same analytical form of χ. This point deserves further attention.

The relativistic equations (3.44) are implied by the invariant *xt* of (3.39), as shown in (3.42); obviously, replacing *xt* with another function $\psi_{nr} \neq \chi$, the Equation (3.37) would not hold. By consequence, in this case *p* and ε in (3.44) would be reasonably replaced by non relativistic quantities p_{nr} and ε_{nr} numerically different but having however an analogous physical meaning by dimensional reasons: the notation emphasizes the non-relativistic character of their classical approximation. Replace thus $x't$ of χ in Equation (3.39) with any function $\psi_{nr} = \psi_{nr}(x,t)$, putting for example $\psi_{nr} = \psi_x(x)\psi_t(t)$: with this Newtonian position where time and space are independent entities defining distinct dynamical variables of classical mechanics, the Equations (3.39) turn respectively into

$$\chi_{nr} = \pm \log(\xi \psi_{nr} + \eta) + \zeta, \quad \psi_{nr} = \psi_x \psi_t;$$

for simplicity of notation, the symbols of the constants have been kept unchanged. Hence the first two Equations (3.44) turn into

$$\frac{i\hbar}{\xi \psi_{nr} + \eta} \frac{\delta \psi_{nr}}{\delta t} = \pm \varepsilon_{nr}, \quad \frac{i\hbar}{\xi \psi_{nr} + \eta} \frac{\delta \psi_{nr}}{\delta x} = \mp p_{nr};$$

Put eventually $\eta = 0$, for example assuming η proportional to c^{-1}; as *c* is infinite in classical physics, these equations take the well known form

$$\pm i\hbar \frac{\delta \psi_{nr}}{\delta t} = \varepsilon'_{nr} \psi_{nr}, \quad \mp i\hbar \frac{\delta \psi_{nr}}{\delta x} = p'_{nr} \psi_{nr}, \quad p'_{nr} = \xi p_{nr}, \quad \varepsilon'_{nr} = \xi \varepsilon_{nr}.$$

Clearly these expressions, suggested by the outcomes (2.23), agree with (3.28) and specify via the limit $\delta \to \partial$ which function is actually involved by the change symbol δ. Hence

$$\pm i\hbar \frac{\partial \psi_{ef}}{\partial t} = \varepsilon' \psi_{ef}, \quad \mp i\hbar \frac{\partial \psi_{ef}}{\partial x} = p' \psi_{ef}, \quad p' = \xi p, \quad \varepsilon' = \xi \varepsilon; \quad (3.47)$$

these results are the well known equations of the old quantum theory; the subscript "*ef*" stands for "eigenfunction". The modern quantum physics was born postulating these crucial equations, whence the importance of having found them as corollaries: the present theoretical approach brings back just to early formulation of quantum mechanics and its basic assumptions.

2) Consider now also the plus sign of (3.36), which yields

$$\hbar \frac{\delta \psi}{\delta t} = \pm \frac{mc^2}{\sqrt{1-(v/c)^2}}, \quad \hbar \frac{\delta \psi}{\delta x'} = \mp \frac{mv}{\sqrt{1-(v/c)^2}}. \quad (3.48)$$

The Equations (3.64) correspond to the Equations (3.28), whereas the Equations (3.48) read $\hbar/(\pm \varepsilon \delta t) = \hbar/(\pm p \delta x)$, *i.e.* $\pm \varepsilon \delta t = \pm p \delta x$; this expression is a

particular case of the Equations (3.1) regarding $\delta p = \pm p - 0$ and $\delta\varepsilon = \pm\varepsilon - 0$, where the reference boundaries of these ranges coincide with zero momentum and zero energy. Considering indeed the particular case $\pm\varepsilon_0\delta t = \pm p\delta x_0$ and subtracting side by side one finds again the expected more general result $\pm(\varepsilon - \varepsilon_0)\delta t = \pm(p - p_0)\delta x$ in agreement with (3.1) as $\delta\varepsilon = \varepsilon - \varepsilon_0$ and $\delta p = p - p_0$. So (3.48) link the operator formalism (3.28) and (3.67) to the uncertainty equations (3.1) and their relativistic implications (3.44).

Note eventually that the Equations (3.37) are well known in the operator formalism $\hat{p} = -i\hbar\partial/\partial x$, where in effect it is taken for granted that $\hat{p}^2 = -\hbar^2\partial^2/\partial x^2$; indeed (3.37) express nothing else but $\hat{p}^2 = (-i\hbar\partial/\partial x)^2 = -\hbar^2\partial^2/\partial x^2$ previously inferred from (3.30).

In conclusion this simple approach has found the operator formalism and contextually the uncertainty equation, both compatible with relativistic concepts. These outcomes have several further corollaries, the most relevant of which are shortly summarized in the following. Final remark to close this section. The range products $\delta x\delta p$ and $\delta\varepsilon\delta t$ characterize the quantum uncertainty (3.1), whereas the product $\delta x\delta t$ characterizes the invariant space time (3.38): the connection between quantum physics and relativistic physics is comprehensible corollary if space and time are mixed in either way. In this respect, what about the other mixed term $\delta\varepsilon\delta p$ also possible in alternative to $\delta x\delta p$ of (3.1)? According to (3.1) it yields

$$\delta\varepsilon\delta p = \hbar\frac{\delta\varepsilon}{\delta x} = \hbar F \tag{3.49}$$

the "new" quantity *F*, so far not explicitly concerned but only anticipated in Section 3.2 for exposition purpose only, takes in this way justification and physical meaning, it is usually known as force. The concept of pressure and energy density also follow from this result dividing both sides by the arbitrary surface Δx^2

$$\hbar\frac{\delta\varepsilon}{\delta x\Delta x^2} = \hbar\frac{F}{\Delta x^2}. \tag{3.50}$$

4. Some Classical Corollaries

Are concerned in this section several interesting outcomes still hidden in the approach hitherto outlined.

4.1. The Fermat and Maupertuis Principles

The key equations are (3.1) and (2.25). Consider an arbitrary time range Δt during which one particle moves between two coordinates x_1 and x_2 defining the total path Δx. It is possible to write

$$\Delta t = t_2 - t_1, \quad \delta\Delta t = 0;$$

the second position expresses that the time interval is arbitrary but fixed by definite time boundaries within which hold the following considerations. Since Δx traveled by the particle can be imagined as the sum of elementary ranges δx corresponding to elementary time steps δt_k, write $\Delta t = \sum\delta t_k$. Being both time

and space steps arbitrary, it is possible to replace the sum with an integral and write the following chain of equations with the help of (2.25)

$$\Delta t = \int_{t_1}^{t_2} \delta t = \int_{t_1}^{t_2} \frac{n\hbar}{\delta\varepsilon} = \int_{t_1}^{t_2} \frac{\delta p \delta x}{\delta\varepsilon} = \int_{t_1}^{t_2} \frac{\delta x}{v}.$$

Hence, integrating along an element $\mathrm{d}x$ of trajectory for $\delta \to \partial$,

$$\delta\Delta t = 0 = \delta \int_{t_1}^{t_2} \frac{\mathrm{d}x}{v}$$

so the Fermat principle, also expressible identically as $\delta\int \mathrm{n}\mathrm{d}x = 0$ with $\mathrm{n} = c/v$, is actually a straightforward corollary of the uncertainty equation.

In an analogous way one finds the Maupertuis principle. Calculate $\delta p \delta x$ for $\delta x \to \mathrm{d}x$ and $\delta\varepsilon\delta t$ for $\delta t \to \mathrm{d}t$; in this way, even considering vanishingly small range sizes still holds the concept of local velocity v_l, *i.e.* $\delta x/\delta t \to \mathrm{d}x/\mathrm{d}t = v_l$. Considering the coordinates $x_1 = x(t_1)$ and $x_2 = x(t_2)$ and integrating both sides, one finds according to (3.1)

$$\int_{x_1}^{x_2} \delta p \mathrm{d}x = \int_{t_1}^{t_2} \delta\varepsilon \mathrm{d}t.$$

The right hand sides involves $\delta\varepsilon_1 = \delta\varepsilon(t_1)$ and $\delta\varepsilon_2 = \delta\varepsilon(t_2)$. Suppose now that $\delta\varepsilon = 0$ because ε is constant itself; then, being $\delta p = p_2 - p_1$ by definition, one finds

$$\int_{x_1}^{x_2} \delta p \mathrm{d}x = \int_{x_1}^{x_2} p_1 \mathrm{d}x - \int_{x_1}^{x_2} p_2 \mathrm{d}x = 0,$$

i.e. $\int_{x_1}^{x_2} p_1 \mathrm{d}x = \int_{x_1}^{x_2} p_2 \mathrm{d}x$ and thus $\int_{x_1}^{x_2} p\mathrm{d}x = const$ for any $p_1 \le p \le p_2$. Hence, along an element $\mathrm{d}x$ of trajectory,

$$\delta \int_{x_1}^{x_2} p\mathrm{d}x = 0, \quad \varepsilon = const.$$

4.2. Further Considerations on the Group Velocity

The reasoning already carried out for a beam of particles, see (2.46), is extended here considering a light beam propagating in a dispersive medium at rate $v < c$. The Equations (2.37) and (2.36) yield

$$v = \frac{\delta\varepsilon}{\delta p} = \frac{\delta\nu}{\delta\lambda^{-1}} = \frac{\delta\nu}{\delta(\mathrm{n}\nu/c)} = \frac{c}{\frac{\delta(\mathrm{n}\nu)}{\delta\nu}}, \quad \mathrm{n} = \frac{c}{v}, \quad \lambda^{-1} = \frac{\nu}{v}; \tag{4.1}$$

of course v_g of a light wave packet is found through the local limit $\delta \to \partial$, *i.e.*

$$v_g = \frac{c}{\frac{\partial(\mathrm{n}\nu)}{\partial\nu}}. \tag{4.2}$$

It is instructive to examine closer the Equation (4.2) in order to evidence that a further aspect of the motion of a corpuscle of mass m is describable by a wave

packet moving as a whole with at rate v_g; the reasoning involves explicitly its energy ε to describe the propagation of the overall shape of the wave packet amplitude through the space. Differentiating the Equation (2.34)

$$\delta\left(\varepsilon/c^2\right)=-\frac{p}{v^2}\delta v+\frac{\delta p}{v};$$

and replacing $p=h/\lambda$ in this equation, trivial calculations yield

$$\delta\left(\varepsilon/c^2\right)=-\frac{h}{\lambda^2 v}\left(\lambda\frac{\delta v}{v}+\delta\lambda\right).$$

Require now purposely $\delta\varepsilon=0$, *i.e.* the wave transports a fixed amount of energy; for example $\varepsilon=const$ could be just that of one free particle. So

$$\frac{\delta v}{\delta\lambda}=-\frac{v_m}{\lambda}=-\nu_m, \tag{4.3}$$

being v_m the particular value of v fulfilling the given condition; the frequency ν_m is then formally implied by dimensional reasons too. Hence

$$-\delta\nu_m=-v_m\delta\lambda^{-1},$$

so that

$$v_m=\frac{\delta\nu_m}{\delta\lambda^{-1}}=\frac{\delta\nu_m}{\delta\left(\nu_m/v_m\right)}=\frac{c}{\frac{\delta\left(\mathrm{n}_m\nu_m\right)}{\delta\nu_m}},\quad \mathrm{n}_m=\frac{c}{v_m};$$

then, for $\delta\to\partial$ once more, $v_m=v_g$.

The key step of the reasoning is the well defined amount energy ε transported at the rate v_g, by consequence of which results defined the frequency ν_m corresponding to the unique v_m. The different definitions of λ in (2.36) and in (4.3) are significant; their comparison yields

$$\frac{\nu_m}{\nu}=\frac{v_m v}{c^2}<1.$$

Think now one Planck frequency (2.37) as that included in a packet of waves of different wavelengths propagating in a dispersive medium with different λ-dependent velocities: in effect, the Equations (3.1) regard $h\nu$ and h/λ as random values within energy and momentum ranges that in turn define various frequencies and momenta corresponding to $\delta\lambda$ and $\delta\nu$. Both statements agree with the fact that the propagation of the particle or its related wave correspond to v_g and not to the single phase velocities $\lambda\nu$. Just for this reason from (3.1) can be inferred the corpuscular and wave aspects of quantum physics.

The equations now obtained directly from the Equation (2.34) emphasize a new implication: neither ε_ω nor p_ω show explicit reference to the mass, which now becomes mere dimensional parameter inherent the definition of $\hbar$. Appears thus the necessity of explaining how and why the mass is apparently waived from the quantum Equation (2.37) of momentum and energy. In other words, a valid reason is required to replace m with $m-m'$, being m' a new mass even compliant with $m-m'=0$ as a limit case. Tentatively this implies

defining m as a velocity dependent variable, as in effect it has been already found in the Equation (2.33) and more specifically in (3.13). On the one hand this strategy seems at least in principle adequate to highlight why a moving mass m could turn into an immaterial wave. On the other hand further confirms should be provided next to validate the following way of describing this subtle point.

4.3. The Refractive Index

According to (3.64) and (2.35), if $m \neq 0$, then p and ε are calculable for $v < c$ only; however even $v \to c$ is admissible if contextually $m \to 0$. Implementing concurrently both limits, p and ε tend to the indeterminate forms 0/0, which admit in principle finite values. Let p' and ε' be these limit values, assumed existing by definition: the reverse question rises now, *i.e.* whether or not $v < c$ requires $m \neq 0$. The answer is negative: as the speed of photons in dispersive materials is lower than that in the vacuum, it is possible in principle that photons travel in a dispersive medium at the same v allowed to a beam of massive particles. The fact that $v < c$ is compatible with both $m = 0$ and $m \neq 0$, suggests that the kinetic mass m should actually be function of v itself: if so, then the separate correspondences $m \neq 0 \to v < c$ and $v < c \leftrightarrow (m = 0, m \neq 0)$ merge into the unique correspondence $m \neq 0 \leftrightarrow v \leq c \leftrightarrow m = 0$ provided that an appropriate function of $m = m(v)$ does exist. In other words it should be true that both $m \neq 0$ and $m = 0$ are compatible with a unique $v \leq c$ via $m = m(\mathrm{n})$. In fact this conclusion has been already inferred in (2.33), where the concept of mass was introduced in the present model as rest mass. The following reasoning represents the extension of this concept to the kinetic mass.

Regard m of (2.33) as a particular case of a general dynamical variable related to p through v and examine how the new concept of mass could tend to zero correspondingly to $v \to c$; is interesting in this respect the position

$$m = m'\sqrt{1 - v^2/c^2}, \quad m' \geq m, \tag{4.4}$$

which regards m as a constant mass while introducing a new mass $m' = m'(v)$. The Equation (2.33) has anticipated this conclusion in the particular case where $m = m'$ for $v \to 0$, whereas a further hint to the concept of rest mass has been provided by (3.13). Replacing formally m of (4.4) in both (3.64) one obtains $p' = m'v$ while contextually $\varepsilon' = m'c^2$; then, eliminating m' from these results, one still finds $p' = \varepsilon' v/c^2$ in agreement with (2.34). So m' fulfills the same relativistic formula of p with initial mass m, despite now the limit for $v \to c$ corresponds to the finite value $p' = \varepsilon'/c$ implemented in (3.14) and (3.15); this relationship between energy and momentum is expected in general for a wave, see Equation (2.24). Hold for m' all steps from (3.14) to (3.20). The wavelike implication of (4.4) is further acknowledged considering $\delta\varepsilon' = c\delta p'$ of (2.14).

In conclusion, according to the quantum uncertainty the behavior of a corpuscle of mass m should inherently have a wave-like propagation too, whereas

the fact that $m = m'$ for $v = 0$ shows that m and m' are rest and kinetic masses. So the Equation (4.4) in fact generalizes the concepts of m and v introduced in (2.33): m' is the particular value pertinent to m at the specific speed v. As a consequence note that $p/\varepsilon = v/c^2 = p'/\varepsilon'$ define a pure number

$$\frac{\varepsilon}{pc} = \mathrm{n} = \frac{\varepsilon'}{p'c}, \quad \mathrm{n} = \frac{c}{v} \tag{4.5}$$

that introduces the refractive index of the medium where propagates an electromagnetic wave at velocity $v \le c$; owing to the Equation (6.2) in fact $\mathrm{n} \ge 1$, as it has been already introduced in (2.46) and (4.1). Moreover the position (4.4) also agrees with (2.45); indeed

$$p' = \frac{\varepsilon' v}{c^2} = \frac{h\nu' v}{c^2} = \frac{h}{\lambda}, \quad \lambda = n\lambda' = \frac{c^2}{v\nu'}, \quad \lambda' = \frac{c}{\nu'}$$

takes into account that $\lambda = \mathrm{n}\lambda'$ depends on the refractive index of the medium through which propagates the electromagnetic wave or the De Broglie pilot wave. The position (4.4) introduces thus the first step to explain how and why the concept of mass does not explicitly appear in (2.45): once having introduced the refraction index, v is in fact eliminated from the equations being replaced by n. Formally this means expressing the displacement rate v of the particle in c units; yet v appears subsequently also as the rate $\lambda\nu$ at which a single wave phase propagates. This fact encourages thinking that somehow it should be possible to infer a formula that specifically emphasizes what the Equations (3.64) and (2.45) already show themselves, *i.e.* the way m and λ replace each other in defining p and ε. The mathematical approach to this task proceeds noting that

$$\delta m = m' - m = \left(1 - \sqrt{1 - v^2/c^2}\right)m' \quad \text{i.e.} \quad c^2\delta m \approx \frac{1}{2}m'v^2 + \cdots, \tag{4.6}$$

so that the right hand side represents kinetic energy. On the one hand m' corresponds to the classical mass defining the kinetic energy, although for $v \ll c$ the deviation of m' from m is irrelevant for practical purposes. On the other hand it is possible to write

$$\frac{m' - m}{m'} + \sqrt{1 - \frac{1}{\mathrm{n}^2}} = 1, \quad \left(\frac{m}{m'}\right)^2 + \left(\frac{v_g}{c}\right)^2 = 1. \tag{4.7}$$

The second equation is direct consequence of the first one; it emphasizes that the concerned velocity v is actually v_g of (2.45), because in general this latter and not v of (2.3) is related to and describable by n. This confirms that m' is the effective value of m when the particle velocity takes up just the specific value v_g pertinent to the group velocity at which propagates the wave packet. As expected $m = m'$ for $v = 0$, whereas $m = 0$ for $\mathrm{n} = 1$; in effect according to (3.13) m/m' is definable even for $v \to c$, so (2.33) and (4.4) are compliant with these limits. Regard thus the addends of (4.7) as probabilities, whose sum represents the certainty of concerning the existing particle through its mass displacement velocity or wave propagation rate. The first addend describes the probability for the particle to loose its classical kinetic mass, till to become an

immaterial propagation wave; the second addend, previously introduced to express the actual velocity v_g of the particle, takes the meaning of reciprocal refraction index n of the resulting wave, being it in effect still related to the propagation rate of the wave/particle. The addends account therefore for the dual behavior of matter in a probabilistic way correspondingly to the probability of energy fluctuation, thanks to which the particle effectively displaces with velocity dependent mass or with frequency dependent propagation rate of a wave packet: indeed, one must also expect an appropriate energy fluctuation to balance the chances of mass energy loss. Obviously to this mass change correspond different p and ε and thus different λ, whence the necessity of linking m with a group of waves that spread with collective v_g given by (2.45). So the worth of (4.7) is that of having emphasized the quantum probabilistic meaning of the relativistic position (4.6).

These considerations rise however three questions.

The first one can be formulated as follows: as (4.7) is made by mass and massless terms, what determines either property of matter? Obviously the immediate answer points to the kind of experiment made on the particles constituting the body of matter. Also this is the non-real essence of quantum mechanics, which actually regards the matter neither as a packet of waves nor as a cluster of corpuscles, but as an undefined state of probabilistic mixing of both states until some experiment "creates" either state. The electron diffraction in the two slit experiment and the Thomson experiment inspired by the Millikan result elucidate the physical meaning of the addends of (4.7). To this equation is also related the physical meaning of the EPR thought paradox, showing that the quantum properties are not pre-definable outcomes according to some principles of classical mechanics, rather they are created by the experiment itself. In effect (3.1) exclude not only the concept of trajectory, but also that of distance and velocity; as shown in 3.2 the local space time coordinates must be replaced by the respective ranges, so concepts like "superluminal" distance are actually unphysical. In this sense the EPR paradox shouldn't even be formulated: replacing systematically $x \to \delta x$ and $t \to \delta t$ are missing the concepts themselves of point to point space distance and time to time lapse needed to define any "superluminal" effect; remember that in effect according to (2.8) and (2.9) $v_0 = c$ is introduced "as such", *i.e.* as a fundamental constant of Nature regardless the ratio δx over δt.

The second one concerns the addition of velocities. Consider an electromagnetic wave that appears in the point where $m = 0$. An example is the annihilation of m by collision with its antiparticle purposely assumed in the vacuum: one would naively expect that the new born electromagnetic wave should propagate at rate classically resulting from its own velocity c summed up to that v initially characterizing the moving center of mass of the annihilating particles. Yet (2.48) has already negatively answered this question.

The third question concerns the energy fluctuation necessary to account for the mass change when $m \to 0$. This point is concerned in the next two subsections.

4.4. Energy Fluctuation

The corpuscle/wave dualism has been accepted as compelling experimental evidence since the early experiments of electron diffraction, simply acknowledging that either behavior depends on the kind of experiment. Yet this shortcut leaves in fact unexplained why mass appears explicitly in (3.64) whereas it is hidden in the proportionality constants (2.36), despite both concern momentum and energy of a free particle. The fact that both equations have been inferred in the frame of a unique model based on the definitions (1.11) and (1.12) stimulates one to think that even this duality could find rational explanation, *i.e.* explainable by a logical physical reasoning in the conceptual frame of the present model, without need of supplementary "ad hoc" hypotheses. This hope is supported by the probabilistic character of (4.7), direct consequence of the concept of velocity dependent mass elucidated in the form (4.4): in effect the chances $m=0$ and $\lambda\nu=c$ or respectively $m=m'$ and $v_g=0$ appear in principle reciprocally consistent and compliant with the unity, *i.e.* the certainty that anyway something travels through the space time as amount of mass or wave: in the former case it is appropriate to think about corpuscle displacement velocity, in the latter about wave propagation rate. The validity of this idea is proposed in this subsection not only by evidencing its self-consistency, but also quoting as a verification further well known results contextually obtainable.

The results of the point 4.3 have been obtained considering initially a particle of mass m that displaces at rate v; next has been considered also its probability of mass, *i.e.* energy, fluctuation, which eventually turns it into massless electromagnetic wave or matter wave traveling at rates n^{-1} or v_g respectively. On the one hand, besides the formal similarity with the propagation of either kind of wave, the Equations (4.5) and (4.4) show that this virtual process scales both p and ε to p' and ε' by a common factor related to the refractive index. On the other hand, this also implies an energy change that occurs in a time range $\Delta t=t-t_0$, being t_0 the arbitrary time at which the mass m starts modifying its value.

Owing to (2.34), consider thus the energy change $\Delta\varepsilon=\varepsilon'-\varepsilon=p'c^2/v'-pc^2/v$ since when the particle starts loosing its initial mass m to when eventually $m\to 0$ according to (4.7). The fluctuation driven energy change is summarized by the following equations

$$\Delta\varepsilon=\frac{c^2}{v''^2}\delta\varepsilon=\mathrm{n}^2\delta\varepsilon,\quad v''^2=v'v,\quad \delta\varepsilon=p'v-v'p,\quad \mathrm{n}=\frac{c}{v''}. \tag{4.8}$$

The energy range $\Delta\varepsilon$ must not be confused with $\delta\varepsilon$ of (3.1): $\delta\varepsilon$ concerns the quantum uncertainty unavoidably constraining the arbitrary variability range allowed to the conjugate dynamical variables of any system, $\Delta\varepsilon$ is instead the specific energy fluctuation allowed in particular to the particle during the mass loss virtual process that "converts" it into a wave. The time length related to $\Delta\varepsilon$ is thus

$$\Delta t = \frac{\hbar}{\Delta\varepsilon} = \frac{\hbar}{\mathrm{n}^2\delta\varepsilon} \quad i.e. \quad \mathrm{n}^2\Delta t = \delta t. \tag{4.9}$$

The Equation (4.9) yields

$$\frac{v^2}{c^2} = \frac{\Delta t}{\delta t} = \frac{\nu_\delta}{\nu_\Delta}, \quad \nu_\delta = \frac{1}{\delta t}, \quad \nu_\Delta = \frac{1}{\Delta t} \tag{4.10}$$

these positions are easily understood; the respective energies proportional to v^2 and c^2 are also proportional to the frequencies ν_δ and ν_Δ. Of course the only way to regard this result in the wave formalism of quantum mechanics is the link between frequencies and energies, which in fact is just the Planck position: precisely in this sense the probabilistic Equation (4.7) introduces the ratio $\left(m/m'\right)^2$, mass addend, and the corresponding ratio ν_δ/ν_Δ, wave addend. This confirms that the corpuscle/wave behavior has probabilistic origin and follows an energy fluctuation of quantum matter.

Are the Equation (4.8) along with its premises and implications true indeed? To support the validity of (4.7) and thus (4.8) itself, is now tested their direct consequence, the Equation (4.9), in three particular cases of major physical interest. Write first with the help of (3.1) and (2.25)

$$\Delta t = \frac{\hbar}{\mathrm{n}^2\delta\varepsilon} = \frac{v\delta x}{c^2} = \frac{\delta x^2}{c^2\delta t}, \tag{4.11}$$

noting that $c^2\Delta t$ has physical dimensions of diffusion coefficient D introduced in (2.19); this suggests that $v\delta x = \sigma D$, being σ an appropriate proportionality coefficient to be determined. So

$$\sigma D = v\delta x = \frac{\delta x^2}{\delta t} \tag{4.12}$$

The coefficient σ is crucial to specify the kind of problem precisely concerned.

1) Putting first $\sigma = 1$ means describing one particle that displaces with diffusion coefficient D through δx at average velocity v. Strictly speaking, as previously remarked about the Equation (3.60), in the present one space dimensional model (3.39) defines C as linear density $mass/length$ instead of the actual $mass/length^3$; yet C regarded in the usual 3-dimensional space allows to define the actual physical dimensions of flux J of matter, *i.e.* $mass/length^2 time$. Multiplying both sides of the first equality $v\delta x = D$ by δC, being C mass per unit volume, yields $v\delta C = D\delta C/\delta x$. This result is more appropriately rewritten as $|v|\delta C = \pm D\delta C/\delta x$; the double sign accounts for the fact that v is actually a velocity component on the x-axis along which is defined δx, correspondingly to the definition $v_0^2\tau$ of the Equation (2.19). Simple dimensional considerations allow defining the equation

$$J = \pm D\frac{\delta C}{\delta x}, \quad J = |v|C, \quad C = \frac{m}{V} \tag{4.13}$$

that introduces with the minus sign the concept of mass flux J, *i.e.* mass transferred per unit surface and time through the volume V; so δC is due to the

diffusion driven matter transfer between the surfaces δx apart of an ideal cube of matter of volume V. The Equation (4.13) completes the Equations (2.19), as it is well known. Anyway, merging both expressions (4.13) of J, one finds $vC = -D\delta C/\delta x$; then, recalling (3.60) and (3.61) as already done in (2.22), the limit $\delta \to \partial$ yields

$$v = \mp D\frac{\partial \log C}{\partial x} = -\frac{D}{kT}\frac{\partial \mu}{\partial x}, \quad \mu = kT\log\left(\frac{C}{C_0}\right). \tag{4.14}$$

The double sign of v is obvious, being it a velocity component. For simplicity and brevity v and D have been regarded not dependent on x, to make quickly recognizable the link of these results with well known concepts of elementary diffusion theory; also, the diffusion process has been assumed at the constant temperature T. With the minus sign in (4.13), positive D, one acknowledges once more the definition of chemical potential μ in agreement with (3.61). Moreover, as the $-\partial\mu/\partial x$ is equivalent to a force $\mathcal{F}$, this yields also the famous Einstein-Smoluchowski relationship between mobility $\mathcal{M}$ and diffusion coefficient D, *i.e.*

$$D = \mathcal{M}kT, \quad \mathcal{M} = \frac{v}{\mathcal{F}}. \tag{4.15}$$

Eventually the plus sign in (4.13), which instead corresponds to negative D, describes phenomena like the spinodal decomposition of alloys of appropriate composition [9].

2) Putting next $\sigma = 2$ and writing thus (4.12) as $v\delta x = 2D$ means describing one particle that travels with diffusion coefficient D the distance $\delta x/2$ at average velocity $\pm v$. The factor 1/2 specifies therefore that the particle displaces around the mean coordinate $\bar{x}$ towards both sides of δx, in which case δx^2 at the right hand side of (4.11) reads $\delta x^2 \to \overline{\left(x-\bar{x}\right)^2}$ and takes thus the statistical meaning of average square displacement $\overline{\delta x^2}$ of the particle traveling through the whole range around $\bar{x}$. So the second equality (4.12) yields

$$\overline{\delta x^2} = 2D\delta t, \tag{4.16}$$

i.e. the famous Einstein equation of one dimensional Brownian motion.

3) The validity of the Equation (4.8) is further checked implementing the property $\mathrm{n} \geq 1$. Consider now a system of particles, the i-th of which has energy ε_i. The fact that $\Delta\varepsilon$ is in general n^2 times greater than $\delta\varepsilon = \varepsilon - \varepsilon_0$ suggests the possible chance of regarding the former as $\Delta\varepsilon = E - E_0$ and the latter as the sum of an appropriate number N of terms $\delta\varepsilon_i = \varepsilon_i - \varepsilon_{0i}$ such that $\Delta\varepsilon = \mathrm{n}^2\delta\varepsilon = \sum\left(\varepsilon_i - \varepsilon_{0i}\right)$; clearly N depends of the value of n^2 and size of all ranges $\varepsilon_i - \varepsilon_{0i}$. Anyway the initial Equation (4.8) is compatible with the position

$$E - E_0 = \mathrm{n}^2\varepsilon - \mathrm{n}^2\varepsilon_0 = \sum\left(\varepsilon_i - \varepsilon_{0i}\right) \tag{4.17}$$

simply requiring

$$E = \mathrm{n}^2\varepsilon = \sum\varepsilon_i, \quad E_0 = \mathrm{n}^2\varepsilon_0 = \sum\varepsilon_{0i};$$

as in principle n can take any value from 1 to ∞, the number of terms of the sum is arbitrary. The Equation (4.17) is well known and reported in all standard textbooks concerning the fluctuations of thermodynamic systems: it yields $\left(E-E_0\right)^2 = \sum\sum\left(\varepsilon_i-\varepsilon_{0i}\right)\left(\varepsilon_j-\varepsilon_{0j}\right)$ and thus $\overline{\left(\mathcal{E}-\bar{\mathcal{E}}\right)^2} = \sum\overline{\left(\varepsilon_i-\bar{\varepsilon}_i\right)^2}$ regarding appropriately E_0 and ε_{0i} as average quantities. So with $E_0 \to \bar{\mathcal{E}}$ and $\varepsilon_{0i} \to \bar{\varepsilon}$, follows then immediately

$$\overline{\left(\mathcal{E}-\bar{\mathcal{E}}\right)^2} = N\overline{\left(\varepsilon-\bar{\varepsilon}\right)^2}. \tag{4.18}$$

4.5. Liouville Theorem

An interesting question concerning (3.1) is the following: is $\delta\varepsilon = \varepsilon - \varepsilon_o$ simply an energy range or is it even compatible with the physical meaning of difference between two diverse forms ε and ε_o of energy? This question, which according to (2.25) involves $\delta p = p - p_o$ too, is answered rewriting identically (3.1) as

$$\delta\varepsilon = \frac{\delta x}{\delta t}\delta p = \delta x\frac{\delta p}{\delta t}. \tag{4.19}$$

The first equality reads

$$\frac{\delta\varepsilon}{\delta\dot{x}} = \delta p, \quad \delta\dot{x} = \frac{\delta x}{\delta t}, \tag{4.20}$$

the second equality reads

$$\frac{\delta\varepsilon}{\delta x} = \delta\dot{p}, \quad \delta\dot{p} = \frac{\delta p}{\delta t}. \tag{4.21}$$

Now fulfill the idea that $\delta\varepsilon$ defines the difference of two distinct energies, specifically T and U introduced in section 2.3, which implies the chance of writing in general $\varepsilon = \mathrm{T} \pm \mathrm{U}$. To highlight this point, concerning in particular the energies already introduced in (2.41) and (2.42), introduce the following positions

$$\delta\varepsilon = \delta\mathrm{T} \pm \delta\mathrm{U}, \quad \mathrm{T} = \mathrm{T}\left(\dot{x}, x\right), \quad \mathrm{U} = \mathrm{U}\left(x\right) \tag{4.22}$$

in this way the sign of δp in (4.20) is uniquely defined since T only depends on $\dot{x}$, whereas is expected the double sign in (4.21) because both energies T and U depend on *x*. As in effect δp is the component of $\delta\boldsymbol{p}$ along the *x*-axis, so that it can actually take in principle both signs, rewrite explicitly (4.21) as

$$\frac{\delta\varepsilon}{\delta x} = \pm\delta\dot{p} = \delta\dot{p}' - \delta\dot{p}_o', \quad \delta\dot{p}' = \frac{p-p_o'}{\delta t}, \quad \delta\dot{p}_o' = \frac{p_o-p_o'}{\delta t} \tag{4.23}$$

the double sign on the one hand emphasizes that both p and p_o are actually components of the vectors $\boldsymbol{p}$ and $\boldsymbol{p}_o$ along the *x*-axis. The last two equations also agree with the fact that in principle

$$\delta\dot{p}' \gtrless \delta\dot{p}_o'$$

in lack of any information about the ranges, both inequalities are actually possible. Regarded in this way, *i.e.* implementing range boundaries arbitrary and independent each other, the notation (4.23) effectively defines $\delta\varepsilon$ as difference of two energies reasonably dissimilar according to (4.22). Taking the ratios side by side of the first Eqs (4.20) and (4.21) one finds

$$\frac{\delta\dot{x}}{\delta x}=\pm\frac{\delta\dot{p}}{\delta p} \tag{4.24}$$

It is immediate to link (4.24) and (4.22), noting that the former defines at both sides ratios with physical dimensions of reciprocal time range. Multiplying both sides by $\hbar$, the equations

$$\hbar\frac{\delta\dot{x}}{\delta x}+\hbar\frac{\delta\dot{p}}{\delta p}=0,\quad \hbar\frac{\delta\dot{x}}{\delta x}-\hbar\frac{\delta\dot{p}}{\delta p}=0 \tag{4.25}$$

define energies that, in agreement with (2.42) and (2.38), correspond respectively to

$$\delta\mathrm{H}=\delta\mathrm{T}+\delta\mathrm{U}=0,\quad \delta\phi=\delta\mathrm{T}-\delta\mathrm{U}=0.$$

Hence simple considerations on the range boundaries imply the concepts of Hamiltonian and Lagrangian according to the previous Equations (2.38) and (2.42): ϕ has been identified with the Lagrangian of a particle, H with the Hamiltonian of the system. In particular, is of interest here

$$\mathrm{H}=\mathrm{T}+\mathrm{U}=const,\quad \frac{\delta\dot{x}}{\delta x}=-\frac{\delta\dot{p}}{\delta p}$$

for the following reason. According to the quantum uncertainty, the left hand side of (3.1) reads $\delta\boldsymbol{x}\cdot\delta\boldsymbol{p}=n\hbar$, where the number of scalars, so far intuitively associated to the three usual space dimensions only, is actually arbitrary, *i.e.* extensible to any number *j* of extra-dimensions required by some theories or, alternatively, to the number of freedom degrees allowed to the system of particles: in fact any freedom degree has its pertinent $\delta\boldsymbol{x}$ and $\delta\boldsymbol{p}$. Thus it is sensible to introduce the dimensionless quantity $\left(\delta x\delta p/\hbar\right)^{j}$ where fall all points in the multidimensional phase space defined by the sizes of all δx and δp of the corresponding particles with respect to $\hbar$. Accordingly

$$\delta\Omega=\left(\delta p\delta x/\hbar\right)^{j}$$

yields

$$\frac{\delta\left(\delta\Omega\right)}{\delta t}=j\frac{\left(\delta p\delta x/\hbar\right)^{j}}{\delta x}\delta\dot{x}+j\frac{\left(\delta x\delta p/\hbar\right)^{j}}{\delta p}\delta\dot{p}=j\left(\frac{\delta\Omega}{\delta x}\delta\dot{x}+\frac{\delta\Omega}{\delta p}\delta\dot{p}\right)$$

the range $\delta\Omega$ includes all points of coordinates x and p falling within $\left(\delta x\delta p/\hbar\right)^{j}$ elementary cells of *j*-dimensional volume $\left(\delta x\delta p\right)^{j}$ in the phase space. So

$$\frac{1}{j}\frac{\delta\left(\delta\Omega\right)}{\delta t}=\frac{\delta\Omega}{\delta x}\delta\dot{x}+\frac{\delta\Omega}{\delta p}\delta\dot{p}$$

yields then

$$\frac{1}{j}\frac{\delta(\delta\Omega)}{\delta t}=\delta\Omega\left(\frac{\delta\dot{x}}{\delta x}+\frac{\delta\dot{p}}{\delta p}\right)=0$$

according to the first (4.25) $\delta(\delta\Omega)/\delta t=0$, *i.e.* the volume $\delta\Omega=const$ along phase space trajectories where $\mathrm{H}=const$.

5. Some Thermodynamic Corollaries

The last results have somehow linked the relativistic Equation (2.25) to important results of classical statistical thermodynamics. The importance of this topic is shortly highlighted in the following three subsections.

5.1. Statistical Sets of Particles

Let us implement once more the Equation (2.26), and calculate the change of $v/c=pc/\varepsilon$ according to the following chain of equations

$$\delta\left(\frac{v}{c}\right)=\frac{\delta(pc)}{\varepsilon}-\frac{pc}{\varepsilon^2}\delta\varepsilon=\frac{pc}{\varepsilon}\delta\log\left(\frac{pc}{\varepsilon}\right).$$

Since by definition

$$\delta\log\left(\frac{pc}{\varepsilon}\right)=\log\left(\frac{pc}{\varepsilon}\right)-\log\left(\frac{p_oc}{\varepsilon_o}\right),$$

being p_o and ε_o arbitrary constants, it is possible to write

$$\delta\left(\frac{v}{c}\right)=\frac{v}{c}\log\frac{v}{c}-\frac{v}{c}\log\frac{v_o}{c}.$$

Consider now preliminarily the case of an ideal gas of non-interacting free particles/atoms/ions/molecules and let p_i and ε_i the momenta and energies of each particle. Then, owing to the last equality, it is possible to write for each i-th particle

$$\delta\Pi_i=\Pi_i\log\Pi_i-\Pi_i\log\Pi_o,\quad \Pi_o=\frac{v_o}{c},\quad \Pi_i=\frac{v_i}{c}<1;$$

moreover it is also possible to sum terms like this of each particle over all particles of the system, so that it is possible to write

$$\sum_i\delta\Pi_i=\sum_i\Pi_i\log\Pi_i-\log\Pi_o\sum_i\Pi_i$$

whence

$$N\overline{\delta\Pi}=\sum_i\Pi_i\log\Pi_i-N\overline{\Pi}\log\Pi_o,\quad \overline{\delta\Pi}=\frac{1}{N}\sum_i\delta\Pi_i,\quad \overline{\Pi}=\frac{1}{N}\sum_i\Pi_i,\qquad (5.1)$$

being N the number of particles of the system. Note that this result is actually more general than prospected here. Suppose first two interacting particles only; in this case we expect p_1' and ε_1' for the first particle and p_2' and ε_2' for the second one because of their interaction: despite the first (2.34) holds for a free particle, it is reasonable to think that changing appropriately $p_i\to p_i'$ and $\varepsilon_i\to\varepsilon_i'$ one can describe at least approximately even an interacting particle.

For example it is possible to replace p_i/ε_i with $p_i'/\varepsilon_i' = \sigma_i\left(p_i/\varepsilon_i\right)$, being σ_i an appropriate correction factor. Anyway, p_1'/ε_1' and p_2'/ε_2' are in principle calculable; summing these terms, the left hand side of (5.1) involves $v_1'/c + v_2'/c$. In the case of three particles mutually interacting one would obtain $p_1''/\varepsilon_1'' + p_2''/\varepsilon_2'' + p_3''/\varepsilon_3''$ defining $v_1''/c + v_2''/c + v_3''/c$, and so on for any number of particles all mutually interacting. On the one hand this means that now the previous Π_i is replaced by $\Pi_i' = \sigma_i v_i c$, whereas the summation is possibly extended to a different number of terms. On the other hand this reasoning holds also for $\Pi_i'' = \sigma_i' v_i' c$ and also for multiple primed probabilities. In fact, summing all v_i/c or all v_i''/c does not change the conceptual statistical meaning of the sum; in other words, whatever v_i'' might be, one could include appropriate correction factors to the various v_i of the allowed states; normalizing the sums, one still obtains an equation like (5.1). To calculate how each v_i turns into v_i' and next into v_i'' because of these interactions, is in general difficult and must be examined case by case; yet, if we content ourselves to describe the evolution of the system as a whole from the non-interacting to the interacting state, the form of the final equation is still similar to the previous (5.1) obtained for free particles. Omitting for simplicity the primed or multiple primed notations for Π_i and v_i in the following, introduce the positions

$$S = N\sigma S_o - N\overline{\delta\Pi}, \quad S_o = -\Pi_o \log \Pi_o, \quad S = -\sum_i \Pi_i \log \Pi_j, \quad \overline{\Pi} = \sigma\Pi_o, \quad (5.2)$$

where σ is now a proportionality constant. The factor N in (5.2) simply shows that S is an extensive property. Introduce the condition expressed by the equivalent positions

$$\overline{v} = \overline{\Pi} = const, \quad \overline{\delta v} = \overline{\delta\Pi}, \quad (5.3)$$

where $const$ can be 0 or more in general $\neq 0$ in this one dimensional model where v is actually a component of $\boldsymbol{v}$ that can take both signs. The first condition regards a completely disordered system of particles regarded as a whole at the equilibrium, whose velocities are randomly distributed both by modulus and direction with equal probability. The second condition assumes a macroscopic system in an unstable situation out of equilibrium, e.g. gas with an internal pressure gradient due to a non uniform distribution of velocities; this can happen for example for a system of charged particles in an inhomogeneous external field. Whatever its particle velocity distribution might be, both chances are assumed compatible with the third position (5.2).

Actually nothing compels these positions, which in effect are purposely introduced to plug the present considerations into the realm of statistical mechanics. In practice $\delta\left(velocity\ distribution\right) = 0$ shows that the equilibrium corresponds to the maximum possible disorder of the system as concerns the velocity distribution of its constituting particles. This statement, assumed valid in general and not in the present one dimensional case only, can be regarded as boundary condition of (5.2) as it implies $\Pi = const$, in this particular case $S = const$. According to (5.2) this constant can be nothing else but the right hand side, *i.e.*

$$S_B = -\sigma' \log \Pi_o, \quad \sigma' = \sigma \Pi_o \tag{5.4}$$

This is nothing else but the Boltzmann definition of dimensionless entropy: note that Π_o constant indeed does not mean that it has one fixed value only, but that it does not depend on the index of summation states *I*, whereas it depends of course on *T*.

This simple procedure has introduced the function *S* as sum of consistent functions $-\Pi_i \log \Pi_i$ of all particles of the system; the summation over *i* has been extended to the velocities v_i of all particles of the system. This summation is surely positive and finite because all $v_i < c$.

It is possible to ask at this point whether this kind of equation is uniquely referable to the property $\sum_i \Pi_i = const$, or it has a more general worth, e.g. in the case of probability distribution function of states such that $\sum_i \Pi_i = 1$. This expectation is sensible, being a particular case of the second (5.3). The next subsection concerns just this point.

5.2. The Entropy

The starting point is now (3.38) with the minus sign. The way to implement this equation is similar to that just described for the Equation (5.1): any space time factor $\delta x \delta t$ is regarded as $\delta x_j \delta t_k$, with notation that goes back to the section 1 in order to specify an arbitrary *j*-th state of a system of particles at the time t_k. The system defined in this way is a statistical set in the sense previously highlighted for each $v/c \to v_i/c$, in agreement with the definition of $v_j = \delta x_j / \delta t_k$: in other words, the variation of configuration of the system implies reasonably the change of local space coordinates of a cluster of *j*-th particles enclosed in δx_j during the time range δt_k: both ranges define a possible state in the phase space as described in the subsection 4.5. Actually both δx and δt were inherent the definition of v_i in subsection 5.1; similarly *W* must be be introduced here in order to describe the non-instantaneous evolution of a local small volume of the system during space time ranges that represent its configuration change rate. The Equations (3.60) and (3.61) show that this way of thinking allowed to infer the chemical potential μ hidden in χ; let us examine here the possibility of extracting further thermodynamic information from this function.

The algebraic steps are listed one by one after rewriting (3.38) as

$$\chi_- = \zeta - \log W, \quad W = \xi \delta x \delta t + \eta. \tag{5.5}$$

1) On the basis of the section 5.1, define

$$u = a_k W \chi_- = E + a_k S, \quad E = \zeta a_k W, \quad S = -W \log W, \quad a_k = a_k\left(t_k\right) > 0, \tag{5.6}$$

being a_k positive factor dependent on the time t_k only;

2) regard $\delta x \delta t \to \delta x_j \delta t_k$, *i.e.* any local space time coordinate *xt* is defined as one that characterizes the *j*-th state of each particle in the space range δx_j during the time range δt_k, which implies $W \to W_{jk}$ while $u \to u_{jk}$ and $S \to S_{jk}$ as well

$$W_{jk} = W_{jk}\left(\delta x_j \delta t_k\right), \quad S_{jk} = S_{jk}\left(\delta x_j \delta t_k\right), \quad u_{jk} = u_{jk}\left(\delta x_j \delta t_k\right) \tag{5.7}$$

3) sum over all allowed states *j* accessible during an assigned δt_k by all particles in the phase space

$$U_k = \zeta a_k \Theta_k + a_k S_k, \quad U_k = \sum_j u_{jk}, \quad \Theta_k = \sum_j W_{jk}, \quad S_k = -\sum_j W_{jk} \log W_{jk}, \tag{5.8}$$

whereas the factor a_k is defined by

$$U_k = U_o + a_k S_k, \quad U_o = \zeta a_k \Theta_k; \tag{5.9}$$

4) the last Equation (5.9) defines a "new" quantity T called temperature

$$U_o = U_k - TS_k, \quad a_k = \frac{\delta U_k}{\delta S_k} = T, \quad T = T\left(t_k\right) \tag{5.10}$$

uniquely defined for a body of matter at the thermal equilibrium. Note that the first equation has been written introducing at the left hand side the summation over W_{jk} only. Also note that

$$\frac{\delta S_k}{\delta W_{jk}} = \frac{\delta S_{jk}}{\delta W_{jk}} = -\log W_{jk} - 1$$

i.e.

$$-\log W - \frac{\delta S_{jk}}{\delta W_{jk}} = 1 \tag{5.11}$$

the *j*-th addend contributing to S_k is to be considered to calculate the right hand side. Multiplying both sides by W_{jk} one finds

$$W_{jk} \frac{\delta S_k}{\delta W_{jk}} = -W_{jk} \log W_{jk} - W_{jk}$$

and then, summing over *j*, owing to (5.8) one finds

$$\Theta_k = S_k^o + S_k, \quad S_k^o = -\sum_j \left(W_{jk} \frac{\delta S_k}{\delta W_{jk}} \right). \tag{5.12}$$

Normalizing via Θ_k, this result reads

$$1 = \frac{S_k^o}{\Theta_k} + \frac{S_k}{\Theta_k} \tag{5.13}$$

If $S_k^o > 0$ this equation emphasizes the certainty resulting from the sum of two positive terms, which therefore can be regarded as probabilities. If S_k measures the disorder of the system, then reasonably S_k^o measures the order: the sum of these probabilities yields the certainty that both order and disorder concur to define the state of any system. In other words, any system can be partially ordered and partially disordered; e.g. some parts of a crystal lattice can contain in general local point and/or line pile up defects inside a surrounding defect free volume.

This probabilistic interpretation is possible if $S_k^o \geq 0$: in other words, the probability of modifying the local order/disorder of the system requires accord-

ing to (5.12) $\delta S_{jk}/\delta W_{jk} \le 0$ inside any δx_j at different δt_k at which is calculated δS_{jk}. Let be therefore $\delta S_{jk} = S_{jk+1} - S_{jk}$ and $\delta W_{jk} = W_{jk+1} - W_{jk}$ the changes allowed to occur within any space range δx_{jk} at any time within δt_k and rewrite W of (5.5) according to the positions (5.7) via (3.1) and (2.28); once more the space ranges, and not the local space time coordinates they represent, are physically appropriate to describe the changes in the system. Replacing $\delta t = \hbar/\delta\varepsilon$ and $\delta x = \hbar/\delta p = v\hbar/\delta\varepsilon$ one finds

$$W_{jk} = \xi \frac{\hbar^2}{\delta\varepsilon^2}\frac{\delta x_{jk}}{\delta t_k} + \eta$$

having expressed $v = \delta x_{jk}/\delta t_k$ according to the current notation; clearly $\delta\varepsilon_{jk}$ is the pertinent energy change corresponding to the configuration change in progress within δx_{jk}. Hence, keeping $\delta\varepsilon_{jk}$ and δx_{jk} constants, write

$$\delta W_{jk} = -\xi \frac{\hbar^2}{\delta\varepsilon_{jk}^2}\delta x_{jk}\left(\frac{1}{\delta t_{k+1}} - \frac{1}{\delta t_k}\right)$$

and thus

$$\frac{\delta S_{jk}}{\delta W_{jk}} = -\left(\xi \frac{\hbar^2}{\delta\varepsilon_{jk}^2}\delta x_{jk}\right)^{-1} \frac{S(\delta t_{k+1}) - S(\delta t_k)}{\delta t_{k+1} - \delta t_k}.$$

In this way S_k and S_k^o describe the changes occurring in δt_{k+1} with respect to δt_k in the given region δx_{jk} of the system. Certainly the local S_{jk} is due to the corresponding local changes of x_{jk} and ε_{jk}; however it is in principle possible that even at δt_{k+1} both these latter remain still included in the same range size δx_{jk} where they were at δt_k; this simply means that δS_{jk} is small enough to imply correspondingly small changes of x_{jk} and ε_{jk} that therefore still remain included within the same δx_{jk}. While acknowledging that this is in principle admissible because all range sizes are in principle arbitrary, it is interesting to compare what happens at δt_{k+1} and δt_k. If for example $\delta t_{k+1} > \delta t_k$ by definition, *i.e.* the former is greater than the latter because it must include increasing values of local time coordinates t_k, then $\delta W_{jk} < 0$ implies $W_{jk} < W_{jk+1}$. The negative sign of δW_{jk} means that on the one hand $S_k^o \ge 0$ fulfils via $\delta S_{jk}/\delta W_{jk} \le 0$ the probabilistic meaning of (5.13) and that on the other hand it also implies all $\delta S_{jk} \ge 0$. Thus summing over *j* all terms δS_{jk} at all δt_k one infers

$$\delta S_k = \sum_j \delta S_{jk} \ge 0. \tag{5.14}$$

Clearly this is just the second law of thermodynamics because, as written, it concerns an isolated system; the conclusion is in effect true if no external action perturbs the system. If not so, then any action altering substantially the configuration of the system modifies by consequence the *j*-th range size too; in general different δx_{jk} and $\delta\varepsilon_{jk}$ are reasonablyimplied before and after the external action. Thus, in particular, it can result that $\delta x_{jk+1} < \delta x_{jk}$ while however (5.13) can be again fulfilled: $\delta S_{jk}/\delta W_{jk} \le 0$ still holds even with $\delta W_{jk} > 0$ but $\delta S_k < 0$. Clearly in the system no longer isolate the external action has modified the spontaneous tendency towards increasing entropy.

It is worth remarking once more that the evolution of the physical system has implemented two subsequent time lapses δt_k and δt_{k+1}, not two deterministic time coordinates t_k and t_{k+1}: these latter and the respective deterministic ε_{jk} and ε_{jk+1} representing the external action would be incompatible with the Heisenberg principle.

5.3. The Statistical Distributions

Let the change δW of W be $\delta W = W \pm w$, being w an arbitrary amount added or subtracted to the initial value of W. On the one hand W can increase or decrease by any physical reason with respect to its initial value; the double sign indicates that no reason is guessable to expect that the change consists of either increase only or decrease only of the initial value W. On the other hand it is also reasonable to expect that $\delta W = qW_0$, being q an arbitrary proportionality factor and W_0 an arbitrary value allowed to W consistent with (5.5); this position means that anyway the change δW implies a new quantity still related to the meaning of thermodynamic probability W_0 coherent with W. In other words, W_0 is such that $W' = W + qW_0$ and $W'' = W - qW_0$ are respectively compatible and physically consistent with $W \pm w$. The fact that both W' and W'' must fulfill (5.8) likewise the initial W, allows expecting the consistency of the following considerations with the equations up to (5.10) as well. If so, then

$$\delta W = W \pm w_{\pm} = q_{\pm} W_0, \quad q_{\pm} > 0 \tag{5.15}$$

yields

$$\frac{1}{q_{\pm}} = \frac{W_0 / w_{\pm}}{W / w_{\pm} \pm 1}; \tag{5.16}$$

put in this form, once more the space time Equation (5.5) of W is implemented via W/w and related W_0/w, similarly to the position (5.8) leading to the results (5.10). The Equation (5.15) has been written in order to emphasize how w is to be regarded in agreement with either sign, *i.e.* $W + w_+ = q_+ W_0$ and $W - w_- = q_- W_0$. In conclusion, recalling the Equation (5.5),

$$\frac{w_{\pm}}{q_{\pm}} = \frac{W_0}{\exp\left(\chi_- - \zeta - \zeta_0\right) \pm 1}, \quad w_{\pm} = \exp\left(\zeta_0\right);$$

also here appears the space time function χ_-. Hence, according to the reasoning to infer the Equation (4.14) via (3.60) and (3.61), at any given time $\chi_- - \zeta = \log\left(C/C_0\right) + const$ and thus $\mu/kT - \mu_0/kT$. In conclusion

$$\frac{w_{\pm}}{q_{\pm}} = \frac{W_0}{\exp\left(\left(\mu - \mu_0\right)/kT\right) \pm 1}. \tag{5.17}$$

This equation follows from the arbitrariness of μ_0, consistent with that of ζ_0; the multiplicative factor w^{-1} has simply included ζ_0 in the constant addend of chemical potential μ together with ζ. Implement now either

$$W = q_- W_0 - w_+, \quad W = q_+ W_0 + w_-$$

of (5.15): being $W > 0$ by definition, there is no constrain to the number w_-

related to the negative sign in the Equation (5.17), whereas the positive sign of this equation requires $w_+ \le q\_W_0$ *i.e.* $w_+/q\_W_0 \le 1$. This constrain suggests the possible physical meaning of $w_\pm/q_\pm$ in (5.17). Let W_0 be the numbers of states with a given energy μ, *i.e.* the degeneracy of the state, and $w_\pm/q_\pm = N_\mu$ the number of particles in the given state; if so, then it is easy to realize that in the latter case N_μ can take only the values 0 and 1 whatever μ might be [10]. All details published elsewhere are omitted here for sake of brevity. Thus (5.17) is the well known formula of statistical distribution of fermions and bosons with degeneracy W_0.

5.4. The Phase Space

Entropy and Liouville theorem, both previously inferred, are the key concepts to introduce the phase space. As this topic is well known, are reported here just a few remarks aimed only to emphasize the link between space time and phase space; *i.e.* the concept of space time is actually the third essential ingredient to introduce "*ab initio*" the statistical mechanics. To this purpose consider in particular the Equations (3.39) and (3.1).

Being *x* and *t* arbitrary and independent variables, which represent for example the space coordinate of a given particle at various times in the space time, any value of *xt* can be obtained keeping constant either factor and allowing appropriate values of the other one; both ways of defining an arbitrary space time coordinate $x_j t_k$ are numerically and conceptually equivalent to describe each one among *N* particles of the system at given time t_k in the range of space coordinates $x_{j1} \le x_j \le x_{jN}$ or at x_j during the time range $t_1 \le t_k \le t_2$. According to (3.1), indeed, the space time coordinate of each particle is defined within allowed variability ranges $\delta x_j = x_{jN} - x_{j1}$ and $\delta t_k = t_{k2} - t_{k1}$. So is physically significant the amount $\delta x_j \delta t_k = \hbar^2 / \delta p_j \delta \varepsilon_k$, whatever these range sizes might be. To highlight this point consider the following equations obtained implementing (3.1) and (2.28)

$$\delta x_j \delta t_k = \frac{\hbar^2}{\delta p_j \delta \varepsilon_k} = \frac{\hbar^2}{\left(\delta \varepsilon_j / v_j\right)\left(v_k \delta p_k\right)},$$

whence

$$\frac{\delta x_j}{v_j} v_k \delta t_k = \frac{\hbar^2}{\delta \varepsilon_j \delta p_k} = \delta t_j \delta x_k$$

and thus, comparing the initial and final ranges of coordinates,

$$\delta x_j \delta t_k = \delta x_k \delta t_j \tag{5.18}$$

the initial equation regards the *j*-th space coordinate x_j at the time t_k, the former defined within the interval δx_j the latter in the time range δt_k; the final equation rewrites the first one with exchanged indexes *j* and *k*. As in effect the first two equations are summarized in the third one, it means that the concerned particle is described at different times t_j and t_k by different space ranges δx_j and δx_k, to which correspond the respective momenta δp_j and

δp_k; moreover it also follows $\delta p_j \delta \varepsilon_k = \delta p_k \delta \varepsilon_j$, as it must be because anyway the Equation (3.1) must be fulfilled no matter how any particle moves in the space time. In other words, as *j* and *k* are not specified or specifiable, the particle moves actually through any random space and time ranges in the phase space according to its position and momentum of the space time. This can be better evidenced and generalized rewriting with trivial manipulations the last equation

$$\begin{aligned} &\delta x_j \delta t_k = \delta x_j \frac{\hbar}{\delta \varepsilon_k} = \delta x_j \frac{\delta p_q \delta x_q}{\delta \varepsilon_k} = \frac{\delta x_q \delta p_q}{F_{kj}} = \delta x_q \delta t_{kjq} \\ &F_{kj} = \frac{\delta \varepsilon_k}{\delta x_j}, \quad \delta t_{kjq} = \frac{\delta p_q}{F_{kj}}, \end{aligned} \tag{5.19}$$

i.e. one particle initially at any random x_j within δx_j at the random time t_k included in δt_k is actually found within another δx_q at the subsequent time t_{kjq} within time range δt_{kjq}. Obviously this chain could be further extended starting again from the last term $\delta x_q \delta t_{kjq}$, which in effect has the same form of the first one but is simply rewritten with different subscripts; the first and last terms of this chain represent different space time coordinates and thus its ability of the particle to fill various coordinate and momentum ranges defining the whole phase space. All accessible local coordinates of space time correspond to the respective local coordinates of space phase, with equal probability.

5.5. Further Comments about the Diffusion Coefficient

The diffusion coefficient *D* introduced in (2.19) is usually concerned in problems of matter displacement under non-equilibrium conditions, essentially due to concentration gradients; the same holds for the heat diffusion coefficient (2.22) in non-thermal equilibrium problems, typically in the presence of temperature gradients. However, the four equations from (4.13) to (4.16), as well as the next (5.20) and (5.21), suggest a more profound physical meaning of *D*. In this respect deserve attention the following three remarks.

1) The dimensional definition of *D* is $\hbar/m$; this yields $\delta D/D = -\delta m/m$, *i.e.* $\delta \log\left(D/D_o\right) = -\delta \log\left(m/m_o\right)$, being of course D_o and m_o arbitrary constants. Then, reasoning likewise in (4.14), the right hand side yields $\delta \log\left(\left(m/V\right)/\left(m_o/V\right)\right) = \delta \log\left(C/C_o\right)$, being *C* the amount of mass in a given volume *V*. Hence, being $\log\left(C/C_o\right) = \mu/kT$, as found in (3.60) and (3.61), one finds $\delta \log\left(D/D_o\right) = -\delta\left(\mu/kT\right)$ and therefore

$$\frac{D}{D_o} = \exp\left(-\mu/kT\right) \tag{5.20}$$

this is the usual form to express the dependence of diffusion coefficient on temperature via the activation energy μ and the reference constant D_o.

2) Assume now a body of matter of mass *m* in equilibrium at temperature *T* and implement the reasonable idea that both *D* and $\mathcal{M}$ take finite ranges of allowed values. Let $D_{\min}$ and $\mathcal{M}_{\max}$ be the respective limit values of interest here; is then significant the particular case where the Equation (4.15) concerns

the minimum temperature $T_{\min}$ defined as follows

$$T_{\min} = \frac{D_{\min}}{k\mathcal{M}_{\max}} = \frac{\epsilon_{zp}}{k}. \tag{5.21}$$

Dimensional considerations are useful to guess an order of magnitude estimate of $T_{\min}$. The reciprocal mobility $\mathcal{M}^{-1}$ has physical dimensions $mass/time$, whereas D is $h/mass$; so their product represents the minimum energy ϵ_{zp}. The notation emphasizes that the energy of interest to calculate $T_{\min}$ excludes the contribution of thermal vibrations, being instead due to the mere confinement of a particle or a body of matter within a finite delocalization range δx; accordingly, it is sensible to define $T_{\min}$ as ε_{zp}/k. In fact the Equations (3.1) justify the existence of this form of energy and related force.

Consider indeed one particle of mass m ideally delocalized between two infinite potential walls δx apart; in a one dimensional model it is possible to write

$$\varepsilon_{zp} = \frac{\delta p^2}{2m} = \hbar\frac{\omega}{2}, \quad \omega = \frac{\hbar}{m\delta x^2} = \frac{\sigma D}{\delta x^2}, \tag{5.22}$$

having expressed $D = \sigma\hbar/m$ via an appropriate proportionality constant σ. This result is understandable thinking an oscillating particle confined in δx, so that m bounces back and forth between the potential walls with frequency $\omega = 1/\delta t$. In fact δt is the time lapse to complete one oscillation cycle; $\delta p = p_{\max} - 0$ is the range defined by the maximum delocalization momentum $p = p_{\max}$ related to the range size and $p = 0$ when the particle inverts its motion on both potential walls. This picture agrees with $\delta p \neq 0$, *i.e.* with the physical impossibility of conceiving a localized particle at rest and thus with $p = 0$ in fixed point exactly defined. The circular frequency here introduced is justifiable from a more realistic three dimensional point of view, where the back and forth one dimensional motion of m reads actually $\hbar = m\omega\delta x^2$ and thus $\epsilon_{zp} = m(\omega\delta x)^2/2$; *i.e.* m describes a closed circular path at tangential velocity $v_{\tan} = \omega\delta x$ inside its confinement delocalization volume V, so far not yet introduced explicitly. In effect it is also possible to evidence the confinement volume writing

$$\epsilon_{zp} \approx \frac{\delta p_{zp}^2}{2m} = \frac{\hbar^2}{2m\delta x_{zp}^2} = \frac{\hbar^2}{2mV_{zp}^{2/3}}, \quad F_{zp} \approx \frac{\hbar^2}{2mV_{zp}}, \quad V_{zp} = \delta x_{zp}^3. \tag{5.23}$$

Anyway it is sensible that $T_{\min}$, being presumably a fixed value, cannot depend on the arbitrary m and specific V_{zp}; rather $T_{\min}$ is to be regarded as a universal property of matter uniquely defined. Both requirements suggest restarting from the relativistic energy equation $\varepsilon = pc^2/v$ of one free particle of arbitrary mass m, which however must no longer appear explicitly. Implementing thus the wave expression of momentum $p = h/\lambda$, which in fact allows introducing the expected oscillation behavior as that related to the concept of wavelength λ, one finds

$$\varepsilon = \frac{pc^2}{v} = \frac{hc^2}{\lambda v}, \tag{5.24}$$

which thus also defines

$$\epsilon_{zp}^{\min} = \frac{hc^2}{\left(\lambda_{zp} v\right)_{\max}}.$$

The 3D generalization of this result is obtained imagining an arbitrary amount of mass delocalized in an appropriate range $2\Delta r$, regarded as the diametric size of a hypersphere of radius Δr to which is related the maximum value of λ_{zp}: the idea is to implement steady matter waves of wavelengths λ_{zp} propagating through the hypersphere at rate $v_{\max}$. Regard thus an arbitrary mass of an isolate free corpuscle ideally bouncing within one diametric distance, whose extent corresponds to one half wavelength; the largest zero point wavelength is that with steady nodes on the opposite boundaries of the hypersphere diameter and is thus $\lambda_{zp} = 4\Delta r$. This implies that $\epsilon_{zp}^{\min}$ defined by the zero point momentum wavelength λ_{zp} corresponds to back and forth delocalization through twice the diametric size $4\Delta r$ of the hypersphere: this is the physical meaning of (5.22) where $\nu_{zp} = v/2\left(2\Delta r\right)$ whence $h\nu_{zp} = \hbar\omega_{zp}$. Also, imagining asymptotically $v \to c$ to simulate $v_{\max}$ and recalling (5.22), it is possible to conclude

$$\epsilon_{zp}^{\min} = \frac{hc}{4\Delta r}, \quad T_{\min} = \frac{\epsilon_{zp}^{\min}}{k} = \frac{hc}{4k\Delta r}, \quad \omega = \frac{2\epsilon_{zp}^{\min}}{\hbar} = \frac{\pi c}{\Delta r}. \tag{5.25}$$

These results will be calculated later; regardless of the numerical values, however, it is possible to remark since now some interesting implications:

- The Nernst theorem is automatically fulfilled, *i.e.* the absolute zero actually does not exist being clearly impossible to remove the zero point energy, which is an intrinsic feature itself of any amount of confined matter.
- As expected, the related zero point temperature $\epsilon_{zp}^{\min}/k$ does not depend on the specific amount and physical nature of *m*.
- Is in principle possible the quantization of temperature, which accordingly should start from $T_{\min}$ and change by discrete steps of the order of $T_{\min}$ itself.

3) The reasoning to infer (5.21) and (5.25) introduces $D_{\min}$, whose physical meaning is relevant: it implies that $T_{\min}$ is somehow linked to the possible granular structure of the space time.

To show this last point, calculate the change $\delta\left(x_j t_k\right)$ of the space time coordinates around any local coordinate x_j and t_k. The result is elucidated by the following chains of equations implementing once more the Equations (3.1):

$$\delta\left(x_j t_k\right) = t_k \delta x_j + x_j \delta t_k = \left(t_k v_{jk} + x_j\right)\delta t_k, \quad v_{jk} = \frac{\delta x_j}{\delta t_k};$$

since the expression at right hand side reads

$$\frac{\left(t_k v_{jk}\right)^2 - x_j^2}{t_k v_{jk} - x_j}\delta t_k = \frac{\delta p_{jk}}{\delta \varepsilon_k}\delta\left(\ell^2\right) = \frac{\delta\left(\ell^2\right)}{v_{jk}}, \quad \delta p_{jk} = \frac{\hbar}{\delta x_{jk}}$$
$$\delta\left(\ell^2\right) = \left(t_k v_{jk}\right)^2 - x_j^2, \quad \delta x_{jk} = t_k v_{jk} - x_j,$$

with notation of (6.4), then

$$\delta\left(x_j t_k\right)=\frac{\delta\left(\ell^2\right)}{v_{jk}}=\frac{v_{jk}^2\delta\left(\ell\right)^2}{v_{jk}^3}=\frac{D_{jk}^2}{v_{jk}^3},\quad D_{jk}=v_{jk}\sqrt{\delta\left(\ell^2\right)}.$$

It is possible to identify here a minimum value of $\delta\left(x_j t_k\right)$ defined by $D_{jk}\to D_{\min}$ and $v_{jk}\to c$, at least asymptotically. Anyway one finds

$$\delta\left(x_j t_k\right)_{\min}=\frac{D_{\min}^2}{c^3},\quad \delta\left(\ell\right)^2\equiv\delta\left(\ell^2\right)_{inv},\quad \delta\left(\ell^2\right)_{inv}=\left(t_k c\right)^2-x_j^2,\quad v_{jk}=c. \quad (5.26)$$

Whatever the specific value of $D_{\min}$ might be, is interesting the conceptual idea of "granular" space time determined by a minimum linear size of cells, in the present one-dimensional model $\delta\left(length\times time\right)$, that define any macroscopic values of *xt* within these cell; the third (5.26) corresponds to the definition of invariant interval, which is known to be the basis of the special relativity [11].

Accordingly, the Lorentz transformations, in particular, should actually be nothing else but the straightforward consequence of the granular nature of space time.

5.6. Further Comments about the Zero Point Energy

This section generalizes the idea of regarding the zero point energy and volume (5.23) as intrinsic properties of matter, rather than as operative thermodynamic parameters related to specific experimental conditions. According to (5.23), think the zero point volume V_{zp} considering for example an atom surrounded by neighbor lattice atoms; V_{zp} corresponds to its free lattice volume, whatever it might be depending on temperature and mobility. The fact that ϵ_{zp} is defined by the confinement lattice volume around a given atom/ion, implies the limit $\epsilon_{zp}\to 0$ simply because for an isolated free particle $V_{zp}\to\infty$. Nonetheless ϵ_{zp} depends itself on *T*, both because of the thermal dilation of matter that modifies the size of lattice spacing and because the *T* dependent mobility allows one lattice atom to spread well beyond its volume at $T_{\min}$ via the so called "self-diffusion" [12]. Moreover V_{zp} also depends on the presence of lattice defects, which affect the free space available for its delocalization. In particular, one expects that the lattice atom is quenched in one lattice site at $T=T_{\min}$ only; in other words the volume *V* is a thermodynamic parameter experimentally set, whereas V_{zp} is determined by the physics of matter. In the absence of external fields at $T=T_{\min}$, therefore, $kT_{\min}$ is the minimum non-thermal energy of the lattice atom/ion, as it results from all possible interactions with lattice neighbors that determine the available free confinement volume; at $T>T_{\min}$, the thermal energy *kT* represents actually the additional contribution to the non-thermal zero point energy $\epsilon_{zp}=\epsilon_{zp}\left(T\right)$, so that it seems reasonable to think that in general the simple *kT* should be implemented as $kT+\epsilon_{zp}$. This holds in particular for the FD and BE statistical distributions. In the case of a single free particle this does not hold, as its delocalization volume is infinity by definition: but in general, when considering the thermodynamic properties of a body of matter, ϵ_{zp} cannot be longer omitted at least in principle.

To justify the legitimacy of this conclusion, consider first an ideal gas inside which energy exchanges occur via direct collisions between its molecules only. Without hypothesizing specific interactions between molecules, e.g. long range Coulomb or dipole interactions between electron shells, holds between p and ϵ of each molecule the general equation

$$\epsilon = \frac{pc^2}{v} = \frac{hc^2\tau}{\lambda\delta x}; \tag{5.27}$$

the second equality introduces the time lapse τ between any successive shocks, during which the molecule travels freely the distance $v\tau = \delta x$. During each τ therefore ϵ remains constant, since inside the gas the energy changes are supposedly due to direct collisions only. In particular, as concerns the zero point energy,

$$\mathcal{E}_{zp} = \frac{hc^2\tau}{\mathcal{V}^{2/3}}, \quad \mathcal{V} = \left(\lambda\delta x\right)^{3/2};$$

the notation is justified by the free volume $\mathcal{V}$ dependence analogous to that of the Equation (P77). The fact that $\mathcal{V}$ involves δx is not surprising; δx^{-1} comes from ϵ in (5.27) and agrees with $\delta p/\hbar$, being δp the total range of momentum change $p-\left(-p\right)$ after one shock between molecules.

Let ℓ be the distance between one molecule just after its last shock and the wall of the recipient containing the gas; in general $\ell \neq \delta x$. Moreover, as $-\partial\varepsilon/\partial\lambda = force$, (5.27) yields

$$\mathcal{F} = \frac{\epsilon}{\lambda} = \frac{hc^2\tau}{\mathcal{V}}, \quad \mathcal{V} = \lambda^2\delta x, \tag{5.28}$$

where $\mathcal{F}$ is the impact force of the concerned molecule against the wall. If A is the surface of the wall, then the pressure due to the shock of one molecule is

$$\mathcal{P} = \frac{\mathcal{F}}{A} = \frac{hc^2\tau}{A\mathcal{V}}. \tag{5.29}$$

If ℓ is such that $\ell A = \mathcal{V}_\ell$, then the former equation becomes $\mathcal{P} = hc^2\tau\ell/\mathcal{V}_\ell\mathcal{V}$; so, since the numerator has physical dimensions $energy \times volume$, the result reads

$$\mathcal{P} = \frac{\epsilon_o\mathcal{V}_o}{\mathcal{V}_\ell\mathcal{V}}, \quad \epsilon_o\mathcal{V}_o = hc^2\tau\ell. \tag{5.30}$$

Note that in general neither τ nor ℓ are necessarily constants independent of time; so the pressure $\mathcal{P}$ inferred in (5.30) could be variable during subsequent time lapses τ. Also note that to infer this result in the present one dimensional model it is enough to think the plane A orthogonal to the space coordinate δx. In a general three dimensional approach one should integrate over all possible incidence angles of the molecule against the wall to obtain the pressure, as it is well known. This would entail a numerical factor, which however can be included in $\epsilon_o\mathcal{V}_o$ and thus is irrelevant for the present purposes. Moreover, it is still possible to define a statistical value of $\mathcal{P}$ averaging the shocks

over several time lapses during a time range $\Delta t \gg \tau$. So is defined the quantum pressure

$$\mathcal{P}_{zp} = \frac{\epsilon_o \mathcal{V}_o}{\mathcal{V}_{zp}^2}, \quad \mathcal{V}_{zp}^2 = \overline{\mathcal{V}_\ell \mathcal{V}}, \quad \mathcal{V}_{zp} = \mathcal{V}_{zp}(T).$$

This result has been obtained considering the delocalization volume of one molecule; it holds in general for any number N of molecules regarding $\mathcal{V}_{zp}$ no longer as volume of a single molecule but as total experimental volume V_{exp} of ideal gas simply with the position $\mathcal{V}_{zp} = V_{exp}/N$. The right hand side is the average confinement size of the molecule. It is significant to conclude that the pressure of the gas must be expressed not only taking into account the variable dynamical parameter P_{exp} experimentally determined but adding to this latter the contribution $\mathcal{P}_{zp}$ having merely quantum nature; in effect the present reasoning waives considerations about any state equation of ideal or real gases. Also, regarding the macroscopic volume V_{exp} as a further dynamical parameter experimentally set, it contains the quantum contribution $N\mathcal{V}_{zp}$. So, extending the reasoning carried out for one gas molecule to the case of N molecules, one finds

$$P_{eff} = P_{exp} + \frac{fN^2}{\mathcal{V}_{zp}^2}, \quad V_{eff} = V_{exp} - N\mathcal{V}_{exp}, \quad f = \epsilon_o \mathcal{V}_o. \tag{5.31}$$

The volume $\mathcal{V}_{zp}$ is easily understandable, being intuitively evident that the molecules have finite size contributing to the total volume V_{exp} experimentally measurable. Even $\mathcal{P}_{zp}$ is guessable: if the zero point energy is simulated by an oscillator characterized by a non-thermal vibrational frequency that determines the zero point energy, see the next Equation (5.22), then the energy of any oscillator in the gas volume defines a confinement non-thermal energy density equivalent to pressure, see next Equations (8.34). Indeed (5.31) shows that even at $P_{exp} = 0$, e.g. the core of a free body of matter in the vacuum with zero applied pressure, there is a residual internal pressure, non-eliminable, e.g. it could act substantially similarly to the repulsion between electron shells of molecules; moreover the latter equation shows even a non-reducible residual volume $\mathcal{V} = N\mathcal{V}_o$. The relationship between energy, pressure and volume follows directly from (5.29) as $\mathcal{P}\mathcal{V} = \hbar c^2 \tau / A = const$ at fixed temperature: indeed at the right hand side of the first equality appear fixed quantities, of course at given ℓ and τ and thus constant T. So

$$P_{eff} V_{eff} = \left(\mathcal{P}_{zp} + \frac{fN^2}{\mathcal{V}^2} \right) \left(\mathcal{V} - N\mathcal{V}_{zp} \right).$$

This equation reminds closely the characteristic terms of the Van der Waals equation, where f and $\mathcal{V}_{zp}$ are approximately regarded as gas constants; this holds also here, even though the pressure and volume terms (5.31) have been inferred considering initially an ideal gas via quantum considerations about its constituting molecules. The interactions between these molecules, even not hypothesized and purposely introduced "a priori", appear as quantum effects re-

gardless of specific considerations about their actual nature, which is in effect "hidden" in parameters like τ or ℓ or f descriptive of the properties of the gas; hence is not surprising that the coefficients $\epsilon_o \mathcal{V}_o$ and $\mathcal{V}_o$ contribute to that characterizing the famous Van der Waals equation. The conceptual reasons underlying this equation are well known; is interesting however that the form of the resulting equation based on the present approach is analogous to that just found.

These considerations are now extended to the concept of temperature once having introduced the quantum meaning of $T_{\min}$ related to ϵ_{zp} to show that the zero point energy, typical quantum effect, affects the macroscopic properties of gases: ideal gas is the one where these quantum effects are approximately neglected along with the long range mutual interactions as well.

If effectively exists a minimum temperature $\neq 0$, then it must be defined by $kT_{\min} = \epsilon_{zp}$. However, the right hand side is in general function of T itself; indeed it has been shown that $\epsilon_{zp} \propto V_l^{-2/3}$, being V_l the lattice volume available around a given lattice site. So the thermal dilation modifies ε_{zp}, whose non-thermal physical meaning however still holds identically. Moreover the local mobility is itself T dependent, as it is intuitive to think; in effect it is known that by self diffusion, atoms in a given lattice point can exchange of place with lattice neighbors, so that the actual volume allowed to a given atom is increased by the number of neighbor elementary cells accessible.

These considerations should be also extended in particular to the statistical distributions of bosons and fermions, usually written as a function of kT only: taking into account the considerations elucidated in the case of the Van der Waals equation, one should conclude that strictly speaking in the case of a solid body the simple term kT should be replaced by $k\left(T+T_q\right)$, where kT_q accounts for the quantum contribution related to the T-dependent zero point energy with $T_q = T_q\left(T\right)$.

$$\begin{aligned} &\frac{w}{q_{\pm}} = \frac{W_0}{\exp\left(\left(\mu-\mu_0\right)/k\mathcal{T}\right)\pm 1}, \quad \mathcal{T} = nT_{\min} + \epsilon_{zp}/k \\ &\epsilon_{zp} = \epsilon_{zp}\left(T\right), \quad n = integer \geq 1. \end{aligned} \tag{5.32}$$

Even though the value of $T_{\min}$ is presumably much lower than the ordinary temperatures today attainable and experimentally measurable, the Equation (5.25) suggests the chance of being tested in a situation where ϵ_{zp} is relevant, *i.e.* in the case of theoretical models of solid state physics. In effect the paper [13] implements the ideas of quantized temperature and statistical distributions (5.32), both introduced as hypotheses; the specific heat calculated agrees very well with the experimental data from very low T up to the melting point for several metals with different crystal structures.

6. Some Relativistic Corollaries

In this section are examined some relevant relativistic corollaries of the previous

results. The importance of the following considerations is shown by the chance of obtaining contextually relativistic results in the same conceptual frame of the quantum results previously obtained. The next two subsections emphasize the importance of the previous equations (3.15) and (4.9), now again under test after their previous validations, see respectively Equations (3.20) and (4.6), (4.7), (4.13) to (4.18).

6.1. The Invariant Interval

Implement the Equation (3.15), once more under test after the result (3.20), re-written as follows

$$\Delta\epsilon = c\Delta(pr), \quad \Delta\epsilon = \sigma_\epsilon \varepsilon - \frac{b\varepsilon^2}{c^2} - \epsilon_a, \quad r = \sigma_p - ap/c \tag{6.1}$$

depending on whether $r \lessgtr 1$ one has $\Delta\epsilon \gtrless c\Delta p$. Hence, squaring both sides, $(\Delta\epsilon)^2 \gtrless (c\Delta p)^2$ reads

$$(c\Delta p)^2 - (\Delta\epsilon)^2 = K, \quad K \gtrless 0, \tag{6.2}$$

where anyway K is the resulting value from the left hand side of the Equation (6.2).

Consider first $K > 0$.

Implementing (3.1), one finds

$$\hbar^2 c^2 (\Delta x)^{-2} - \hbar^2 (\Delta t)^{-2} = K,$$

whence

$$\frac{\hbar^2}{(\Delta t \Delta x)^2}\left(c^2\Delta t^2 - \Delta x^2\right) = K; \tag{6.3}$$

this equation reads identically

$$\Delta(s^2) = \frac{K}{\hbar^2}(\Delta x \Delta t)^2 = (c\Delta t)^2\left(1 - \frac{v^2}{c^2}\right), \quad \Delta(s^2) = s'^2 - s''^2 = c^2\Delta t^2 - \Delta x^2, \tag{6.4}$$

having implemented once more $v = \Delta x/\Delta t$, whereas

$$\Delta x^2 = \frac{(\hbar c)^2}{K}\left(1 - \frac{v^2}{c^2}\right)$$

reads by dimensional reasons, whatever the value of K might be,

$$\Delta x'^2 = \frac{\Delta x^2}{1 - v^2/c^2}, \quad \Delta x'^2 = \frac{(\hbar c)^2}{K}. \tag{6.5}$$

It is easy to recognize the Lorentz transformation of the intervals $\Delta x'$ and Δx in two different inertial reference systems R and R', hence in (6.4) both $\Delta(s^2)$ and $\Delta x \Delta t$ must be invariants, as found in particular in (3.62); indeed K is invariant itself if r of (6.1) is calculated via invariant forms of ε and p, see Equations (3.64).

Consider now $K < 0$.

The interval defined in (6.3) reads $\Delta x^2 - c^2\Delta t^2 > 0$; moreover (6.2) reads $\Delta\epsilon^2 = (c\Delta p)^2 - K$, whence (3.20) with an appropriate value of *K*. Also this result holds regardless of the local limit condition $\Delta \to \partial$; so this is not a local property, but a feature of the whole space time. It is significant the fact that these results have been obtained with the help of the quantum uncertainty relationships (3.1).

6.2. The Gravity Force

Consider again the Equation (4.9), now once more under test after its early validation via the Equations (4.6), (4.7) and (4.13) to (4.18). Implementing the Equations (3.1) and the initial definition (2.2) of *v* one finds

$$\Delta t = \frac{v^2}{c^2}\frac{\delta x \delta p}{\delta\varepsilon} = \frac{\delta x^3}{c^2\delta t^2}\frac{\delta p}{\delta\varepsilon}$$

that yields, after multiplying and dividing the right hand side by an arbitrary mass *m*,

$$mc^2 = \mathcal{G}\frac{m}{\ell}, \quad \frac{1}{\ell} = \frac{\delta p/\Delta t}{\delta\varepsilon} = \frac{F}{\delta\varepsilon}, \quad \mathcal{G} = \frac{\delta x^3}{m\delta t^2}; \tag{6.6}$$

by dimensional reasons ℓ is an arbitrary length. The function $\mathcal{G}$ consists in general of a constant term *G* plus a correction term $\mathcal{R}$; indeed $\mathcal{G}$ can be expanded in series around arbitrary reference ranges δx_0, δt_0 and m_0 defining *G, i.e.*

$$\mathcal{G} = G + \mathcal{R}, \quad G = \frac{\delta x_0^3}{m_0\delta t_0^2}, \quad \mathcal{R} = \sum_{j=1}^{\infty} a_j u^j, \quad u = \frac{\delta x^3}{m\delta t^2} - \frac{\delta x_0^3}{m_0\delta t_0^2}, \tag{6.7}$$

being a_j appropriate coefficients of the power series expansion and $\mathcal{R}$ the sum of the higher order terms of the series. This means calculating $\mathcal{G}$ around an arbitrary value $\mathcal{G}_0 \equiv G$. So (6.6) reads

$$c^2 = (G + \mathcal{R})\frac{m}{\ell}, \tag{6.8}$$

It is evident that the general relativity appears in this result: the Newtonian potential Gm/ℓ recognizable in this equation is the approximate particular case of a more general equation involving $\mathcal{R}$ too. It is known for example that the simple addition of a further term to the Newtonian potential is enough to calculate correctly the perihelion precession even without implementing the basic assumptions and tensor calculus of general relativity [14]. Unfortunately the Newton physics does not justify itself this additional term. Nonetheless the mere series expansion of the last (6.6) around an arbitrary space time constant term *G* legitimates the chance of generalization without introducing "ad hoc" hypotheses. In this respect some further considerations exposed below regard in particular the additional non-Newtonian terms due to the series (6.7), still in the conceptual frame introduced by (1.11) and (1.12) as done throughout this paper. In effect, some papers among which [15] show that the quantum uncertainty allows

to infer the most relevant results of general relativity in a unique frame that includes of course the quantum physics. The presence of c^2 at the left hand side of (6.8) suggests the chance of multiplying both sides by a further arbitrary mass $\pm m'$; one finds then

$$\mp\mathcal{R}\frac{m'm}{\ell}\pm m'c^2=\epsilon_G=\pm G\frac{m'm}{\ell}, \tag{6.9}$$

which defines two important quantities

$$\phi_G=\frac{\epsilon_G}{m'}=\pm\frac{mG}{\ell},\quad F_G=-\frac{\delta\epsilon_G}{\delta\ell}\approx\pm G\frac{m'm}{\ell^2},\quad \mathcal{R}\ll G. \tag{6.10}$$

At the left hand side of (6.9) appears the rest energy $\pm m'c^2$ of m' plus a correction term resulting from the series expansion of G, at the right hand side the related potential energy ϵ_G to which correspond the pproximate gravitational potential ϕ_G and force F_G of (6.10) via the constant zero order term $\mathcal{G}$ only. Note that the Equations (2.35) and (3.64) have introduced the concept of states of negative energy; hence the left hand sides of (6.9) and (6.10) have physical meaning even regarding $\pm m'$ as unique mass $m_\pm$ with positive and negative energy states $\pm m'c^2=m_\pm c^2$. So the Equations (4.9) and (3.1) prospect a possible chance for positive sign of m', *i.e.* a repulsive gravity force as already found in [16] [17] [18], whereas (6.9) yields

$$\phi_G^*=\frac{mG}{\ell},\quad F_G^*=-\frac{\delta\epsilon_G^*}{\delta\ell}\approx G\frac{m'm}{\ell^2}. \tag{6.11}$$

Consider here the negative value $-m'$ and suppose $\mathcal{R}\ll G$; (6.9) and (6.10) yield

$$\phi_G=-\frac{mG}{\ell},\quad F_G\approx -G\frac{m'm}{\ell^2}. \tag{6.12}$$

The Equations (6.12) and (6.11) will be further explained just below, see the next Equation (8.6), after having first validated the results achieved in this subsection.

1) The Equation (6.8) reads also

$$\ell=\frac{G+\mathcal{R}}{G}\frac{mG}{c^2},$$

which in the particular case $\mathcal{R}=G$ implies a specific value of ℓ given by

$$\ell_{bh}=\frac{2mG}{c^2}. \tag{6.13}$$

To understand the physical meaning of this result, rewrite identically (6.8) multiplying both sides by an arbitrary factor $\sigma\le 1$; then

$$v_e^2=\mathcal{R}_oGm/\ell,\quad \mathcal{R}_o=(1+\mathcal{R}/G)\sigma,\quad v_e^2=\sigma c^2.$$

For $R_o=2$, in particular, one recognizes the well known escape velocity v_e of an arbitrary mass m' at a distance ℓ from the gravitational center of mass of m, also inferable via (6.12) under the condition of null total energy (potential plus kinetic) of m' at infinity. As $R_o=2$ is compatible with $\sigma=1$ simply

putting $\mathcal{R} = G$ in (6.8), this limit condition for the chance of m' at distance ℓ of escaping from the gravity field of m holds also for c. This velocity can be nothing else but that of a photon, so (6.13) shows that at any $\ell' \le \ell_{bh}$ even light cannot escape from the gravity field of m. Trivial manipulations of the early Equations (4.8) and (4.9) yield therefore the black hole limit condition between mass m and distance ℓ.

2) The present way to introduce the gravity force explains why any test mass m' behaves in the same way in the field of a source mass m: the masses m of (6.6) and m' of (6.9) have been introduced subsequently and independently each other. Note the conceptual difference between the present reasoning and that of the Newtonian approach: Newton has contextually introduced two masses to define their mutual interaction law, here the masses have been consequentially introduced starting from (6.6) because c^2 requires the concept of mass to introduce that of gravitational energy. It is not surprising that once having decided either mass as a field source, the behavior of the other is uniquely fixed: as both masses are independent and arbitrary, once having fixed m the behavior of m' is uniquely determined. In other words there is no reason to expect that any other mass m'' behaves in a different way from m' in the gravitational field of m because the law governing its dynamics has already been independently fixed.

3) Here m and m' are regarded in fact as gravitational and inertial masses; the first one defines the gravitational potential ϕ_G, the second one defines their mutual interaction force F_G. As they are interchangeable, their gravitational and inertial role is physically equivalent and indistinguishable. Thus gravitational and inertial masses must be equal.

4) Consider that (6.9) and (6.12) yield

$$\frac{\ell_{bh}}{\ell} = -2\frac{\phi_G}{c^2}, \quad \delta\left(\frac{\ell_{bh}}{\ell}\right) = -\frac{\ell_{bh}}{\ell^2}\delta\ell = -2\frac{\delta\phi_G}{c^2}. \tag{6.14}$$

Identify now the distance ℓ with one wavelength of a light beam propagating in the vacuum; thus

$$\ell = \frac{c}{\nu}, \quad \delta\ell = -\frac{c}{\nu^2}\delta\nu, \quad \frac{\delta\ell}{\ell^2} = -\frac{c\delta\nu}{\nu^2}\frac{\nu^2}{c^2} = -\frac{\delta\nu}{c}.$$

Hence, given a light wave propagating at distance $\ell > \ell_{bh}$, one finds

$$-\frac{\ell_{bh}}{\ell^2}\delta\ell = -2\frac{\delta\phi_G}{c^2} = -\ell_{bh}\frac{\delta\nu}{c};$$

then

$$\frac{\delta\phi_G}{c^2} = \frac{\delta\nu}{\nu_r}, \quad \nu_r = \frac{2c}{\ell_{bh}} = \frac{c^3}{mG}, \quad \delta\nu = \nu - \nu_r, \tag{6.15}$$

being ν_r a reference frequency of the wave. The first equation defines the famous red shift $(\nu - \nu_r)/\nu_r$ of a light wave due to the gravitational potential field change $\delta\phi_G$.

5) It is instructive to consider now two photons freely moving in the vacuum

on the diametric plane of two concentric hyperspheres: the inner photon just at radial distance ℓ_{bh} from the gravity center of m, given by (6.13), the outer photon at any radial distance $\ell > \ell_{bh}$ from m. The previous result shows that the inner photon cannot escape from the gravity field of m, so it can move on the surface of the inner hypersphere only, whereas any photon moving at very large distance $\ell \gg \ell_{bh}$ from m is free to travel unperturbed as a limit case from minus infinity to infinity or vice-versa. Is reasonable the idea that the outer photon moving at the closest approach distance $\ell > \ell_{bh}$ should follow an intermediate behavior, *i.e.* a curved space time path bent by m. This preliminary consideration justifies why the problem of light beam bending is tackled here with reference to the previous Equation (6.13). The standard way to approach the problem considers the curved trajectory traveled by the photon that follows the space time curvature along an arc δs around m at distance ℓ; the position of the photon before and after its closest approach to m defines the characteristic deflection angle $\varphi = \varphi(\ell, m)$ equal to that formed by the tangents to the osculating circumference at boundaries of δs. From a quantum point of view, however, the concepts of position and trajectory are missing, rather the approach must be similar to that followed to infer (4.16). Just for this reason the present reasoning is instructive to highlight how the quantum requirements are plugged into and provide information on this typical relativistic context.

Regard the arc δs of osculating circumference of radius ℓ defined by $\delta s = \ell\varphi$ conceptually according to of (3.1), *i.e.* as an uncertainty range where the photon is delocalized. Accordingly δs is actually given by two half angles $\delta s_{cw} = \ell\varphi/2$ and $\delta s_{ccw} = -\ell\varphi/2$ traveled by the photon along clockwise and counterclockwise paths around the middle point $\delta s/2$; indeed the photon displacements implied by δs_{cw} and δs_{ccw} are physically indistinguishable, because nothing is known about the motion features of any particle within an uncertainty range. This point of view skips the idea of a photon entering in δs through one boundary and exiting from the other boundary, which in fact would define δs as an element of trajectory. So $\delta s = \delta s_{cw} - (-\delta s_{ccw}) = \ell\varphi$ waives the whole φ, which would imply discriminating the events where the photon travels through δs coming from $-\infty$ towards ∞ or from ∞ towards $-\infty$; actually these events are indistinguishable likewise the boundaries of δs themselves. So, with respect to gravity center of m, the angle of physical interest is $\varphi/2$ and not φ to account for the sought total δs. In other words the Equations (3.1) compel merging two half-paths into a unique travel path without discriminating either of them.

Consider then an angle $\varphi/2$ on a circumference of radius ℓ and its related length $(\delta s/2)/\ell$ to describe the uncertainty range δs where is delocalized the photon. Rewrite the second (6.14) as follows

$$\frac{\ell_{bh}}{\ell} - \frac{\ell_{bh}}{\ell_o} = -\frac{\delta s}{\ell} = -2\left(\frac{mG/c^2}{\ell} - \frac{mG/c^2}{\ell_o}\right), \quad -\frac{\ell_{bh}}{\ell}\delta\ell = \delta s;$$

here the uncertainty ranges $\delta(\ell_{bh}/\ell)$ and $\delta(\phi_G/c^2)$ have been simply re-

written with the usual notation of any $\delta y = y - y_o$ by definition, whatever the concerned y might actually be. In this specific case ℓ and ℓ_o are two different distances of the photon from the gravity center of m. It is clear that it is convenient to put here $\ell_o \to \infty$ because, as previously stated, we are interested to describe the situations where one photon initially unperturbed passes at a finite distance ℓ from m. Hence the last equation reads

$$\frac{\ell_{bh}}{\ell} = -\frac{\delta s}{\ell} = -2\frac{mG/c^2}{\ell}.$$

If, for the aforesaid quantum reasons $\delta s/\ell = \varphi/2$ on the osculating circle, one finds immediately

$$\frac{\varphi}{2} = \frac{l_{bh}}{\ell} = 2\frac{mG}{c^2\ell}.$$

These simple considerations emphasize the actual quantum character of one of the most representative relativistic predictions, the gravitational lensing; the famous factor 4 defining φ at the right hand side appears to be actually the fingerprint of the quantum uncertainty.

6) Consider the physical dimensions of the gravity constant inferred from (6.6): according to (6.7), space and time range sizes δx_0 and δt_0 concur to its macroscopic value together with the arbitrary mass m_0. From a quantum point of view, (6.7) does not exclude the chance of mass fluctuation, *i.e.* according to (4.6)

$$\begin{aligned} \delta G &= -\frac{\delta x_0^3}{m_0^2 \delta t_0^2}\delta m_0 + \frac{1}{m_0}\delta\left(\frac{\delta x_0^3}{\delta t_0^2}\right), \\ \delta m_0 &= m_0' - m_0 = m_0\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right). \end{aligned} \tag{6.16}$$

Even considering preliminarily the fluctuation of m_0 only, and thus energy fluctuation $\delta\epsilon^* = c^2\delta m_0$ only, the constancy of G is expressible as

$$\delta G = 0, \quad \frac{\delta x_0^3}{m_0 \delta t_0^2}\delta m_0 = m_0\delta\left(\frac{\delta x_0^3}{\delta t_0^2}\right)$$

i.e. $\delta\epsilon^*$ implies the change of $\delta x_0^3/\delta t_0^2$ as well. Note that δx_0 and δt_0 are not usual dynamical variables characterizing physical laws, rather they define the structure of G itself. In effect, this quantum standpoint implies that transient fluctuations of values of G are in principle possible, being compatible with corresponding space time energy quantum fluctuations $\delta\epsilon^*$. Making explicit the right hand side of this last equation, trivial algebraic steps yield

$$\begin{aligned} \delta G &= 3\delta x_0 \frac{x_0^2}{m_0 t_0^2} - 2\delta t_0 \frac{x_0^3}{m_0 t_0^3} - \delta m_0 \frac{x_0^3}{m_0^2 t_0^2} \\ &= \frac{x_0^2 \delta x_0}{m_0 t_0^2}\left(3 - 2\frac{x_0}{t_0}\frac{\delta t_0}{\delta x_0} - \frac{x_0}{m_0}\frac{\delta m}{\delta x_0}\right) = 0 \end{aligned}$$

so that $G = const1$ simply if

$$\frac{x_0}{t_0}=\sigma_1\frac{\delta x_0}{\delta t_0},\quad \frac{m_0}{x_0}=\sigma_2\frac{\delta m_0}{\delta x_0},\quad \frac{m_0}{t_0}=\frac{\delta m_0}{\delta t_0},\quad 2\sigma_1+\frac{1}{\sigma_2}=3;$$

otherwise, e.g. a different correspondence between σ_1 and σ_2 whatever their values might be, the quantum definition of G admits $\delta G\neq 0$. All this has clearly to do with the existence of inflationary era of the early Universe: an appropriate fluctuation $\delta G\neq 0$ of G can contribute in principle to the sudden increase of expansion rate of the early Universe. Among the implications of these assertions, one deserves particular attention: the possible fluctuations of G affect the black hole length ℓ_{bh} of (6.13) of a given m. Unfortunately further discussion on this crucial point is outside the aims of the present paper.

7) Returning now to the Equations (5.25), the maximum value of momentum wavelength λ_{zp} has been related to a suitable space range Δr defined as the radius of a hypersphere within which is delocalized an arbitrary amount of mass. As the steady wavelength appropriate to calculate the zero point energy $\epsilon_{zp}^{\min}$ defining $T_{\min}$ corresponds to the maximum delocalization extent physically conceivable in our Universe, is reasonable to relate λ_{zp} to the diametric size $2\Delta r_u$ of the Universe, regarded here as a hypersphere with diameter $2\Delta r_u$. So think the mass ideally bouncing within one diametric distance, whose maximum space extent corresponds to one half wavelength; this wave has thus steady nodes at the diametric boundaries of the Universe, regarded in effect as a hypersphere. Replacing Δr of the Equation (5.25) with Δr_u and putting $v\to c$ asymptotically, one finds

$$\epsilon_{zp}^{\min}=\frac{hc}{4\Delta r_u},\quad T_{\min}=\frac{\epsilon_{zp}^{\min}}{k}=\frac{hc}{4k\Delta r_u},\quad \omega_u=\frac{2\epsilon_{zp}^{\min}}{\hbar}=\frac{\pi c}{\Delta r_u}. \tag{6.17}$$

The current estimate $\Delta r_u=4.35\times10^{28}$ cm [19] yields the numerical values

$$\epsilon_{zp}^{\min}=1.2\times10^{-45}\text{ erg},\quad T_{\min}=8.3\times10^{-30}\text{ K},\quad \omega_u=2.2\times10^{-18}\text{ s}^{-1} \tag{6.18}$$

In addition to the preliminary remarks about the Equation (5.25) previously emphasized, these numerical results suggest further implications:

- Finite Universe means identically $\epsilon_{zp}^{\min}\neq 0$.
- The limit $T_{\min}\to 0$ would hold in an infinite universe only.
- To guess the physical meaning of the small value of ϵ_{zp}, note that $1/\omega=4.6\times10^{17}$ s fits surprisingly well the estimated age of the universe 4.35×10^{17} s reported in [19].
- The fact that the energy corresponding to λ_{zp} agrees reasonably with the estimated order of magnitude of the age of the universe, suggests that $\epsilon_{zp}^{\min}$ with which has been calculated $T_{\min}$ could be a possible vacuum energy fluctuation, still in progress, of the whole Universe.
- It is interesting that ω_u that determines $T_{\min}$ agrees surprisingly well with the Hubble constant.

8) It is possible to implement these results to calculate another important property of the Universe, *i.e.* the vacuum energy density η_u. In general, the energy density corresponds from the dimensional point of view to

$mass \times frequency^2 / length$. Consider now that according to (6.13) nothing, even the light, can escape from a range size ℓ_{bh} enclosing the mass m; hence, the Equation (6.13) represents a significant opportunity to describe how to trap inside a volume of space time energy that cannot be irradiated nor lost outside it. After having inferred that an energy $\hbar\omega_u$ pervades all Universe, whose zero point value $\hbar\omega_u/2$ determines the zero point energy defining $T_{\min}$, is attracting the idea that η_u can be calculated just with the value of ω_u controlling $T_{\min}$. This idea links the vacuum energy density to the zero point energy (5.22) related to $kT_{\min}$. If so, then calculating $m/\ell = c^2/2G$ according to (6.13) and implementing ω_u just calculated,

$$\eta_u = \frac{c^2}{2G}\omega_u^2 = 3.3\times10^{-8}\ \text{erg}/\text{cm}^3 = 3.3\times10^{-9}\ \text{J}/\text{m}^3. \tag{6.19}$$

The sensible value of vacuum energy density further supports the way to calculate the values (6.18). This means that the concept of vacuum does not imply that of "nothing": rather the vacuum must consist of virtual particles whose energy, *i.e.* mass, governs the residual vacuum energy density in agreement with the third law of thermodynamics.

9) But there is more. Implement the mass M_u of the part of visible Universe [16] estimated counting the stars only to calculate

$$M_u = 3\times10^{55}\ \text{g}, \quad \frac{2M_uG}{c^2} = 4.5\times10^{27}\ \text{cm}. \tag{6.20}$$

On the one hand one expects that M_u is estimated by defect, without taking into account that other forms of energy distributed in the Universe could in principle increase this value; whatever this additional energy ϵ^* might be, it concurs with M_u by ϵ^*/c^2 to the total mass of the Universe. On the other hand are visible only the stars whose distance does not overcome the observation limit posed by the light speed; assuming that the distribution of galaxies and thus stars is uniform in the Universe, the actual mass due the total number of stars should be

$$M^* \approx 3\times10^{55}\left(\frac{\Delta r_u}{c\Delta t_u}\right)^3 = 1.1\times10^{57}\ \text{g}.$$

This value still estimates the total visible matter of the Universe as if the light speed would be infinite; as such, however, it does not tell anything about other possible contributions inherently "dark", *i.e.* non-luminous, for example the vacuum energy/c^2 or the zero point energy/c^2. Compare thus just this value M^* with the total mass related to the whole vacuum energy density η_u calculated above. It is

$$\frac{4}{3}\pi\Delta r_u^3\frac{\eta_u}{c^2} = 1.3\times10^{58}\ \text{g}$$

The total vacuum energy of the Universe is still about 12 times higher than that of all visible objects. Make at this point a hypothesis:

The energy density of vacuum and that of matter are equal, *i.e.* regard matter

and vacuum as two different thermodynamic systems at global equilibrium.

So M^* must be incremented just by this factor to make equal the respective densities. Write thus the total mass balance as a function of the true visible mass as

$$2M^* + M_p + M_\varepsilon = M_{tot} = 12M^* \tag{6.21}$$

the factor 2 accounts for the antimatter, wherever it might be in the Universe, whereas M_p and M_ε are the missing masses also concurring to the factor 12 assumed true. The notations account for the fact that the concept of mass can be defined in general via p/v and ε/v^2. Rewrite now these positions with the help of (2.36)

$$M_p = \frac{h}{\lambda_p v}, \quad M_\varepsilon = \frac{h\nu_\varepsilon}{v^2}, \quad \nu_\varepsilon = \frac{v}{\lambda_\varepsilon} \tag{6.22}$$

where λ_p and ν_ε are the pertinent momentum wavelength and energy frequency. Thus

$$M_p = \frac{v_o}{v}\frac{h}{\lambda_p v_o}, \quad M_\varepsilon = \left(\frac{v_o}{v}\right)^2 \frac{h\nu}{\lambda_\varepsilon v_o^2}. \tag{6.23}$$

being v_o arbitrary velocity. Moreover rewrite identically (6.21) as

$$\begin{aligned} &2M^* + q_p M^* + q_\varepsilon M^* = 12M^*, \quad M_p = q_p M^*, \\ &M_\varepsilon = q_\varepsilon M^*, \quad M^* = \frac{h}{\lambda_p v_o} = \frac{h\nu}{\lambda_\varepsilon v_o^2}, \end{aligned} \tag{6.24}$$

where q_p and q_ε are appropriate coefficients able to express numerically M_p and M_ε via M^*. Actually the physical meaning of these positions is to establish a relationship between visible mass and the other contributions to M_{tot}. The last position, in particular, is possible because λ_p and λ_ε are arbitrary. One equation to determine these coefficients is obviously

$$q_p + q_\varepsilon = 10, \quad q_p = q_p(x,t), \quad q_\varepsilon = q_\varepsilon(x,t). \tag{6.25}$$

Moreover, as (6.23) reads

$$\frac{v_o}{v} = q_p, \quad \left(\frac{v_o}{v}\right)^2 = q_\varepsilon = q_p^2, \quad \frac{v}{\lambda_\varepsilon} = \frac{v_o}{\lambda_p}, \tag{6.26}$$

(6.24) yields

$$q_p M^* + q_\varepsilon M^* = \left(q_p + q_p^2\right)M^*, \quad q_p + q_p^2 = 10, \tag{6.27}$$

and thus $q_p = 2.7$ and $q_\varepsilon = 7.3$. Being the third position (6.26) certainly fulfilled via the arbitrary v_o, which however does not appear in (6.27), (6.24) and (6.21) become

$$2M^* + 2.7M^* + 7.3M^* = 12M^*, \quad M_p = 2.7M^*, \quad M_\varepsilon = 7.3M^*,$$

which are expressed more significantly in relative %:

$$(8.3 + 8.3)\%M^*, \quad 22.5\%M_p, \quad 60.8\%M_\varepsilon. \tag{6.28}$$

The papers [10] propose a possible explanation, here omitted for brevity,

about why matter and antimatter are separated. It is more important to note that there is no "ad hoc" hypothesis in this reasoning, rather a further implication of the fundamental concept of uncertainty repeatedly invoked throughout all this paper and again exploited here through the assumption of vacuum/matter equilibrium. In effect the mass densities calculable via the terms M_p and M_ε additional to M^* in (6.21) correspond to the respective energy densities and thus to pressure terms inside the Universe. The next Equations (8.28) to (8.34) elucidate this point.

10) Multiply (6.6) by the mass m_a introduced in (3.20) and (3.21); recalling the second (3.26) it is possible to write

$$m_a c^2 = \frac{2\epsilon_a}{n'\sigma_\varepsilon^2 + n^{-2}} = G\frac{m_a m}{\ell}, \tag{6.29}$$

which links via c^2 the Newtonian gravity and quantum energy at the left hand side. Whatever σ_ε might be, the arbitrary distance ℓ is repalced by arbitrary quantum numbers n and n'. In this sense (3.20) is reasonably defined in [6] as equation of quantum gravity.

It is true that actually G should be replaced by $\mathcal{G}$ to plug all considerations carried out in the subsection 6.2 into the relativistic realm via (6.7); but it is also true that actually m_a has been defined in order to make σ_ε and σ_p of Equation (3.15) consistent with the harmonic oscillator form (3.25) consequent (3.24). Modifying the definitions (3.23) in order to define non-harmonic oscillations would mean adding additional correction terms corresponding to the higher order terms of the series (6.7). It is possible to say shortly that the Newton gravitational energy corresponds to quantum harmonic oscillators at the left hand side of (3.25), the relativistic gravity is described by non-harmonic oscillator terms replacing the mere $n'h\nu$ of (3.25).

7. Klein Gordon, Proca and Maxwell Equations

Implement now the Equations (3.30) introducing a function of $\psi^{(0)}$ defined as follows

$$\psi^{(0)} = Q\exp(\sigma\psi), \quad \psi^{(0)} = \psi^{(0)}(x',t), \tag{7.1}$$

where σ and Q are arbitrary constants. So the first Equation (3.30) yields

$$\frac{1}{\psi^{(0)}}\frac{\delta^2\psi^{(0)}}{\delta t^2} = \sigma\left(\frac{\delta^2\psi}{\delta t^2} + \sigma\left(\frac{\delta\psi}{\delta t}\right)^2\right) = \sigma(1\pm\sigma)\frac{\delta^2\psi}{\delta t^2}; \tag{7.2}$$

proceeding in analogous way with the help of (3.31), the second Equation (3.30) yields

$$\frac{1}{\psi^{(0)}}\frac{\delta^2\psi^{(0)}}{\delta x'^2} = \sigma(1\pm\sigma)\frac{\delta^2\psi}{\delta x'^2}. \tag{7.3}$$

Hence, replacing (7.3) and (7.2) in the right hand side of (3.32), the result is an equation expressed as a function of the functional $\psi^{(0)}$ of ψ

$$\left(\frac{m'c}{\hbar}\right)^2\psi^{(0)} = -\frac{\delta^2\psi^{(0)}}{\delta t^2} + \frac{\delta^2\psi^{(0)}}{\delta x'^2}, \quad m' = m\sqrt{\sigma(1\pm\sigma)}; \tag{7.4}$$

as m appears at the left hand side only of (3.32), it has been included in m' together with the factor $\sigma(1\pm\sigma)$ without loss of generality. Taking the limit $\delta \to \partial$, which by consequence implies $\psi^{(0)} \to \psi_{KG}$ as well, this one dimensional result is actually the well known Klein Gordon equation

$$\left(\frac{m'c}{\hbar}\right)^2\psi_{KG} = -\frac{\partial^2\psi_{KG}}{\partial t^2} + \frac{\delta^2\psi_{KG}}{\partial x'^2} \to \Box\psi_{KG} + \left(\frac{m'c}{\hbar}\right)^2\psi_{KG} = 0,$$
$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2}.$$

On the one hand it confirms the validity of the positions (3.30) and thus of (3.39) too; on the other hand this result shows that the Klein Gordon equation is inferred from the local functional (7.1) of the space time function (3.39). The latter equation simply rewrites the former according to the usual 4D d'Alabertian.

This result can be further generalized taking advantage that the last equation is actually expressed as a function of m' and not of the initial m. The fact that m' can take two values depending on either sign in (7.3) suggests that in fact two equations are tacitly implied by the unique Equation (7.1); for example one scalar equation and one vector equation should be compatible with (7.3). This is very easily proven showing that the scalar and vector fields of the Proca equations can be combined into one resulting Klein Gordon-like equation. The most straightforward way to demonstrate this assertion starts just from the Proca equations

$$\Box\phi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = -\left(\frac{mc}{\hbar}\right)^2\phi,$$
$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = -\left(\frac{mc}{\hbar}\right)^2\boldsymbol{A},$$

which actually, owing to the definition of the operator $\Box$, read respectively

$$\nabla^2\phi + \frac{\partial(\nabla\cdot\boldsymbol{A})}{\partial t} = \left(\frac{mc}{\hbar}\right)^2\phi, \quad \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla\phi)}{\partial t} = -\left(\frac{mc}{\hbar}\right)^2\boldsymbol{A}; \tag{7.5}$$

thus ϕ and $\boldsymbol{A}$ are the sought scalar and vector fields linking the two equations. Multiply the former equation by an arbitrary function $f = f(x,y,z,t)$ and the latter by an arbitrary velocity vector $\boldsymbol{v}_o = \boldsymbol{v}_o(x,y,z,t)$, *i.e.*

$$f\nabla^2\phi + f\frac{\partial(\nabla\cdot\boldsymbol{A})}{\partial t} = \left(\frac{mc}{\hbar}\right)^2 f\phi,$$
$$\boldsymbol{v}_o\cdot\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} + \boldsymbol{v}_o\cdot\frac{1}{c^2}\frac{\partial(\nabla\phi)}{\partial t} = -\left(\frac{mc}{\hbar}\right)^2\boldsymbol{v}_o\cdot\boldsymbol{A}.$$

Subtracting side by side these equations,

$$f\left(\nabla^2\phi + \frac{\partial(\nabla\cdot\boldsymbol{A})}{\partial t} - \left(\frac{mc}{\hbar}\right)^2\phi\right) = \boldsymbol{v}_o\cdot\left(\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla\phi)}{\partial t} + \left(\frac{mc}{\hbar}\right)^2\boldsymbol{A}\right), \tag{7.6}$$

trivial algebraic steps yield the following scalar equation

$$f\nabla^2\phi + f\frac{\partial(\nabla\cdot\boldsymbol{A})}{\partial t} - \boldsymbol{v}_o\cdot\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{v}_o\cdot\frac{1}{c^2}\frac{\partial(\nabla\phi)}{\partial t} = \left(\frac{mc}{\hbar}\right)^2(f\phi + \boldsymbol{v}_o\cdot\boldsymbol{A}). \quad (7.7)$$

As f is an arbitrary function, it can be defined in order that

$$f\frac{\partial(\nabla\cdot\boldsymbol{A})}{\partial t} - \boldsymbol{v}_o\cdot\frac{1}{c^2}\frac{\partial(\nabla\phi)}{\partial t} - \boldsymbol{v}_o\cdot\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -f\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}; \quad (7.8)$$

hence

$$f\nabla^2\phi - f\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \left(\frac{mc}{\hbar}\right)^2(f\phi + \boldsymbol{v}_o\cdot\boldsymbol{A}). \quad (7.9)$$

Eventually it is possible to infer from this equation

$$f\nabla^2\phi - f\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \left(\frac{mc}{\hbar}\right)^2(1+q)f\phi, \quad \boldsymbol{v}_o\cdot\boldsymbol{A} = qf\phi, \quad q \gtreqless 0; \quad (7.10)$$

both f and $\boldsymbol{v}_o$ are arbitrary, thus the last position just introduced is in fact possible. Actually the first (7.10) does not depend on f and has still the form of a Klein Gordon-like equation for the previous scalar function ϕ, where appears however the factor $1+q$ at the right hand side corresponding to the previous $\sigma(1\pm\sigma)$ of (7.3). Just this factor is the key to split this double valued equation: indeed $q=0$ is one scalar equation, whereas the additional chance $q\neq 0$ allows inferring the couple of Proca equations via the position $\boldsymbol{v}_o\cdot\boldsymbol{A} = qf\phi$ that introduces the vector field $\boldsymbol{A}$. It is enough to revert the steps from (7.10) to (7.5) still via the key position (7.8). The profound physical meaning of this position, here purposely introduced to obtain (7.9), is shortly outlined below, to show that it is not merely a useful algebraic step.

The Equation (7.8) reads

$$f\left(\frac{\partial(\nabla\cdot\boldsymbol{A})}{\partial t} + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right) = \frac{\boldsymbol{v}_o}{c^2}\cdot\left(\frac{\partial(\nabla\phi)}{\partial t} + \frac{\partial^2\boldsymbol{A}}{\partial t^2}\right) \quad (7.11)$$

i.e.

$$f\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \frac{\boldsymbol{v}_o}{c^2}\cdot\frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial\boldsymbol{A}}{\partial t}\right). \quad (7.12)$$

In this equation it is possible to put formally

$$\nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0, \quad \nabla\phi + \frac{\partial\boldsymbol{A}}{\partial t} = -\boldsymbol{E}, \quad \frac{v_o}{c^2}\cdot\frac{\partial\boldsymbol{E}}{\partial t} = 0 \quad (7.13)$$

the first position is the Lorentz condition, the second equation is the definition of the new quantity $\boldsymbol{E}$, the third equation is obtained replacing $\boldsymbol{E}$ at the right hand side of (7.11) and simply means that $\boldsymbol{v}_o$ is orthogonal to $\partial\boldsymbol{E}/\partial t$ to get $0=0$ at both sides. Now give $\boldsymbol{A}$ physical meaning introducing the following positions

$$\boldsymbol{H} = \nabla\times\boldsymbol{A}, \quad \nabla\cdot\boldsymbol{H} = 0, \quad \nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} \quad (7.14)$$

the first position is simply the definition of a new quantity $\boldsymbol{H}$, whence follows the second equation by consequence; the third equation is obtained taking $\nabla\times$ of both sides of the second (7.13). Note now that the first and second (7.13) yield

$$-\nabla\cdot\left(\nabla\phi+\frac{\partial\boldsymbol{A}}{\partial t}\right)=-\left(\nabla^2\phi+\frac{\partial\nabla\cdot\boldsymbol{A}}{\partial t}\right)=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi=\nabla\cdot\boldsymbol{E}=\rho \qquad (7.15)$$

Eventually it is possible to infer a fourth equation considering the continuity condition $\delta g=0$ of an arbitrary function $g=g(x,t)$, which reads $\delta g=(\partial g/\partial t)\delta t+(\partial g/\partial x)\delta x=0$; thus the one dimensional expression $\partial g/\partial t+(\partial g/\partial x)v=0$, where v must be intended of course as $v_x=\delta x/\delta t$, reads in general $\boldsymbol{v}\cdot\nabla g=-\partial g/\partial t$. Hence $\boldsymbol{v}\cdot\nabla g=\nabla\cdot(g\boldsymbol{v})-g\nabla\cdot\boldsymbol{v}$ yields

$$g\nabla\cdot\boldsymbol{v}=0=\frac{\partial g}{\partial t}+\nabla\cdot(g\boldsymbol{v}).$$

Since g has not yet been defined, it is possible to rewrite this equation implementing the scalars already inferred, to obtain a self-contained set of equations. Put

$$g=\frac{1}{c}\nabla\cdot\boldsymbol{E}=\rho,\quad g\boldsymbol{v}=\rho\boldsymbol{v}=\boldsymbol{J},$$

which yields therefore

$$\nabla\cdot\nabla\times\boldsymbol{X}=0=\frac{1}{c}\frac{\partial\nabla\cdot\boldsymbol{E}}{\partial t}+\nabla\cdot\boldsymbol{J};$$

It is convenient to utilize the vector property $\nabla\cdot\nabla\times(\,)=0$ in agreement with $\nabla\cdot\boldsymbol{v}=0$ of solenoidal flux of ρ to simplify this last equation whatever the arbitrary field $\boldsymbol{X}$ might be. Thus one obtains

$$\nabla\times\boldsymbol{X}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}+\boldsymbol{J},\quad \boldsymbol{J}=\rho\boldsymbol{v}. \qquad (7.16)$$

Despite the lack of specific information to identify the actual nature of $\boldsymbol{X}$, it is reasonable to put $\boldsymbol{X}\propto\boldsymbol{H}$ with $\boldsymbol{H}$ magnetic field to be introduced just in the next section: this position is in fact possible without introducing a new field, hardly justifiable in the present context. Hence the position (7.8) is valuable as it implies four relevant equations, whose importance appears by answering questions like: flux of what? what kind of fields are $\boldsymbol{E}$ and $\boldsymbol{H}$? The next section clarifies these points that clearly allow to obtain the Maxwell equations (7.14), (7.15) and (7.16).

These fields allow modifying appropriately the functional (7.1), in order to describe one particle even via a possible interaction potential; of course such a calculation is omitted here for brevity, being clearly beyond the purposes of the present paper.

8. The Fundamental Interactions of Nature: Force Laws

Some considerations about the gravity force in the subsection 6.2 were inferred starting from the Equation (4.9). Now the concept of force is reexamined start-

ing from the more general Equations (3.1) and (3.69). This section consists of three subsections.

8.1. Preliminary Considerations

The Equations (3.1) provide several chances of defining the general concept of force, directly related to (3.69) and (3.70)

$$\frac{\delta p}{\delta t}=\frac{\delta \varepsilon}{\delta x}=F \tag{8.1}$$

whence, implementing once more $\delta p=\hbar/\delta x$, one infers in general

$$\delta \dot{p}=-\frac{\hbar}{\delta x^2}\delta \dot{x},\quad \delta \dot{x}=\frac{\delta x}{\delta t},\quad \delta \dot{p}=\frac{\delta p}{\delta t}. \tag{8.2}$$

To understand intuitively the correlation between F and $\delta \dot{x}$, think that the effect of any force is to modify the state of motion of matter on which it acts. Consider a body of mass m under the action of a force F. If m is actually a system of particles, then F perturbs the dynamics of all particles of m: for example they move faster. According to (3.1), modifying the kinetic energy of a system of particles implies changing in principle their delocalization extent and thus the range size δx able to include each one of them. The greater the force altering the status of a system, the greater must be the size change rate $\delta \dot{x}$ to account for the altered delocalization extent of matter in δx. This is the intuitive way to justify in general the quantum link between F and the related $\delta \dot{x}$. More specifically (8.1) also imply

$$v=\frac{\delta x\delta \varepsilon}{\hbar},\quad \delta x\delta \varepsilon \le \hbar c; \tag{8.3}$$

the inequality is direct consequence of the first equation with δt^{-1} replaced by $\delta\varepsilon/\hbar$. Again, the position $\delta \to \partial$ yields the usual definition of generalized local force $F=\dot{p}$. These equations are directly referable to long range interactions, because δx is defined even at infinity in lack of specific constrains; being directly inferred from the general Equations (3.1), F is expected to hold for charged and neutral particles. In effect the Newton and Coulomb forces represent an important class of forces that vanish at infinity as δx^{-2}, justifiable simply assuming $\delta \dot{x}=const$; is evident the analogy of (8.2) with F_G of (6.10).

Beside (8.2), particularly interesting are further dimensional considerations about characteristic space and time ranges related some specific forces. Write for example

$$F=\frac{\hbar^2}{mV}=\frac{(\hbar c)^2}{mc^2V}=\frac{\hbar}{V}\frac{\delta x^2}{\delta t},\quad F=\frac{\hbar}{\delta x\delta t} \tag{8.4}$$

the first definition follows from $\delta p^2/m\delta x=\hbar^2/m\delta x^3$. The Equations (8.4) and (8.3) prospect the chance of introducing the concept of short range force as that related to characteristic lengths, times and possibly volume consistently with the Equations (3.1). An example in this respect is the zero point energy resulting from the confinement of matter in a finite volume of space time, already intro-

duced in (5.23); another example is the Casimir force, shortly sketched below.

The ranges $\delta\varepsilon$ and δx of (8.4) help to fix the energy scale or the distance scale characteristic of specific interactions. Are useful in this respect the values (3.12) of length ranges; the Equations (8.4) are implementable with these scale lengths of prospective interest to estimate the strength of short range forces.

The few remarks exposed here highlight how to proceed in various cases. First of all, it is necessary to examine how long range and short range forces are included in the general definition (8.1) of *F*: the key point is the deformation rate $\delta\dot{x}$ of the space time range δx.

Let two interacting partners be δx apart and expand in series $\delta\dot{x} = \delta\dot{x}\left(\delta x^{-1}\right)$: this position ensures that *F* vanishes at $\delta x \to \infty$. Write thus in general

$$\delta\dot{x} = \delta\dot{x}_0 + \frac{Ac}{\delta x} + \frac{Bc}{\delta x^2} + \cdots \tag{8.5}$$

where *A* and *B* are the constant coefficients of the series expansions. So (8.2) splits into the sum of various terms

$$F_0 = \pm\frac{\hbar\left|\delta\dot{x}_0\right|}{\delta x^2}, \quad F_1 = \pm\frac{\hbar\left(\ell_0 + \ell_1\right)c}{\delta x^3}, \quad F_2 = \pm\frac{\hbar c\left(\ell_2 + \ell_3\right)^2}{\delta x^4},$$
$$\ell_0 + \ell_1 = A, \quad \left(\ell_2 + \ell_3\right)^2 = B,$$

Here $\delta\dot{x}_0 = const$ by definition, with signs of F_0 dependent on whether $\delta\dot{x}$ swells or shrinks during δt. The Equation (8.5) is more general than (6.7): the latter concerns specifically $\mathcal{G}$ and thus the gravity, *i.e.* F_0 corresponds to F_G, whereas (8.5) instead concerns more in general $\delta\dot{x}$; the higher order terms expressing $\delta\dot{x}$ include $\mathcal{R}$ of (6.7). Just to show this point the coefficients *A* and *B* have been split introducing constants lengths ℓ_i that characterize various kinds of forces in fact included in (8.2) and made explicit by the respective terms of the series expansion of *F*. Examine thus one by one the forces resulting from the first three terms of (8.5) to show that in effect this approach is interpretable in a sensible way. The first force identifiable is

$$F_{NC} = \pm\frac{\hbar\left|\delta\dot{x}_0\right|}{\delta x^2} + \frac{\hbar c\ell_1}{\delta x^3} + \frac{\hbar c\ell_2^2}{\delta x^4} + \cdots, \tag{8.6}$$

where the subscript *NC* stands for Newton Coulomb. The second and third forces easily identifiable are

$$F_{zp} = \frac{\hbar\ell_0 c}{\delta x^3} = \pm\frac{\hbar^2}{2mV_{zp}}, \quad \ell_0 = \frac{\hbar}{2mc},$$

having inferred ℓ_0 by comparison with (5.23), and

$$P_{Ca} = \frac{F_2}{\ell_2\ell_3} = \pm\frac{\hbar c\sigma}{\delta x^4}, \quad \sigma = \frac{\ell_3}{\ell_2} + 2.$$

One finds again the zero point force controlled by the Compton length ℓ_0 of the mass *m*. Moreover one recognizes the Casimir force per unit surface $\ell_2\ell_3$ given by $\hbar c$ times the pertinent numerical factor σ whose value is controlled

by ℓ_3 whatever the value of ℓ_2 contributing to F_G might be. In this way neither ℓ_0 nor ℓ_1 and ℓ_2 result fixed; so the second and third terms of (8.6) are the higher order terms of the zero order approximation F_0 in principle definable by these lengths independently of the coefficients characterizing F_{zp} and force per unit surface P_{Ca}. The zero point energy has been already concerned; a detailed discussion of the Casimir term is clearly outside the purposes of the present paper. It is really crucial the fact that various kinds of forces are nested in the general uncertainty Equation (8.2), in turn direct consequence of (2.10). Owing to the importance of (8.6), the following consideration will be focused on this equation for sake of brevity only; this allows to complete the information in Section 6.2.

8.2. Long Range Gravity Force

To verify how (8.6) implies more specifically the space time curvature, examine both sides of the general Equation (8.2). The Equations (3.1) imply the following chain of equations implementing the left hand side of (8.2)

$$\frac{\delta p}{\delta t}=\frac{\hbar}{\delta t}\frac{1}{\delta x}=\frac{\hbar}{\delta t}\frac{\delta x'}{\delta x'\delta x}=\frac{\hbar}{\delta t}\frac{\delta x'}{\delta x+\delta x'}\left(\frac{1}{\delta x}+\frac{1}{\delta x'}\right)=\frac{\delta p'}{\delta t}r_c\delta x' \tag{8.7}$$

$$\delta p'=\frac{\hbar}{\delta x+\delta x'},\quad r_c=\frac{1}{\delta x}+\frac{1}{\delta x'},$$

being r_c the Laplace average curvature radius of an ideal surface such that $F\to 0$ for $\delta x\to\infty$ and $\delta x'\to\infty$.

Note that r_c is formally similar to $K_L=1/r_1+1/r_2$, where the addends are defined on two orthogonal planes called curvature sections; it refers to flat space time. According to its classical derivation, the local value of K_L is calculable as both radii r_1 and r_2 are assumed in principle exactly knowable. Here, instead, the quantum derivation of r_c does not allow any information about size and even mutual orientation of δx and $\delta x'$, the only indication available being that it is conceptually defined by two curvature sections in a 4D space time; so r_c is not calculable in practice, it is compatible with all combination of values included within δx and $\delta x'$. The relativistic curvature K_E is instead self-defined in a Gauss curvilinear coordinates in a covariant way regardless of the reference system. According to (3.2), however, even r_c defined by two uncertainty ranges actually waives the link to a specific reference system; hence the impossibility of calculating uniquely r_c prevents comparing it to K_E, whose local value is instead in principle calculable. Hence is meaningless to enquiry whether or not the Equation (8.7) fits the standard definition of general relativity or not. In other words K_E must be necessarily covariant to have physical meaning, because effectively it can be calculable locally; r_c instead fulfills the quantum concept of covariance required by (3.2) and cannot take any deterministic value. Hence, in lack of numerical assessment, the quantum reasoning implied by (8.7) allows conceptual comparison only: the quantum definition of space time curvature, although symbolic, is still related to its relationship with *F*,

i.e. $F \neq 0$ for δx and $\delta x'$ both finite. The non trivial implication of this reasoning is that in this way the relativity becomes a corollary of (3.1) and (8.2) via the series expansion (8.5) that generalizes the mere Newtonian term. This is not surprising because the Section 1 has evidenced the 4D holistic character of the present model; the purpose of the next considerations is to clarify further this idea.

Recalling (2.36) $p = h/\lambda$, the right hand side of (8.2) yields, owing to (4.24) and (2.36),

$$-\frac{\hbar}{\delta x}\frac{\delta \dot{x}}{\delta x} = -h\frac{\delta \dot{\lambda}}{\delta \lambda^2} = \mp\frac{\delta p}{\delta t}; \tag{8.8}$$

so, merging (8.7) and (8.8), one finds

$$-\frac{h\delta\dot{\lambda}}{\delta\lambda^2} = \mp\frac{\delta p'}{\delta t} r_c \delta x' = \mp r_c \delta\varepsilon', \quad \delta\varepsilon' = v'\delta p', \quad v' = \frac{\delta x'}{\delta t}. \tag{8.9}$$

No apparent reference to the mass is explicitly evident in this formula: the force $r_c\delta\varepsilon'$ and its approximation F_0 are due to the mere deformation rate of the range $\delta\dot{x}_0$.

On the one hand (8.2) results consistent with this equation that links force $\delta\dot{p}$ and energy $\delta\varepsilon'$ via the curvature radius r_c of space time: as expected the momentum, and thus its time change as well, can be expressed via corpuscle formalism, see e.g. (3.64), and via wave formalism inferred in (2.36). Accordingly $F = \dot{p}$ waives the concept of mass if just (2.36) is implemented to calculate $\dot{p} = -h\delta\dot{\lambda}/\delta\lambda^2$. The analogy with (8.6) and (8.5) appears because also now it is possible to write

$$\delta\dot{\lambda} = \pm\delta\dot{\lambda}_0 + \sum_j k_j \delta\lambda^{-j} \tag{8.10}$$

this is the meaning of the Equations (8.7) to (8.9), where the mass is mere dimensional property of *h*.

On the other hand, the connection of (8.6) with (6.12) implied by (6.7) requires writing

$$F_0 = \pm G\frac{m_1 m_2}{\delta x^2}, \quad \delta\dot{x}_0 = m_1 m_2 \frac{G}{\hbar} \tag{8.11}$$

the curvature r_c is linked to one mass, that defining ε' and $\delta\dot{p}$ of (8.9), the zero order deformation rate $\delta\dot{x}_0$ of δx is given by the constant $G/\hbar$ times the product $m_1 m_2$, *i.e.* $\delta\dot{x}_0$ is proportional to $m_1 m_2$ that in turn is directly proportional to the force. The Equations (8.6) and (8.11) are the space time/matter formulations of the gravity force analogous to the wave/corpuscle formulations of the energy and momentum in quantum mechanics; in effect (3.70) has shown that in general the force is proportional to $\delta\varepsilon\delta p$.

It turns out therefore from the previous considerations with the help of (2.36)

$$\left|\frac{\delta p}{\delta t}\right|_{Ein} = |F| = \left|\frac{\delta p}{\delta t}\right|_{New} \leftrightarrow r_c\delta\varepsilon' = \frac{h\delta\dot{\lambda}}{\delta\lambda^2} = G\frac{m_1 m_2}{\delta x^2}, \tag{8.12}$$

whence the correlations

$$\text{force field} \rightarrow \text{deformation rate of space time ranges} \rightarrow \text{acceleration} \qquad (8.13)$$

Comparing the left and right hand sides of both (8.12), is evident why Einstein has successfully replaced the concept of force with that of space time curvature, while skipping the more intuitive Newtonian correspondence between mass and force: thus, in His intuition, the mass appears directly related to the space time curvature. So the first correlation (8.13) is understandable. The second correlation deserves attention, as the concept of acceleration has been not yet introduced; it will be concerned in the next subsection in particular to explain what have to do $\delta\dot{x}$ or $\dot{\lambda}$ with the acceleration implied by *F*. It will be shown that just the ranges, which link the concept of force to the quantum uncertainty, also plug the Newton and Coulomb forces into the realm of quantum mechanics.

8.3. The Equivalence Principle

Implement the Equations (8.11) and (8.8) to understand why the mass appears in the former and not in the latter. Also this topic, shortly sketched here for completeness, has been concerned in [10].

Position and size of any $\delta x = x - x_o$ in an arbitrary *R* are respectively definable considering the distance of either range boundary, say x_o, from the origin *O* and the distance of *x* from x_o; of course the opposite choice would be identically admissible. Being both boundaries arbitrary, in general it is possible to regard $x = x(t)$ and $x_o = x_o(t)$. A force *F* arises inside δx because in general $\dot{x} \neq \dot{x}_o$, *i.e.* when the range size of δx shrinks or stretches as a function of time during $\delta t = t - t_o$ with respect to its initial size at the time t_o. To simplify the reasoning it is enough to examine the cases where: 1) $x_o = const$ and $x = x(t)$ only or 2) $x = const$ and $x_o = x_o(t)$ only; as anyway the size of δx changes, for example because of energy fluctuation of a particle inside δx, both cases imply in general $F \neq 0$ and the following considerations about inertial and accelerated reference systems.

Imagine an observer sitting on x_o and assume for simplicity that one particle only is delocalized in δx; the rising of any *F* is detected observing the dynamical behavior of this test particle.

In the case (1) the observer is by definition at rest in *R* with respect to *O*; yet he acknowledges a force $F = -\left(\hbar/\delta x^2\right)\delta\dot{x}$ acting on the particle, actually due to $\dot{x} \neq 0$. To justify in principle why the motion of the particle is perturbed, the observer reasonably thinks to the presence of an external force, e.g. a gravitational mass outside δx.

In the case (2) the observer no longer at rest in *R* necessarily accelerates with respect to *O*, whereas the force $F_o = \left(\hbar/\delta x_o^2\right)\delta\dot{x}_o$ again appears in δx according to $\dot{x}_o \neq 0$; now F_o governs the dynamics of the particle delocalized in δx. The observer concludes that its own acceleration is due to F_o.

Of course the analytical forms of F and F_o are in principle analogous, although their strengths are in general different if $\dot{x} \neq \dot{x}_o$; indeed the forces only differ by the time dependence of either boundary coordinate of δx with which is calculated the overall $\delta \dot{x}$. However, despite the boundaries of δx are in general arbitrary and independent each other, nothing hinders to assume in particular $\dot{x}_o / x_o^2 = -\dot{x}/x^2$: *i.e.* stretching of δx occurs via forwards displacement of x only or backwards displacement of x_o only with respect to O. So locally $F = -F_o$. The acceleration experienced by the observer and the presumed force F arising outside δx perturb in the same way the test mass because they have actually a unique background, the deformation rate (8.5) of the space time range δx itself that in effect implies (8.6): in (2) this deformation rate is perceptible by the observer as force F_o whereas in (1) it does not, although in both cases the observer can anyway record the same change of local dynamics of the particle inside δx. Clearly the observer reference system R_o with origin on x_o is at rest or inertial in (1) but non-inertial in (2) with respect to R, in agreement with the aforesaid correlation.

One key point of the reasoning is that these conclusions hold exchanging the role of x and x_o, because both range boundaries are arbitrary and physically equivalent; no physical property characterizes specifically either space time boundary displacement. As (1) and (2) are physically indistinguishable, the unique information available is the overall deformation rate $\delta \dot{x}$ and its related F; this means that the concept of acceleration implied by F_o holds identically for F as well. Another key point is that the acceleration does not necessarily imply the concept of mass, but that of force in turn due to inertial and non-inertial reference systems.

One could also say that the concept of force is redundant, what in fact exists is the stretching/shrinking rate of δx which in turn implies space time curvature according to (8.7). But now this statement has quantum foundation only.

Anyway the dualism wave/corpuscle of quantum mechanics has relativistic analogy in the "corpuscular" Equation (6.9) and "wave" Equation (8.9) properties of matter; the latter originates from the space time curvature, the former from the necessity of defining the change of delocalization extent of massless or massive particles both contextually implied by the probabilistic Equation (4.7). In this sense quantum and relativistic physics are perfectly symmetrical, which is not surprising because both are rooted on the quantum uncertainty. From this analogy follow the correspondences (8.12) along with the Equations (6.28) and the various forces implied by (8.5).

Is evident the analogy of δx with the elevator of Einstein's thought experiment: the cases of inertial and non-inertial reference systems merge here into the unknown and unknowable behavior of the boundaries of a unique space time uncertainty range δx only. Implementing space time ranges rather than local space time coordinates plugs a typical relativistic reasoning about inertial and non-inertial reference systems into the quantum uncertainty (3.1). In this sense

the relativity is conceptually compatible with quantum requirements; any reasoning via local coordinates, e.g. the tensor calculus, wouldn't. The present approach shows what the mere wave formalism to quantum mechanics cannot emphasize itself: quantum and relativistic theories are conceptually rooted in the unique concept of uncertainty, the operator formalism exposed in the subsection 3.5 is instead less general being actually itself a corollary of the quantum uncertainty.

8.4. Long Range Electromagnetic Interactions

Start from the Equation (4.8) $\Delta\varepsilon = \mathrm{n}^2\delta\varepsilon$ and the position (4.9) $\Delta\varepsilon = \hbar/\Delta t$, which now are once more under test besides to the results (4.13), (4.14), (4.16) and (4.17) already obtained; the Equation (4.9) was also the starting point of the section 6.2. Recalling the definition (4.5) of refractive index n, elementary manipulations yield

$$\delta\varepsilon = \frac{v^2}{c}\frac{\hbar}{\ell} = \varepsilon_2 - \varepsilon_1, \quad \ell = c\Delta t, \tag{8.14}$$

being of course ℓ an arbitrary length. The second equality is the mere definition of energy range $\delta\varepsilon$ with arbitrary boundary values ε_2 and ε_1. A possible way to split accordingly $v^2\hbar/\ell c$ too, is to rewrite (8.14) defining ε_1 and ε_2 as follows

$$\delta\varepsilon = \frac{\varepsilon_1^2}{\varepsilon_2}\frac{q^2}{\hbar c}, \quad \varepsilon_1 = \frac{\hbar v}{\ell}, \quad \varepsilon_2 = \frac{q^2}{\ell} \tag{8.15}$$

to obtain next (8.14) rewritten as

$$\frac{v^2}{c}\frac{\hbar}{\ell} = \varepsilon_2 - \varepsilon_1 = \frac{q^2 - \hbar v}{\ell} \tag{8.16}$$

the positions (8.15) convert thus the unique term at the left hand side of (8.14) into the difference of energies q^2/ℓ and $\hbar v/\ell$ defining $\delta\varepsilon$. In general $q = q(v)$. Let ℓ measure the distance between two interacting partners. The fact that ℓ is defined by c means that the carriers of the force are massless particles, photons. Also, v that defines the energy range (8.14) characterizing this kind of interaction must be consistently identified by $v = c$; as any $v < c$ could not be enough to travel through ℓ, a coherent way to characterize the peculiar value of q consistent with ℓ requires the boundary condition $\mathrm{n} = 1$ in (8.16). Put thus the resulting value of q, now uniquely defined, proportional to a "new" quantity called e. So, calling $2\alpha_0$ the dimensionless proportionality constant linking q^2 and e^2, (8.16) yields

$$\frac{\hbar c}{\ell} = \frac{2\alpha_0 e^2 - \hbar c}{\ell}, \quad q^2 = 2\alpha_0 e^2, \quad v = c. \tag{8.17}$$

Multiplying both sides by $\ell/\hbar c$ one finds

$$\alpha_0 = \frac{1}{\alpha}, \quad \alpha = \frac{e^2}{\hbar c}$$

as a result we have obtained the definition of fine structure constant α via three universal constants of the nature, whereas α_0 is to be regarded as a pure number corresponding to the numerical value of $1/\alpha$.

Note in this respect that the electric charge *e*, so far not yet explicitly introduced, appears in the model via α. This introductory reasoning outlines the next task to be concerned just now: to show how the Equation (4.9) implies the electromagnetic interaction too. This subsection links therefore the following considerations to the Section 7. Usually *e* is introduced by postulating the Coulomb force; here instead *e* and thus the electromagnetic forces are introduced starting from α. By analogy with (8.6), the Equation (8.2) is rewritten as follows

$$\pm\frac{\hbar c}{\delta x^2}=F_C=\pm\alpha^{-1}\frac{e^2}{\delta x^2},\quad \delta\dot{x}_0=c, \tag{8.18}$$

i.e. the physical meaning of α introduces itself also the long range Coulomb force component F_C.

Is evident the formal analogy between F_0 of (8.11), concerning the mass linear density $m/\delta x$, and F_C of (8.18), introducing the charge linear density $e/\delta x$: both regard in particular $\left|\delta\dot{x}_0\right|\times\hbar$ as constant characterizing the lowest order term (8.6) of the series expansion (6.7). This suggests that the first order approximations of Newton and Coulomb laws should be both deducible from $\left|\delta\dot{x}_0\right|\times\hbar$ through a dimensional constant. In effect the connection between *G* and *e* via a dimensional proportionality constant is easily proven; indeed

$$\left|e\right|=\left(\xi_{eG}\alpha\right)G$$

yields numerically

$$e=4.80\times10^{-10}\ \text{ues},\quad \alpha G=4.88\times10^{-10}\ \text{cm}^3/\text{s}^{-2}\cdot\text{g},$$
$$\xi_{eG}\approx1.01\ \text{ues}\times t^2\ \text{g}/\text{cm}^3=\left(\text{g}/\text{cm}\right)^{3/2}\cdot\text{s}.$$

Of course the signs of the component F_C correspond to equal or opposite charges defining α. Implement first (8.18) for both charges at rest in *R*; so

$$F_C=\frac{e_r^2}{\delta x^2},\quad v=0, \tag{8.19}$$

where the subscript stands for "rest"; in this equation there is no explicit reference to *v*, which from now on denotes the relative motion of the charges. As δx is arbitrary, α has been included in it, to simplify notations like $\delta x^*=\sqrt{\alpha}\delta x$.

The second way to implement (8.18) assumes constant the rate *v* with which moves either charge with respect to the other at rest in *R*; as according to (2.34) $\delta p=\hbar/\delta x=\left(v\delta\varepsilon+\varepsilon\delta v\right)/c^2$, (8.18) yields

$$\frac{e^2}{\delta x^2}=\frac{\hbar c}{\delta x^2}=\frac{\delta\left(pc\right)}{\delta x}=\frac{1}{c}\left(\frac{v\delta\varepsilon}{\delta x}+\frac{\varepsilon\delta v}{\delta x}\right). \tag{8.20}$$

Thus

$$\frac{e_m^2}{\delta x^2} = \frac{v}{c}\frac{\delta\varepsilon}{\delta x}, \quad v = const, \tag{8.21}$$

where the subscript stands for "mobile", whence

$$\frac{e_m^2}{\delta x^2} = \frac{\delta\varepsilon}{c\delta t} = \frac{\delta\varepsilon\sqrt{1-v^2/c^2}}{c\delta t'}. \tag{8.22}$$

The last equality has introduced the time range $\delta t_r'$ defined in a reference system R' of the mobile charge moving at rate $-v$ with respect to R; an observer ideally sitting on the moving charge in R' sees the other charge at rest, as the backwards motion of R' in R balances exactly the forwards motion of the charge. So, in practice (8.22) can be regarded likewise (8.19); obviously (8.22) reduces anyway to (8.19) in particular for $v = 0$. Hence it is possible to write the second equality as

$$F_{em} = \frac{\delta\varepsilon}{c\delta t'} = \frac{e^2}{\delta x'^2\sqrt{1-v^2/c^2}}, \quad F_{Lor} = vH\frac{e}{c}; \tag{8.23}$$

the first equation emphasizes the link between $e^2/\delta x^2$ in R and R', the second equation also follows directly from (8.21) according to the following chain of elementary steps

$$\frac{v}{c}F = \frac{v}{c}e\frac{F}{e} = \frac{e}{c}vH, \quad F = \frac{\delta\varepsilon}{\delta x}, \quad H = \frac{F}{e}.$$

Regarding $F = F_{em}$ one finds a "new" quantity called magnetic field already introduced as a final step (7.16) of the reasoning in the section 7. The one dimensional scalar approach followed throughout this paper hides the actual vector character of $\boldsymbol{v}$ and thus of $\boldsymbol{F}_{em}$ and $\boldsymbol{F}_{Lor}$. Simple considerations allow however to surrogate this missing information acknowledging that F_{Lor} and F_{em} are two different corollaries of a unique information to describe the charge dynamics: both Equations (8.23) follow from different ways of rewriting the unique Equation (8.21), as in effect it is physically sensible. So, to avoid that the energy of a mobile test charge in both fields is counted twice summing separately F_{em} and F_{Lor}, it is necessary that the former only performs work on the mobile charge, likewise as in the particular case of charge at rest, whereas the latter doesn't; the vector properties of these forces follow from these considerations, *i.e.* $\boldsymbol{H}$ and $\boldsymbol{v}$ must be such that $\boldsymbol{F}_{Lor} \propto \boldsymbol{H}\times\boldsymbol{v}$.

The third way to handle α concerns the case where v is not constant; owing to (8.20), an additional force term F_ω is expected because of the addend $\varepsilon\delta v/c\delta x$ previously omitted in (8.21). This additional term reads

$$F_\omega = \frac{\varepsilon\delta v}{c\delta x}\frac{\delta t}{\delta t} = \frac{\varepsilon\delta t}{\delta x}a = \frac{\varepsilon\hbar}{\delta x\delta\varepsilon}a, \quad a = \frac{1}{c}\frac{\delta v}{\delta t},$$

where the acceleration a of the charge appears as a reciprocal frequency because of the factor c^{-1}. It is possible to extract from this chain of equation the frequency ω defined by the energy $F_\omega\delta x$, *i.e.*

$$\frac{F_\omega\delta x}{\hbar}\frac{\delta\varepsilon}{\varepsilon} = \omega = a \tag{8.24}$$

A fraction $\delta\varepsilon/\varepsilon$ of energy $F_\omega \delta x$ is thus converted into and appears as electromagnetic radiation, whose energy $\hbar\omega$ increases of course with $\delta\varepsilon/\varepsilon$. The fact of having found ω for $\delta v \neq 0$ means that an accelerated charge implies emission of e.m. radiation. Of course $F_\omega = 0$ and thus $\omega = 0$ for $v = 0$ or $v = const$, so even this contribution to the right hand side of (8.20) vanishes for charges at rest. In principle $\varepsilon\omega$ prospects the chance of calculating the power irradiated by an accelerated charge regarding appropriately F_ω and $\delta\varepsilon$ of (8.24). This chance has been exemplified in section 7 in a more complete and rigorous way via the Maxwell equations.

Is still proven useful here the initial idea of implementing uniquely the early Equations (1.11) and (1.12) without any further physical hint but simply including α among the fundamental constants of Nature.

8.5. Quantum Charges

Dividing both sides of the inequality (8.3) by e^2 and next by δx too, one finds

$$\delta\varepsilon = \frac{\sigma^2}{\alpha}\frac{e^2}{\delta x}, \quad \sigma^2 \le 1, \tag{8.25}$$

being σ an appropriate factor. According to (3.69) and (3.70) this yields also

$$F_{e'} = \frac{\delta\varepsilon}{\delta x} = \frac{1}{\alpha}\frac{e'^2}{\delta x^2}, \quad e' = \sigma e, \tag{8.26}$$

Now the question rises: does e' have mere numerical meaning or it actually generalize the concept of usual charge e according to $e' \le e$? To answer this question consider first some implications of (8.26) based on (3.70). As $F_{e'} = PA$, where A is an arbitrary area and P pressure, one finds the dimensional relationships

$$P = \frac{F_{e'}}{A} = \frac{1}{\alpha}\frac{e'^2}{A\delta x^2} \tag{8.27}$$

the Equations (8.26) imply the rising of a pressure P related to the energy density ε/V due to charged particles enclosed in the volume V. Start thus just from the dimensional identity between pressure and energy density and write

$$P = \sigma'\frac{E}{V}, \quad V = \Delta x^3 = \sigma' A\delta x, \tag{8.28}$$

where σ' is a proportionality factor necessary to define in general the volume V as a function of A defining the pressure. Owing to the dimensional character of this equation, although P and E have been defined specifically by (8.27) and (8.28), the following reasoning holds in general for any E, *i.e.* also for atoms, ions, elementary charges and even photons. The second equation yields

$$\frac{\delta V}{V} = 3\frac{\delta\Delta x}{\Delta x}. \tag{8.29}$$

Consider now that if in V are contained photons or matter, e.g. gas particles, it

is possible to implement the wave properties of matter and write

$$V=\left(\frac{v}{\nu_\lambda}\right)^3=\left(\frac{c}{\nu}\right)^3,\quad \nu=\nu_\lambda\frac{c}{v}=\frac{c}{\lambda},\quad \lambda=\frac{v}{\nu_\lambda},\quad \nu\geq\nu_\lambda \tag{8.30}$$

the matter particles are assumed moving at average rate v with De Broglie momentum h/λ and thus frequency ν_λ; for calculation purposes, ν_λ has been rewritten as a function of ν as indicated here to include also photons. The presence of steady waves in V requires

$$\frac{\delta V}{V}=-3\frac{\delta\nu}{\nu}=-3\frac{\delta E}{E} \tag{8.31}$$

the first equality is directly deducible from (8.30) that expresses the steadiness condition, *i.e.* the change of λ requires that of Δx as $\lambda=n\Delta x$ with n integer, the second equality expresses the proportionality between ν and E. So the change of E inside V is related to that of ν of the matter/light waves propagating in V. Implement now the idea that (8.29) regards a number of corpuscles inside V, whose change of energy density is uniquely definable by δV anyhow it might be obtained; instead (8.31) regards waves, whose energy changes are presumably related to how the early V is modified by a given δV because of steadiness condition. Reasonably the steady propagation of waves is different depending on whether one side only or two sides or even three sides of V are modified. Consider thus the three possible ways to deform the initial V, whose ΔV remains however uniquely defined in all cases: V can be equivalently rewritten as $\Delta x_0^2\Delta x$ or $\Delta x_0\Delta x^2$ or Δx^3. Being both Δx and the constant Δx_0 arbitrary, it is certainly possible to define them in order to fit a given value V of course arbitrary itself. Hence

$$\frac{\delta V}{V}=n_\nu\frac{\delta\Delta x}{\Delta x},\quad n_\nu=1,2,3.$$

To make consistent both ways of defining $\delta V/V$ merge this result with (8.31) to obtain

$$n_\nu\frac{\delta\Delta x}{\Delta x}-3\frac{\delta E}{E},$$

whence

$$\frac{\delta E}{\delta\Delta x}=-\frac{n_\nu}{3}\frac{E}{\Delta x},\quad \frac{n_\nu}{3}=\frac{1}{3},\frac{2}{3},1. \tag{8.32}$$

Assume that the left hand side defines an average force such that $\delta E=\langle F\rangle\delta\Delta x$; dividing both sides by the surface Δx^2 one finds

$$\frac{\langle F\rangle}{\Delta x^2}=P=-\frac{n_\nu}{3}\frac{E}{\Delta x^3}. \tag{8.33}$$

At the right hand side appears an energy density ρ_E defined by an amount of energy δE, arbitrary, in the volume $\Delta x^2\delta\Delta x$, arbitrary as well, and recall the initial position (8.28); then, as the dimensions of $E/\Delta x$ and Δx^2 are force and surface, one finds eventually

$$P_1 = \frac{1}{3}\rho_E, \quad P_2 = \frac{2}{3}\rho_E, \quad P_3 = \rho_E, \quad \rho_E = \frac{E}{\Delta x^3}. \tag{8.34}$$

The minus sign in (8.33) and (8.34) has been omitted, it simply establishes whether an internal or external pressure expands or shrinks V. These results, which hold for photons or gases because no specific hypothesis has been made, are well known: P_1 holds for a light beam completely absorbed by the internal surface of V, whereas P_2 when waves or corpuscles bounce elastically; P_3 yields the well known law $P_3\delta V = \delta E$. As P_3 is due itself to elastic shocks of corpuscles against the internal walls of V, then $E = 2E'/3$; *i.e.* E should be 2/3 of another energy E' that yields $PV = (3/2)E'$; is evident the connection of this last conclusion with the elementary kinetic theory of gases, where E' is easily demonstrable to be the average kinetic energy of molecules.

Skipping further considerations on this well known topic, return now to the Equations (8.25) and (8.26) to specify the result (8.34) in order to explain σ in the equation $e' = \sigma e$. The comparison of (8.32) and (8.26) suggests the correspondences

$$E \to \varepsilon, \quad \delta\Delta x \to \delta x, \quad \langle F \rangle \delta x = \varepsilon, \tag{8.35}$$

where the third position is the usual definition of force. So, owing to (8.26) and (8.18),

$$\frac{\delta E}{\delta \Delta x} \to \frac{\delta\varepsilon}{\delta x} = \frac{1}{\alpha}\frac{e'^2/\delta x}{\delta\Delta x}, \quad \frac{E}{\Delta x} = \frac{\varepsilon}{\Delta x} = \frac{1}{\alpha}\frac{e^2}{\delta x^2}, \quad \Delta x = \frac{n_v}{3}\delta x;$$

the third equation makes (8.33) compliant with (8.35). So the Equation (8.32) reads

$$\frac{e'^2}{\Delta x} = \frac{n_v}{3}\frac{e^2}{\delta x}, \tag{8.36}$$

whence, recalling the second (8.26), *i.e.*

$$e' = \pm e, \quad e' = \pm\frac{2}{3}e, \quad e' = \pm\frac{1}{3}e. \tag{8.37}$$

The second and third charges are consistent with quark charges, all with both signs correspondingly to $\pm e$, in the nuclear volume V. It is amazing the fact that even the quark charges appear here as a consequence of the dual wave/corpuscle behavior of matter and light, whereas their fractional character is reminiscent of the radiation/matter wave pressure in the volume enclosing them.

As a final remark, note that all charges can take both signs because e can be found in negative and positive energy states, as previously shown.

Now let us return to (8.24), to exemplify in a simple case how it is in fact calculable.

Let ε be the energy of a charged particle; (8.24) provides in principle the energy radiated per unit time with the help of (2.28)

$$\varepsilon\omega = \frac{F_\omega}{\hbar}\delta x\delta\varepsilon = F_\omega\frac{\delta\varepsilon}{\delta p} = F_\omega v.$$

It is easy to find the total power radiated $E/\delta t$ by such an accelerated particle, knowing that its charge is e and the change rate of its momentum is $\delta \dot{p}$. To solve this problem, however, more information is necessary about the link between radiation pressure and energy density. Once having found

$$P = \frac{2}{3}\frac{\varepsilon}{V} = \frac{2}{3}\frac{\varepsilon}{(c/\nu)^3},$$

being $\nu = 1/\delta t$ the wave frequency, the non relativistic result as a function of the acceleration $a = \delta \dot{p}/m$ is obtained after having multiplied both sides by $\delta x^3/\delta t$ via the following chain of equations

$$\begin{aligned} P\frac{\delta x^3}{\delta t} &= \frac{2}{3}\frac{\varepsilon}{\delta t}\frac{\nu^3 \delta x^3}{c^3} = \frac{2}{3}\frac{e^2}{\delta x \delta t}\frac{\nu^3 \delta x^3}{c^3} = \frac{2}{3}\frac{e^2}{c^3}\frac{(\nu \delta x)^2}{\delta t^2} \\ &= \frac{2}{3}\frac{e^2}{c^3}\frac{\delta v^2}{\delta t^2} = \frac{2}{3}\frac{e^2}{c^3}\frac{(\delta p/m)^2}{\delta t^2} = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{\delta \dot{p}}{m}\right)^2. \end{aligned} \tag{8.38}$$

8.6. Short Range Nuclear Interactions

The Equations (3.12) and (8.4) suggest specific orders of magnitude significant to introduce short range forces. Skipping all theoretical details outside the purposes of the present paper, a few short comments are exposed below to highlight at least the essential features of these forces implementing only results so far obtained. Consider thus the following chances introduced by the general Equation (8.1)

$$F = \frac{\delta p}{\delta t} = v\frac{\delta p}{\delta x} = \frac{v}{c}\frac{\delta (pc)}{\delta x}$$

to infer, in agreement with (3.14) and the reasons therein explained,

$$F_s = \frac{\delta (pc)}{c\delta t}, \quad F_w = \frac{v\delta (pc)}{c\delta x} \tag{8.39}$$

only in the first case δx has been replaced by $v\delta t$. The question that rises now is whether these expressions are mere equivalent ways of rewriting the same Equation (8.19) or they represent actually different force laws.

Preliminary inspection evidences that in both cases the force is defined via $\delta\varepsilon/length$ but in two different ways: F_w depends explicitly on v and δx, whereas F_s on $\delta x_s = c\delta t$ only. In the former case the interaction force is inversely proportional to δx; in the latter case the interaction energy $F_s \delta x_s$ increases with the distance δx_s, whatever this latter might be. This suggests the concept of "asymptotic freedom". Moreover the messenger particles of F_w carrying the interaction through $\delta x_w = v\delta t$ should be massive, owing to v; the messenger particles of F_s should be massless, as the force carried through δx_s involves c only. Consequently, one expects that for assigned δt the characteristic range measured by δx_s is greater than δx_w.

Taking according to (8.4) $\delta t \sim \delta x/c$ and $\delta\varepsilon \lesssim \hbar c/\delta x$, order of magnitude estimates of space ranges and time ranges that characterize F_s and F_w can be

calculated utilizing the values (3.12). Particularly interesting is in this respect the range

$$\delta x_n = r_e - r_N = r_B \alpha^2 (1-\alpha).$$

1) Consider first F_s assuming preliminarily that the concerned interactions occur at the nuclear or sub-nuclear scale: *i.e.* reasonably δx_n concerns the interaction between different nucleons and in the nucleons themselves. Accordingly define the length

$$\sqrt{r_N r_e} = r_B \alpha^{2.5},$$

which is clearly an average value within δx_n, and introduce two complementary subranges δx_{s1} and δx_{s2} of the whole δx_n as follows

$$\delta x_{s1} = \sqrt{r_N r_e} - r_N = r_B \left(\alpha^{2.5} - \alpha^3\right), \quad \delta x_{s2} = r_e - \sqrt{r_N r_e} = r_B \left(\alpha^2 - \alpha^{2.5}\right). \quad (8.40)$$

It yields

$$\sqrt{r_N r_e} \approx 2.4\times 10^{-14} \text{ cm}, \quad \lambda_C^{nucl} = 2.1\times 10^{-14} \text{ cm};$$

moreover both subranges expressible through electron and nuclear range sizes yield

$$\begin{aligned} \delta x_{s1} &= r_B \left(\alpha^{2.5} - \alpha^3\right) \approx 2.2\times 10^{-14} \text{ cm}, \\ \delta x_{s2} &= r_B \left(\alpha^2 - \alpha^{2.5}\right) \approx 2.6\times 10^{-13} \text{ cm}. \end{aligned} \quad (8.41)$$

Is relevant the fact that that δ_{s1} is surprisingly close to the nucleon Compton lengths λ_C^{nucl} of both proton and neutron, which have in effect a similar order of magnitude. Also $\sqrt{r_N r_e}$ does so, which means that the nucleon mass represents the boundary value discriminating the interaction lengths δx_{s1} and δx_{s2} inside and outside the respective nucleon; this also explains the order of magnitude of the nucleon mass, indeed $\hbar c/\delta x_{s1} = 1.6\times 10^{-24}$ g differs from the experimental nucleon mass by about 5.5% only. Here we take advantage of the fact that proton and neutron masses differ by less that 0.14% only. Hence the forces (8.4) defined by these ranges could concern both nucleons and their mutual interactions at distances consistent with the inequality (8.3); indeed, as expected, $\hbar c/\delta x_{s1} = \delta\varepsilon = 1.6\times 10^{-3}$ erg $= 1$ GeV is related just to the order of magnitude of the nucleon mass, whereas $\hbar c/\delta x_{s2} = \delta\varepsilon_N = 1.2\times 10^{-4}$ erg $= 0.08$ GeV is related to the binding energy between nucleons. In fact F_s is attractive, in agreement with the concept of "asymptotic freedom" already emphasized for quarks in nucleons less than δx_{s1} apart. The characteristic times are $\tau_1 \approx 7\times 10^{-25}$ s and $\tau_2 \approx 9\times 10^{-24}$ s. At this point it is possible a rough estimate of the stability of the nucleus comparing this energy $\delta\varepsilon_N$ with the Coulomb repulsion energy calculated via (8.23) approximately as $\varepsilon_{rep} \approx e^2/\delta x_{s2}$ between two protons δx_{s2} apart; $\varepsilon_{rep} \approx 9\times 10^{-7}$ erg is negligible with respect to the attractive field in $\delta\varepsilon_N$. The fact that $\delta x_{s1} \gtrsim \lambda_C$ suggests that F_s should concern sub-nuclear particles that form protons and neutrons, which therefore are not elementary particles themselves. Hence the whole charge of proton and the null charge of the neutron can be due to nothing else but appropriate combinations

of the e' fractional charges (8.37).

These considerations, well known and here shortly sketched only, are enough to conclude that F_s concerns the strong force. Further considerations are clearly outside the scopes of the present paper, merely aimed to show how to identify the fingerprints of the short range forces (8.39) in the conceptual frame hitherto outlined. Some more details are reported in [20].

2) Consider now F_w, noting that with the help of (2.28) the second Equation (8.39) reads also

$$F_w = \frac{v\delta(pc)}{c\delta x} = \frac{\delta\varepsilon}{\delta x} = \frac{\hbar}{\delta x \delta t}. \tag{8.42}$$

Being $v < c$ one expects $\delta x_w < c\delta t$; in other words, once having fixed δt, a shorter interaction rang $\delta x_w < \delta x_s$ is to be expected for δx_w of F_w. According to (8.4), $\delta x \delta\varepsilon \lesssim \hbar c$ implies information about δx_w compatible with $\delta\varepsilon_w$ in this case. To estimate v in the same reference system of (8.23), implement (8.23) supposing that an appropriate δx fulfills the condition

$$F = \frac{\hbar}{\delta x \delta t} \approx \frac{e^2}{\delta x^2 \sqrt{1 - v^2/c^2}}. \tag{8.43}$$

Elementary manipulations show that this position yields $\hbar v\sqrt{1 - v^2/c^2} = e^2$ *i.e.* $v/c\sqrt{1 - v^2/c^2} = \alpha$, whence the solutions $v/c \lesssim \alpha$ and $v/c \lesssim 1$. Hence, taking the same value of δt of (8.41) by comparison purposes, one finds two possible corresponding ranges

$$\delta x_\alpha \lesssim 7.7\times10^{-17} \text{ cm}, \quad \delta x_c \lesssim 1\times10^{-14} \text{ cm}, \quad \delta t_w \approx 3.5\times10^{-25} \text{ s}. \tag{8.44}$$

In effect, extending the Equations (3.12) to the fourth power of α one finds the further length $\delta x_w = r_B\alpha^4 \approx 1.5\times10^{-17}$ cm necessary to include δx_α in the whole range of F_w. The first and second results are acceptable, as both yield space ranges shorter than that of both (8.41); the first value, in particular, yields according to the fourth (8.4)

$$\delta\varepsilon_w \lesssim \frac{\hbar c}{\delta x_\alpha} \approx 0.4 \text{ erg} = 255 \text{ GeV} \tag{8.45}$$

The existence of two range sizes (8.44) of δx compatible with (8.43) suggests that F_w should imply two different kinds of massive force carriers, reasonably with and without charges; if so, then the charges must have opposite signs. This kind of interaction needs thus three kinds of carriers. Assuming charged and neutral carriers of masses $m_\pm$ and m_0, just a few considerations are enough to infer significant information on the masses of these messengers.

Implementing this assumption to establish the energy balance governing the formation of the carriers, the results are in full agreement with the experimental data.

Is reasonable the idea of regarding the cluster of messengers as a system of particles interacting themselves in order that the gain of binding energy of the charges accounts not only for their own masses m_+ and m_- but also for that

of m_0. Is interesting the energy balance of the charged carriers according to the electromagnetic Equation (8.43). Start with the energy of a Coulomb system with a nucleus of mass $m_{\pm}$ formed by either charge, e.g. m_+, in the field of which interacts the other charge, e.g. m_-, at average distance r_B; the subscript "B" stands for "bound", whereas the simplest hypothesis on the masses is $m_+ = m_-$. Consider preliminarily that such a system can be described as shown in subsection 3.2: the reasoning introduced to describe the electron charge around the nuclear charge holds in principle also for integer spin charged particles. The early hydrogenlike atom was introduced before the concept of spin, which became essential to account for the electron pile up in many electron atoms [7] according to the exclusion principle and for the possible presence of an external field. The success of Bohr's idea was allowed by the fact that the spin-orbit and spin-spin interaction between electron and nucleus are both small with respect to the Coulomb interaction. Consider at this point uniquely (3.1) that has general validity and skips, as shown in section 3.3, the operator formalism implementing wave functions along with all related implications: e.g. it is known that a 0 spin particle requires a 4 dimensional scalar wave function, whereas a spin 1 particles requires a three component wave function. On the one hand (3.1) has 4 dimensional character as it merges space and time coordinates through the respective uncertainty ranges, to which are related energy and momentum ranges too. On the other hand the necessity of describing the particle in any reference system is in fact ensured by (3.1) according to (3.2). Implement thus the electromagnetic interaction only to describe via (3.1) even a system of spin 1 charged bosons m_+ and m_- trusting that the steps from (3.3) to (3.9) still hold at least approximately also now; the comparison with the experimental data will be the decisive benchmark to assess the validity of these considerations. So it is possible to write for the system of boson charges $\varepsilon_n = e^4 m_r / 2n^2\hbar^2$, see Equation (3.7) with $Z = 1$, where m_r is the reduced mass of the concerned system. Accordingly (3.9) yields

$$\varepsilon_B = -\frac{1}{2}\frac{\alpha^2}{n^2}m_r c^2 = -\frac{1}{2}\frac{\hbar c\alpha}{r_B}, \quad r_B = \frac{n^2\hbar c}{\alpha m_r c^2}, \quad m_r = m_{\pm}/2 \tag{8.46}$$

for charges of equal mass. Putting $n = 1$ and including α into m_r, this equation reads

$$\varepsilon_B = -\frac{1}{2}m_r' c^2 = -\frac{1}{2}\frac{\hbar c}{r_B'}, \quad m_r' = m_r\alpha^2, \quad r_B' = \frac{r_B}{\alpha} = \frac{\hbar}{m_r' c}; \tag{8.47}$$

with these positions ε_B depends explicitly on r_B' and m_r' only, no longer on r_B and m_r. Is of interest now an appropriate r_B' compliant with ε_B and such that

$$\left(m_0 + 2m_{\pm}\right)c^2 = \frac{1}{2}\frac{\hbar c}{r_B'} = \frac{\pi\hbar c}{n'\lambda}, \quad 2\pi r_B' = n'\lambda \tag{8.48}$$

with n' integer, the second equation is a well known condition of the wave mechanics already implemented in (3.8) with the same physical meaning. Let the

shortest wavelength λ be the Compton length of either $m_{\pm}$, as suggested by the Equation (3.6) and [7], *i.e.*

$$\lambda = \frac{\hbar}{m_{\pm}c};$$

then, replacing $m_{\pm}c^2 = \hbar c/\lambda$ into (8.48) one finds

$$m_0 c^2 = (\pi - 2n')\frac{\hbar c}{n'\lambda}.$$

In conclusion, comparing with (8.48) and putting $n' = 1$, it is possible to write

$$\frac{m_0 c^2}{(m_0 + 2m_{\pm})c^2} = \frac{\pi - 2}{\pi}, \quad \frac{m_{\pm}c^2}{(m_0 + 2m_{\pm})c^2} = \frac{1}{\pi}.$$

These results are verifiable by comparison with the experimental masses $m_{\pm}$ and m_0:

$$m_0 = 91.19 \text{ GeV}, \quad m_{\pm} = 80.39 \text{ GeV},$$
$$\frac{m_0 c^2}{(m_0 + 2m_{\pm})c^2} = 0.36, \quad \frac{m_{\pm}c^2}{(m_0 + 2m_{\pm})c^2} = 0.32,$$

which in effect compare well with $(\pi - 2)/\pi = 0.36$ and $1/\pi = 0.32$ respectively.

Moreover note that (8.47) regards by definition r_B' as average distance between m_+ and m_-, whereas δx_a of (8.44) is by definition the total range of F_w; therefore one infers that reasonably $r_B' = \delta x_a/2$. Hence, according to (8.48), $\varepsilon_B = -\hbar c/\delta x_a$ is the binding energy gain available to create the masses $(m_0 + 2m_{\pm})c^2$. In effect one finds that the total energy ε_w related to F_w is

$$\varepsilon_w = -\varepsilon_B = \frac{\hbar c}{\delta x_\alpha} = 255 \text{ GeV} = (m_0 + m_+ + m_-)c^2, \tag{8.49}$$

in agreement with (8.45) and with the experimental masses. The Equations (8.47) and (8.46) differ in fact only formally; once having removed α merely including it in the reduced mass of the system as a numerical scale factor, someway analogous to (3.12), it appears that F_w is different from but closely related to the electromagnetic interaction constant via the linked energy scale factor.

In effect, considering m_r' and r_B', and not m_r and r_B, one calculates experimental masses of the force carriers and reasonable estimate of the interaction range $\delta x_\alpha \approx 7.3\times 10^{-17}$ cm that agree with the total energy (8.45); these values support the idea that a hydrogenlike system bound by electromagnetic interaction via photon carriers turns into a short range interaction system via massive carriers. There appears in this way the link between electromagnetic and weak interactions.

At this point, something else about δx_α can be still inferred to confirm that F_w corresponds to the weak interaction. Helps in this respect the first (8.4) $FV = \hbar\delta x^2/\delta t$ that reads

$$\varepsilon_w V = \frac{\hbar}{\delta t}\delta x^3, \quad \varepsilon_w = \frac{\hbar c}{\delta x_\alpha}$$

whence, with the help of the values (8.44),

$$\varepsilon_w V = \frac{\hbar c}{\delta x_\alpha}\delta x_\alpha^3 = 2\times 10^{-49}\ \text{erg}\cdot\text{cm}^3; \tag{8.50}$$

this is the Fermi constant characterizing the weak interactions.

This result is more than mere fingerprint of weak interactions; interesting information can be inferred from it expressing appropriately energy and volume inherent this result. Write

$$2\times 10^{-49} = \varepsilon_F \lambda_F^3 = \varepsilon_F\left(\frac{\hbar}{m_F c}\right)^3 = \varepsilon_F\left(\frac{\hbar c}{\varepsilon_F}\right)^3, \quad \varepsilon_F = m_F c^2 \tag{8.51}$$

the energy ε_F that defines the characteristic Fermi constant has been expressed via Compton length of the characteristic mass m_F that in turn defines ε_F too. Hence

$$\varepsilon_F = \sqrt{\frac{(\hbar c)^3}{2\times 10^{-49}}} = 0.397\ \text{erg} = 255\ \text{GeV}. \tag{8.52}$$

It is not surprising that one finds once more the value of total energy of this kind of interaction. Implement now the idea that actually the energy (8.49) is degenerate: it consists of m_- moving in the field of m_+ or, identically, from m_+ moving in the field of m_-. As both configurations can coexist consistently with the unique reduced mass m_r (8.46) that calculates e_B of (8.47), it is reasonable to regard the value (8.49) as the sum of both allowed chances; this means that the total energy refers to the total volume calculated via (8.51), so that each configuration has energy

$$\varepsilon_H = \frac{1}{2}\varepsilon_F = 0.2\ \text{erg} = 127\ \text{GeV}. \tag{8.53}$$

Note that this value is also consistent with that inferred through a characteristic range similarly as done in (8.40) and (8.41)

$$\begin{aligned} \delta x_H &= r_B\left(\alpha^{3.5} - \alpha^4\right) = 1.6\times 10^{-16}\ \text{cm}, \\ \frac{\hbar c}{\delta x_H} &= 0.197\ \text{erg} = 123\ \text{GeV}. \end{aligned} \tag{8.54}$$

8.7. The Dirac and Lamb Equations

This section generalizes the results of the Section 3.2 obtained implementing the non-relativistic equation $p^2 = p_r^2 + M^2/2r^2$. The following considerations show how to describe a relativistic hydrogenlike system replacing the classical position (3.4) with the series expansion

$$\varepsilon = \ell_0 \delta\dot{p} = \sigma_0 + \frac{\sigma_1}{\Delta r} + \frac{\sigma_2}{\Delta r^2} + \frac{\sigma_3}{\Delta r^3} + \cdots \tag{8.55}$$

expressing $\delta\dot{p} = \delta\dot{p}\left(\Delta r^{-1}\right)$ as $\delta\dot{p} = \sum_{i=0}\sigma_i \Delta r^{-i}$ similarly as done in the Equations (8.5) or (8.6). In effect even the term $\Delta p_r^2/2m$ of (3.4) can be written as $\sigma_1'/\Delta r^2$ with coefficient $\sigma_1' = (n\hbar)^2/2m$; so (3.4) is actually a particular case of

the series (8.55) truncated at the second order. Of course ℓ_0 is an arbitrary constant length that introduces the energy corresponding to force $\delta\dot{p}$. Although $\delta\dot{p}$ vanishes at the infinity, the arbitrary constant $\sigma_0 = mc^2$ accounts for the electron rest mass energy. Hence it is possible to write

$$\begin{aligned}&\delta\varepsilon = \varepsilon_2 - \varepsilon_1 = \ell_0\delta\dot{p} - mc^2 = \frac{\sigma_1}{\Delta r} + \frac{\sigma_2}{\Delta r^2} + \frac{\sigma_3}{\Delta r^3} + \cdots,\\ &\Delta r = r_2 - r_1, \quad \Delta p_r = p_{r2} - p_{r1},\end{aligned} \tag{8.56}$$

being Δp_r the radial momentum range conjugate to Δr. All range boundaries are of course arbitrary. In this way we deliberately waive introducing explicitly radial and angular momenta exploited in section 3.2, but implement directly the fundamental Equation (3.1). Multiply both sides of (8.56) by mc^2 so that

$$\left(\ell_0\delta\dot{p} - \frac{\sigma_1}{\Delta r}\right)mc^2 - \left(mc^2\right)^2 = \frac{\left(\Delta p_r c\right)^2 m\sigma_2}{\left(n\hbar\right)^2} + \frac{\left(\Delta p_r c\right)^3 m\sigma_3}{\left(n\hbar\right)^3 c} + \cdots,$$

The series truncated at the third order yields

$$\begin{aligned}&\delta\left(\varepsilon^2\right) = \varepsilon_2^2 - \varepsilon_1^2 = \frac{\left(\Delta p_r c\right)^2 m\sigma_2}{\left(n\hbar\right)^2} + \left(mc^2\right)^2 + \frac{\left(\Delta p_r c\right)^3 m\sigma_3}{\left(n\hbar\right)^3 c},\\ &\delta\left(\varepsilon^2\right) = mc^2\ell_0\delta\dot{p} - mc^2\frac{\sigma_1}{\Delta r}.\end{aligned}$$

As the coefficient σ_2 has not yet been defined, it is convenient to turn this equation into

$$\delta\left(\varepsilon^2\right) = \left(\Delta p_r c\right)^2 + \left(mc^2\right)^2 + \frac{\left(\Delta p_r c\right)^3 m\sigma_3}{\left(n\hbar\right)^3 c} + \cdots, \quad \sigma_2 = \frac{\left(n\hbar\right)^2}{m};$$

then, dividing both sides by $\left(mc^2\right)^2$, one finds

$$\begin{aligned}&\frac{\varepsilon_2^2}{\left(mc^2\right)^2} = \left(\frac{\Delta p_r c}{mc^2}\right)^2 + 1 + \Theta,\\ &\Theta = \frac{\varepsilon_1^2}{\left(mc^2\right)^2} + \frac{\left(\Delta p_r c\right)^3 m\sigma_3}{\left(mc^2\right)^2\left(n\hbar\right)^3 c} = \frac{\varepsilon_1^2}{\left(mc^2\right)^2} + \frac{\sigma_3}{mc^2\Delta r^3}.\end{aligned} \tag{8.57}$$

Next write

$$\frac{\Delta p_r c}{mc^2} = \frac{p_{r2}c}{mc^2} - \frac{p_{r1}c}{mc^2} = si\sqrt{\left(\frac{\varepsilon_2^2}{\left(mc^2\right)^2} - \Theta\right) - 1}, \quad si = \pm 1 \tag{8.58}$$

so that, subtracting ε_2/mc^2 at both sides, one finds

$$\frac{p_{r2}c - \varepsilon_2}{mc^2} = b + si\sqrt{\left(a^2 - \Theta\right) - 1} - a, \quad b = \frac{p_{r1}}{mc^2}, \quad a = \frac{\varepsilon_2}{mc^2} \tag{8.59}$$

and then

$$\frac{\left(mc^2\right)^2}{\left(p_{r2}c - \varepsilon_2\right)^2} = \left(b + si\sqrt{\left(a^2 - \Theta\right) - 1} - a\right)^{-2}.$$

According to the first (8.57) $\varepsilon_2^2-(1+\Theta)(mc^2)^2=(\Delta p_r c)^2$; thus the last equation reads

$$\frac{\varepsilon_2^2}{(1+\Theta)(p_{r2}c-\varepsilon_2)^2}-\frac{(\Delta p_r c)^2}{(1+\Theta)(p_{r2}c-\varepsilon_2)^2}=\left(b+si\sqrt{(a^2-\Theta)-1}-a\right)^{-2}. \quad (8.60)$$

Put now

$$p_{r2}c-\varepsilon_2=\pm\Delta p_r c. \quad (8.61)$$

This position has two implications: the first replacing it in (8.57)

$$\varepsilon_2^2-(1+\Theta)(mc^2)^2=(p_{r2}c-\varepsilon_2)^2 \quad (8.62)$$

and the second replacing in (8.60)

$$\frac{\varepsilon_2^2}{(1+\Theta)(p_{r2}c-\varepsilon_2)^2}=\left(b+si\sqrt{(a^2-\Theta)-1}-a\right)^{-2}+\frac{1}{1+\Theta}. \quad (8.63)$$

Note that we have introduced four conditions: a and b in (8.59) plus (8.62) and (8.63); the unknowns in these equation are $p_{r1}, p_{r2}, \varepsilon_2, \Theta$. In principle the system appears solvable.

Taking the reciprocal of both sides one finds

$$si'\left(\frac{p_{r2}c}{\varepsilon_2}-1\right)=\frac{1}{\sqrt{1+\Theta}}\left[\left(b+si\sqrt{(a^2-\Theta)-1}-a\right)^{-2}+\frac{1}{1+\Theta}\right]^{-1/2}, \quad si'=\pm 1. \quad (8.64)$$

The notations si and si' have been introduced to allow that the upper and lower signs in (8.58) and (8.56) are independent each other. Then it is possible that

$$-\frac{p_{r2}c}{\varepsilon_2}=\frac{1}{\sqrt{1+\Theta}}\left[\left(b+si\sqrt{(a^2-\Theta)-1}-a\right)^{-2}+\frac{1}{1+\Theta}\right]^{-1/2}-1, \quad si'=-1 \quad (8.65)$$

or

$$\frac{p_{r2}c}{\varepsilon_2}=\frac{1}{\sqrt{1+\Theta}}\left[\left(b+si\sqrt{(a^2-\Theta)-1}-a\right)^{-2}+\frac{1}{1+\Theta}\right]^{-1/2}+1, \quad si'=1 \quad (8.66)$$

Subtracting (8.65) from (8.66) with the same sign si, one finds $(p_{r2}c/\varepsilon_2)_+-(p_{r2}c/\varepsilon_2)_-=2$: this suggests that the left hand side of these equations must have the form E/mc^2, so that

$$E_+-E_-=2mc^2.$$

The minus sign at the left hand side of (8.65) represents binding energy of the electron to the nucleus in mc^2 units; the first addend at the right hand side represents the energy gain with respect to that of the free electron in either energy state.

The previous algebraic steps aimed just to find an equation introducing the ratio $p_{r2}c/\varepsilon_2$. In effect this ratio is significant because, according to (8.61), if $p_{r2}c\to\varepsilon_2$ then $\Delta p_r\to 0$ and thus $\Delta r\to\infty$ whereas p_r becomes constant; this is the limit case of free electron. Indeed $\Delta p_r\neq 0$ implies binding energy, since the electron takes random values of radial momentum between p_{r1} and

p_{r2} depending on its finite random distance from the nucleus.

Examine the result (8.65) putting first $\Theta = 0$: in fact, according to the second (8.57), this occurs putting $\sigma_3 = 0$ and in (8.55) and $\varepsilon_1 = 0$, *i.e.* considering the energy ε_2 only instead of the energy range $\delta\varepsilon$ (Heisenberg compliant quantum case). Now require that

$$\frac{E_-}{mc^2} = \left[\left(b + si\sqrt{a^2 - 1} - a\right)^{-2} + 1\right]^{-1/2} - 1 \tag{8.67}$$

must be compatible with the non-relativistic quantum Equations (3.5) and (3.9); in other words, in (8.67) must somehow appear not only αZ but also n and $l(l+1)$ as well. To fulfill this boundary condition as a limit case for small values of αZ, put in (8.67))

$$a = \frac{a_o}{\alpha Z}, \quad b = \frac{b_o}{\alpha Z}; \tag{8.68}$$

in effect, replacing and expanding in series around $\alpha Z = 0$, (8.67) becomes

$$\left.\frac{E_-}{mc^2}\right|_{\alpha Z \to 0} = \frac{(\alpha Z)^2}{2\left(b_o + sia_o - a_o\right)^2}.$$

Considering in particular $si = 1$, the boundary condition requires $b_o = n$. Once having identified b_o, regard then a_o and a_o^2 in order to be compliant with l and $l(l+1)$ of Equation (3.5), while also fulfilling (3.9) and (3.10). This suggests reasonably $a_o = l + 1/2 \pm s$, being s the electron spin; in effect, depending on the sign, a_o becomes l or $l+1$. In conclusion

$$\frac{E_-}{mc^2} = \left[\left(\frac{\alpha Z}{n + \sqrt{(j+1/2)^2 - (\alpha Z)^2} - (j+1/2)}\right)^2 + 1\right]^{-1/2} - 1, \quad j = l \pm s, \quad \Theta = 0 \tag{8.69}$$

This is the Dirac equation, which however becomes in the present approach particular case of an even more general equation including Θ as well. The Equation (8.65) reads indeed

$$\frac{E_-}{mc^2} = \frac{1}{\sqrt{1+\Theta}}\left[\left(\frac{\alpha Z}{n + \sqrt{(j+1/2)^2 - (1+\Theta)(\alpha Z)^2} - (j+1/2)}\right)^2 + \frac{1}{1+\Theta}\right]^{-1/2} - 1$$

which removes the degeneracy of states with equal n and j of the Dirac equation and also suggests that a further physical effect related to Θ not concerned in (8.69) is still hidden in this result. In effect the Dirac equation becomes in this approach the zero order approximation of a more complex energy function whose series expansion reads

$$\frac{E_-}{mc^2} = E_{Dir} + \left.\frac{\partial E_-}{\partial \Theta}\right|_{\Theta=0} \Theta + \cdots \tag{8.70}$$

It is evident that (8.70) removes the degeneracy of the $2p_{1/2}$ and $2s_{1/2}$ states: indeed, whatever the actual analytical form of Θ might be, calculating the energy difference of these states one finds

$$\left.\frac{E_-}{mc^2}\right|_{2p_{1/2}} - \left.\frac{E_-}{mc^2}\right|_{2s_{1/2}} = \left(\left.\frac{\partial E_-}{\partial\Theta}\right|_{\Theta=0}\Theta\right)_{2p_{1/2}} - \left(\left.\frac{\partial E_-}{\partial\Theta}\right|_{\Theta=0}\Theta\right)_{2s_{1/2}} \neq 0.$$

Owing to the physical dimensions $energy \times length^3$ of the coefficient σ_3 in the second (8.57), it is easy to guess the order of magnitude of the second addend of Θ according to the following reasonable positions

$$\sigma_3 \approx \varepsilon_{bohr} \times \lambda_C^3, \quad \Delta r \approx r_{bohr}, \quad \frac{\sigma_3}{\Delta r^3} \approx \frac{\varepsilon_{bohr}\lambda_C^3}{r_{bohr}^3},$$

being λ_C the electron Compton length. So it follows from the second (8.57) with the help of (3.4) and (3.8)

$$\frac{\varepsilon_{bohr}\lambda_C^3}{mc^2 r_{bohr}^3} = -\frac{1}{2}\left(\frac{\alpha Z}{n}\right)^5, \quad \Theta \approx \frac{\varepsilon_1^2}{\left(mc^2\right)^2} - \frac{1}{2}\left(\frac{\alpha Z}{n}\right)^5. \tag{8.71}$$

The second addend of Θ in (8.71) is the signature of the radiative energy displacement due to the interaction of the electron with the quantum vacuum, known as electron driven vacuum polarization effect; the analytical form of the first addend represented by ε_1, not yet concerned explicitly, is at present still under investigation. So, even without detailed calculations in this respect, appear two relevant facts: 1) the first three terms of (8.55) are enough to infer the Dirac equation; 2) the cubic term and the implementation of the energy range $\delta\varepsilon$ instead of a unique energy term ε are essential to infer contextually the Lamb energy shift too.

9. Discussion

The present model has concerned several topics of fundamental physics self-consistently inferred uniquely from the concept of evolution inherent the definitions (1.11) and (1.12). The concepts of mass, momentum, energy and electric charge, obviously missing in these equations, have been uniquely and self-consistently introduced through the fundamental constants of Nature. It appears also significant the chance of describing the Universe according to laws inferred from the change of a unique primordial function ψ, even regardless of a specific and detailed knowledge about the function that is changing itself: it is instead crucial how it changes.

The fact of having introduced an initial function and its actual space time evolution, has been proven enough to infer contextually quantum uncertainty and relativistic results in a surprisingly straightforward way even regardless of any deterministic metric and without hypotheses "ad hoc".

On the one side the necessity of quantized physical laws is implied by the concept itself of uncertainty, Equations (3.1) and (3.2), on the other side special and general relativity are implied by the space time frame under the condition of its holistic evolution. The reverse reasoning is also true: the foundation of quantum and relativistic theories are the fingerprint of an evolving Universe, whose evolution is governed by a few constants in which are nested the essential dy-

namical variables of interest for the everyday experimental activity, mass, energy charges and so on.

The modern physics is essentially wave physics. This is because the Bohr atom first opened the way to the hydrogenlike atoms and thus to the probabilistic interpretation of the wave functions. Next Schrödinger further enhanced this conceptual path including in the wave function the potential term and thus the electron correlation in many electron atoms and ions. Eventually a further step ahead was accomplished by Dirac: with its relativistic hydrogen atom, He has in fact introduced the quantum field theory. Yet all these physical models implemented wave formalism. The present paper, instead, introduces and contextually exploits the corpuscle nature of the particles constituting the matter, appropriately integrated with their wave nature when necessary. The subsection 3.2 has been reported just to clarify this point. In this way is irrelevant the theoretical problem raised by many physicists about why $\Psi^{*}\Psi$, and not Ψ itself, has physical meaning [21]; moreover the approach to the various equations of quantum and relativistic physics appears not only simpler but also the equations themselves are more interconnected. The Heisenberg principle has negative content; the statistical formulation of the space time uncertainty has instead a highly positive content as it shares both quantum and relativistic theories. Regarding a fundamental statement the uncertainty and following the approach shortly sketched in Section 3.2 the EPR paradox would be meaningless because the concept of distance is missing; the uncertainty ranges waive since the beginning conceptually, and not as a sort of approximation useful to simplify calculations, the concepts of local space time coordinates necessary to define "superluminal" distances.

As concerns the quantum way of describing the reality, these basic concepts can be summarized as follows

$$\begin{array}{ccc} -\dfrac{\hbar}{i}\dfrac{\partial}{\partial x} & & \dfrac{\hbar}{i}\dfrac{\partial}{\partial t} \\ \text{operator} & & \text{operator} \\ \updownarrow & & \updownarrow \\ p & \text{dynamical variables} & \varepsilon \\ \updownarrow & & \updownarrow \\ \text{uncertainty range} & & \text{uncertainty range} \\ \widetilde{\Delta p} & & \widetilde{\Delta \varepsilon} \\ \Delta x & \hbar, c, G, \alpha & \Delta t \end{array} \tag{9.1}$$

The upper part deals with differential equations that by definition describe the local properties of the solution of the pertinent wave equation; the lower part describes instead the system regardless of its local properties and thus without need of solving the pertinent differential equations. In principle both approaches are equivalent, although the operator formalism is a byproduct of the quantum uncertainty; in practice, however, the problem is to see which approach is more effective in describing the quantum properties of the Universe regardless of the local and deterministic tensor formalism. It is worth recalling that all papers

based only on the Equation (3.1) only, allowed to obtain the most significant results of both general relativity and quantum physics [20]; in the latter case, in particular, the usual positions (3.28) introducing the operator formalism of wave mechanics according to (3.66) and (3.67) are systematically replaced by the uncertainty positions

$$x \to \delta x, \quad p \to \delta p, \quad t \to \delta t, \quad \varepsilon \to \delta \varepsilon, \tag{9.2}$$

while obtaining results identical to that of the standard wave formalism, as shortly shown in subsection 3.2.

The classical dynamical variables p and ε are to be regarded equivalently as quantum differential operators or quantum uncertainty ranges: this implies that actually it is necessary neither to solve the Schrödinger equation of wave mechanics nor the tensor calculus of relativity. The form (3.2) expressing the quantum uncertainty, apparently weird, shows the quantum equivalent of the relativistic covariance: the Equations (3.1) could seem defined in some particular reference system, instead (3.2) show that whenever the dynamical variables are replaced by the respective uncertainty ranges about which nothing is known in the sense highlighted in the subsection 3.2, the dependence of any formula on a particular reference system, inertial or not, is lost. So the independence of formulas on any particular R is ensured by (3.1), despite their different forms in R and R'; however holds the more substantial fact the any formula inferred from (3.1) has validity in any R': this is the profound reason why relativistic formulas can be inferred from (3.1).

Yet, the rational foundation of everything is just the conceptual impossibility of knowing everything.

10. Conclusions

As stated in Section 1, part of this paper aimed to find known results as a test of validity of the present theoretical model. Besides well known results, explicitly quoted throughout the exposition, the model has also provided original results:

- Evolutionary imprinting and derivation of physical laws.
- Possible granular structure of the space time.
- Possible quantization of the temperature.
- Lnk between entropy, phase space and space time.
- Link between Van der Waals equation and quantum zero point state of matter.
- Link between relativity and quantum gravity.
- Probabilistic link between corpuscle and wavelike behavior of matter.
- Link between operator and uncertainty driven approach to quantum problems.
- Generalization of Dirac equation to include the Lamb effect.

Moreover:

- The Equations (4.4) and (3.13) show that even a small mass m_0 can take large values of kinetic mass m for $v \to c$; also, (4.6) shows that just m is the

classical mass.

- The Equation (6.19) has shown the existence of finite vacuum energy density η, to which corresponds according to (8.34) a pressure $P_{vac} = 2.2 \times 10^{-8}$ dyn/cm^2.
- The Equation (4.7) has shown that the corpuscle/wave behavior of matter has probabilistic character and that this probability involves the ratios m_0/m and v_g/c.
- The velocity dependence of mass shows that $m \to m_0$ implies $v \to 0$, whatever m_0 might itself be; contextually, increasing m_0 to m' means decreasing v_g from c to a smaller value v'.
- Short notes, although necessarily incomplete, emphasize the essential fingerprints of the strong and weak interactions, Equations (8.37), and contextually also the gravity force and Maxwell equations.
- The model explains why un upper limit of velocity, c, must necessarily exist.

These short remarks are enough to conclude that the present model fits the basic concepts of thermodynamics and fundamental forces of nature merging concept of quantum and relativistic physics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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