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Jacobi Elliptic Function Solutions for (2 + 1) Dimensional Boussinesq and Kadomtsev-Petviashvili Equation

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Abstract

(2 + 1) dimensional Boussinesq and Kadomtsev-Petviashvili equation are investigated by employing Jacobi elliptic function expansion method in this paper. As a result, some new forms traveling wave solutions of the equation are reported. Numerical simulation results are shown. These new solutions may be important for the explanation of some practical physical problems. The results of this paper show that Jacobi elliptic function method can be a useful tool in obtaining evolution solutions of nonlinear system.

Keywords: Jacobi Elliptic Function, Traveling Wave Solution, Kadomtsev-Petviashvili Equation, Jacobi Elliptic Function Expansion Method, Numerical Simulation

1. Introduction

It is well known that the nonlinear physical phenomena are related to nonlinear partial differential equations, which are employed in natural and applied science such as fluid dynamics, plasma physics, biology, etc. Their solution spaces are infnite-dimensional and contain diverse solution structures. In the past few years, wide variety of the powerful and direct methods to find all kinds of analysis solutions of nonlinear evolution equations had been developed [1-13]. The basic purpose of them is to construct new solitary wave solutions and periodic solutions. (2 + 1) dimensional Boussinesq and Kdomtsev-Petviashvili (BKP) equation is an important nonlinear partial differential equation in mathematical physics, which had been mentioned in literatures [14-16]. The equation belongs to a symmetry and integrable system. The aim of this paper is to apply the Jacobi elliptic function expansion method [11] to solve (2 + 1) dimensional BKP equation. The general BKP equation has the form

$$U_y = W_x$$

$$V_x = W_y$$

$$W_t = W_{xxx} + W_{yyy} + 6(WU)_x + 6(WV)_y \quad (1)$$

where $U(x,y,t)$, $V(x,y,t)$ and $W(x,y,t)$ are the functions about x,y, and t. U_y, V_x, W_y, are the derivatives of x,y and t, respectively.

2. Transformed Boussinesq and Kadomtsev-Petviashvili Equation and Jacobi Elliptic Function Expansion Method

In this section, we will apply the Jacobi elliptic function expansion method to BKP equation.

Using a wave variable, we obtain the transformed wave solutions as: $U(x,y,t) = U(\xi)$, $V(x,y,t) = V(\xi)$, and $W(x,y,t) = W(\xi)$, where $\xi = x + y - ct$, c is nonzero constant.

Plugging $U(\xi)$ $V(\xi)$, and $W(\xi)$ and integrating (1) once with respect to ξ and considering the constants of integration to be zero, we obtain the transformed BKP equation:

$$U - W = 0,$$

$$V - W = 0,$$

$$2W_{\xi\xi} - cW - 6UW - 6VW = 0. \quad (2)$$

The above equations are an ordinary differential equation. Fu and Liu gave an example in Jacobi elliptic function expansion method [11] to find the solutions for the ordinary differential equation. According to the method, the ansatz solutions for (2) are supposed as

$$U = a_0 + a_1 sn\xi + a_2 (sn\xi)^2; \quad (3a)$$

$$V = b_0 + b_1 sn\xi + b_2 (sn\xi)^2; \quad (3b)$$

$$W = c_0 + c_1 sn\xi + c_2 (sn\xi)^2; \quad (3c)$$

where $sn\xi$ is Jacobi elliptic function, $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1$, and c_2 are the expansion coefficients to be determined later. Substituting (3) into (2) yields a system of algebraic equations.

$$a_0 + a_1 sn\xi + a_2 (sn\xi)^2 - c_0 - c_1 sn\xi - c_2 (sn\xi)^2 = 0 \quad (4a)$$

$$b_0 + b_1 sn\xi + b_2 (sn\xi)^2 - c_0 - c_1 sn\xi - c_2 (sn\xi)^2 = 0 \quad (4b)$$

$$\begin{aligned}&2\left(c_0 + c_1 sn\xi + c_2 (sn\xi)^2\right)_{\xi\xi}\\ &-c\left(c_0 + c_1 sn\xi + c_2 (sn\xi)^2\right)\\ &-6\left(c_0 + c_1 sn\xi + c_2 (sn\xi)^2\right)\\ &\left(a_0 + a_1 sn\xi + a_2 (sn\xi)^2 - b_0 - b_1 sn\xi - b_2 (sn\xi)^2\right)\\ &= 0\end{aligned} \quad (4c)$$

Solving the above first two Equations (4a) and (4b), we obtain $a_0 = b_0 = c_0; a_1 = b_1 = c_1; a_2 = b_2 = c_2$.

Equating the coefficients of Jacobi elliptic function $sn\xi$ (4c) for the to zero with the above results and the follow relation

$$(cn\xi)^2 = 1 - (sn\xi)^2;\ (dn\xi)^2 = 1 - m^2 (sn\xi)^2, \quad (5)$$

we have

$$\begin{aligned}&ca_0 + 12a_0^2 - 4a_2 = 0;\\ &2a_1 + ca_1 + 2ma_1 + 24a_0 a_1 = 0;\\ &12a_1^2 + 8a_2 + ca_2 + 4ma_2 + 4m^2 a_2 + 24a_0 a_2 = 0;\\ &2ma_1 + 2m^2 a_1 - 24a_1 a_2 = 0;\\ &4ma_2 + 8m^2 a_2 - 12a_2^2 = 0;\end{aligned} \quad (6)$$

where $dn\xi$, $cn\xi$ and m $(0 < m < 1)$ are different kinds of Jacobi elliptic functions and modulus of Jacobi elliptic function, respectively.

Solving (6), we obtain

$$a_0 = \frac{-8 - c - 4m - 4m^2}{24}; a_1 = 0;$$

$$a_2 = \frac{m + 2m^2}{3}; c = 4\sqrt{4 - 3m^2 + 2m^3 + m^4};$$

and

$$a_0 = \frac{-8 - c - 4m - 4m^2}{24}; a_1 = 0;$$

$$a_2 = \frac{m + 2m^2}{3}; c = -4\sqrt{4 - 3m^2 + 2m^3 + m^4}.$$

With the help of (5), we rewritten (3) as the follow three kinds of Jacobi elliptic functions:

Case [I]

$$\begin{aligned}&U = V = W;\\ &U = a_0 + a_2 (sn\xi)^2;\end{aligned} \quad (7)$$

Case [II]

$$\begin{aligned}&U = V = W;\\ &U = a_0 + a_2\left(1 - (cn\xi)^2\right);\end{aligned} \quad (8)$$

Case [III]

$$\begin{aligned}&U = V = W;\\ &U = a_0 + a_2\left(\frac{1 - (dn\xi)^2}{m^2}\right).\end{aligned} \quad (9)$$

3. Periodic Traveling Wave Solutions for Boussinesq and Kadomtsev-Petviashvili Equation

When the modulus of Jacobi elliptic function $m \to 0$ and $m \to 1$, Jacobi elliptic functions asymptotically transformed into periodic trigonometric and hyperbolic traveling wave solutions:

$$m \to 0,$$

$$\text{sn}\,\xi \to \sin\xi; cn\xi \to \cos\xi; dn\xi \to 1 \quad (10)$$

$$m \to 1$$

$$sn\xi \to \tanh\xi; cn\xi \to \text{sech}\xi; dn\xi \to \text{sech}\xi \quad (11)$$

With the relation (10) and (11), Equations (7)-(9) are transformed into

$$\begin{aligned}&m \to 0; U = V = W;\\ &U = a_0 + a_2 (\sin\xi)^2;\end{aligned} \quad (12)$$

$$\begin{aligned}&m \to 1; U = V = W;\\ &U = a_0 + a_2 (\tanh\xi)^2;\end{aligned} \quad (13)$$

The simulation results of (7) and (13) are given with the help of Mathematica software in **Figures 1** and **2** for some special local parameter. From the **Figures 1** and **2**, we know that the amplitude of the wave is stable. These wave solutions may further help us to find some new physical phenomena.

4. Discussions and Conclusions

Some new analytical solutions of BKP equation are obtained by successfully employing Jacobi elliptic function expansion method in this paper. When the modulus of Jacobi elliptic function $m \to 0$ and $m \to 1$, the Jacobi elliptic functions asymptotically transformed into periodic trigonometric and hyperbolic traveling wave solutions. The results obtained in this paper are new solutions in the representation of Jacobi elliptic function and solitary wave solutions. These new solutions may be important for the explanation of some practical physical problems. These solutions may help us to learn more about the complex nonlinear evolutions systems. The Jacobi elliptic function expansion method in its present form is a successful, direct and concise tool in obtaining a serials

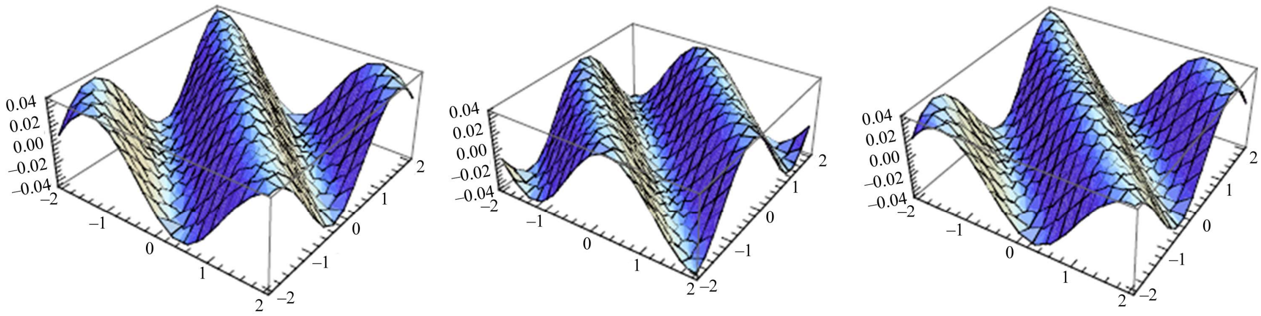

Figure 1. Jacobi elliptic function solutions Equation (7) is shown for *U* with *x* ∈ (–2, 2), *y* ∈ (–2, 2), *m* = 0.2, from left to right *t* = –0.2, *t* = 0, *t* = 0.2, respectively.

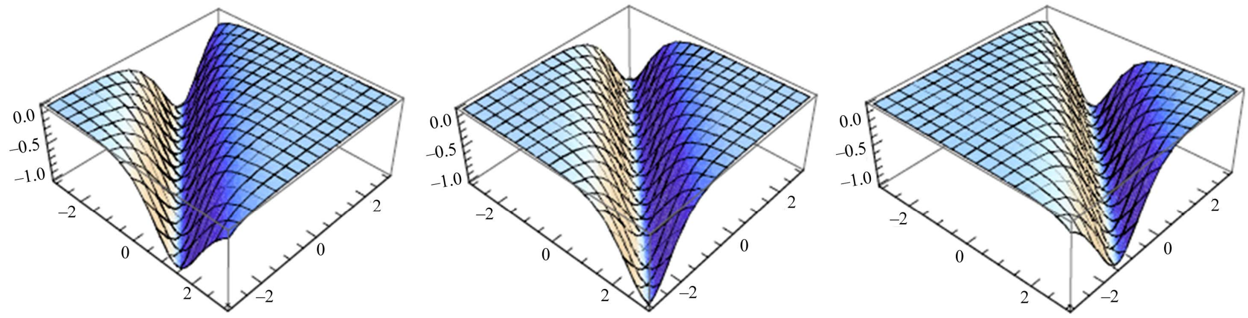

Figure 2. Hyperbolic function solutions Equation (13) is shown for *U* with *x* ∈ (–3, 3), *y* ∈ (–3, 3), *m* = 1, from left to right *t* = –0.2, *t* = 0, *t* = 0.2, respectively.

of nonlinear equations. Of course, this method can be also applied to other nonlinear wave equations. Seeking new more general traveling wave solutions of nonlinear equation is still an interesting subject and worthy of further study.

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