Theoretical Economics Letters, 2018, 8, 2893-2913
http://www.scirp.org/journal/tel
ISSN Online: 2162-2086
ISSN Print: 2162-2078

Optimal Foreign Exchange Risk Hedging: Closed Form Solutions Maximizing Leontief Utility Function

Yun-Yeong Kim

Department of International Trade, Dankook University, Yongin-si, South Korea
Email: yunyeongkim@dankook.ac.kr

How to cite this paper: Kim, Y.-Y. (2018) Optimal Foreign Exchange Risk Hedging: Closed Form Solutions Maximizing Leontief Utility Function. *Theoretical Economics Letters*, **8**, 2893-2913.
https://doi.org/10.4236/tel.2018.814181

Received: January 24, 2018
Accepted: October 16, 2018
Published: October 19, 2018

Abstract

In this paper, we extend Kim (2013) [9] for the optimal foreign exchange (FX) risk hedging solution to the multiple FX rates and suggest its application method. First, the generalized optimal hedging method of selling/buying of multiple foreign currencies is introduced. Second, the cost of handling forward contracts is included. Third, as a criterion of hedging performance evaluation, there is consideration of the Leontief utility function, which represents the risk averseness of a hedger. Fourth, specific steps are introduced about what is needed to proceed with hedging. There is a computation of the weighting ratios of the optimal combinations of three conventional hedging vehicles, *i.e.*, call/put currency options, forward contracts, and leaving the position open. The closed form solution of mathematical optimization may achieve a lower level of foreign exchange risk for a specified level of expected return. Furthermore, there is also a suggestion provided about a procedure that may be conducted in the business fields by means of Excel.

Keywords

Foreign Exchange, Risk, Optimal Hedging, Closed Form Solution

1. Introduction

Recently, foreign currency fluctuations are one of the key sources of risk in multinational business/investment operations because of the widespread adoption of the floating exchange rate regime in many countries after the breakdown of the Bretton Woods system.[1] The U.S. Department of Commerce has also warned that "The volatile nature of the FX market poses a great risk of sudden and drastic

[1]The number of countries with floating and free floating arrangements are 36 and 29 by 2014, respectively, according to IMF (https://www.imf.org/external/pubs/nft/2014/areaers/ar2014.pdf).

FX rate movements, which may cause significantly damaging financial losses from otherwise profitable export sales" (*Trade Finance Guide*, Ch. 12).[2] Furthermore, that *Guide* also suggested three FX risk management techniques that are considered suitable for small and medium-sized enterprises companies: non-hedging FX risk management techniques, FX forward hedges, and FX options hedges.

However, for practical use by businesses or individuals, there has not been an analytical method with a closed form solution to choose from among the various available hedging tools to reduce the risk optimally, as correctly pointed out by Khoury and Chan (1988) [8]. For further studies on this issue, see Sercu and Uppal (1995) [11].

Khoury and Chan (1988) [8] gauge the preferences of finance officers in terms of the specific characteristics of a hedging tool, by relying on a questionnaire survey. Bodie, *et al.* (2002) [2] and Nancy (2004) [1] illustrate the technique of computerized optimization and simulation modeling to manage foreign exchange risk. However, their techniques are not a closed form optimal hedging solution that requires additional computational burden. So its application is limited in the real business world. In this regard, Kim (2013) [9] introduced the optimal foreign exchange risk hedging solution by exploiting a standard portfolio theory.[3] Hsiao (2017) [7] applies the framework of Kim (2013) [9] to investigate the effects of foreign exchange exposures on the performance of Taiwan hospitality industry and try to propose some hedging strategies and strengthen their corporate risk management.

In this paper, we extend Kim (2013) [9] for the optimal single FX risk hedging solution and theory to the multiple FX rates and suggest its application method in the business fields. First, the generalized optimal hedging method of selling/buying of multiple foreign currencies is introduced. Second, the cost of handling forward contracts is included. Third, as a criterion of hedging performance evaluation, we consider the Leontief utility (or profit for a firm) function, which represents the risk averseness of a hedger. Fourth, steps are introduced about what is needed to proceed with hedging. There is a computation of the weighting ratios of the optimal combinations of three conventional hedging vehicles, *i.e.*, call/put currency options, forward contracts, and leaving the position open. As in the standard portfolio theory, the closed form solution of mathematical optimization may achieve a lower level of foreign exchange risk for a specified level of expected return. There is also a suggestion provided for a procedure that may be conducted in the business fields by means of Excel.[4]

The rest of this paper is as follows. Section 2 derives the expected return and return variance of the hedging vehicles. Section 3 analyzes the optimal hedging selection. Section 4 is on application of developed method, and Section 5 is the conclusion.

[2]U.S. Department of Commerce, *Trade Finance Guide*, Ch. 12, "Foreign Exchange Risk Management," http://trade.gov/publications/pdfs/tfg2008ch12.pdf.

[3]Kim (2013) [9] considered a single currency case and just for selling case. So its practical applications are very limited.

[4]American currency option and currency future are not considered in this paper because of the speculative nature. Thus, the focus is solely on the hedging of the FX risk.

2. Expectation and Variance of Hedging Tools' Returns

In this section, we construct an efficient hedging frontier composed of the expected value and variance of each hedging vehicle's return for the multiple foreign exchanges. So, it is exactly matched with the portfolio possibilities curve in modern portfolio theory. Note an optimal combination of hedging vehicles is one that maximizes the expected return given a desired level of risk. For this objective, there is a need to compute the mean and variance of each tool.

Before proceeding, we assume that a foreign investor needs to buy or sell m-different currencies $\Theta \equiv (\theta_1, \theta_2, \cdots, \theta_m)'$ $[m \times 1]$ at a future time T where θ_i is represented by the unit of i-th currency. He is worrying about the foreign exchange risk of domestic currency (e.g., US dollar) term translated value of Γ and to hedge it optimally at time 0. The m-foreign exchange rates at time t in terms of domestic currency, is denoted as $S_t \equiv (e_{1t}, e_{2t}, \cdots, e_{mt})'$. For instance, e_{it} is the dollar price of one euro or yen where the dollar is the domestic currency. It is presupposed that there are three hedging tools, *i.e.*, European currency put (or call) option, forward contracts, and leaving the position open.[5] Furthermore, there are the following definitions: a forward contract rate vector $F_t \equiv (\overline{e}_{1t}, \overline{e}_{2t}, \cdots, \overline{e}_{mt})'$, a striking price vector $K \equiv (\kappa_1, \kappa_2, \cdots, \kappa_m)'$, and its premium $P \equiv (p_1, p_2, \cdots, p_m)'$ at time t of a European put (or call) option with the common maturity T.[6] Finally, $C \equiv (c_1, c_2, \cdots, c_m)'$ is a per unit handling cost vector for the forward contract F_t if a bank is used.

Define the log of domestic currency term translated value of FX asset Γ at time t is given as $s_t \equiv \ln(\Theta' S_t)$ which is a FX value. For instance, if $\Theta = (10 \text{ Euro}, 50 \text{ Yen})$ and $S_t = (1.5 \text{ Dollar/Euro}, 0.1 \text{ Dollar/Yen})$, then $s_t = \ln(20 \text{ Dollar})$. We assume the ($s_t$) follows a random walk process:

Assumption 2.1. *We suppose*

$$s_{t+1} = s_t + u_{t+1}, t = 1, 2, \cdots, n \tag{2.1}$$

where $\{u_t\}$ *is independent, identically and normally distributed sequence with the mean zero and variance* $\sigma^2 > 0$.

Note $E_t s_{t+1} = s_t$ where E_t is a conditional expectation. Thus Assumption 2.1 just a variant of the efficient FX market hypothesis.[7] σ^2 is consistently estimated by $\hat{\sigma}^2 = \frac{\sum_{t=1}^{n} (\Delta s_t)^2}{n}$.

Now we derive the return and its variance of different hedging tools, where the return is compared with the selling (or buying) a foreign currency (as a bench mark) by the spot rate s_0.

2.1. FX Selling Case

First, we derive the expected return R_n and its variance V_n^2 of the non-hedging (leaving the position open), as follows.

[5]It is a non-hedging and to buy the foreign currency at time T.

[6]The value of the put option was derived by Garman and Kohlhagen (1983) [5].

[7]See Diebold and Nason (1990) [3] for this issue.

Theorem 2.2. *Suppose Assumption* 2.1 *holds. Then the expected return for non-hedging of FX asset* Θ *is* $R_n = 0$ *and its variance during time T is* $V_n^2 = T\sigma^2$.

All proofs of the theorems are in the Appendix.

Second, we derive the expected return R_f and its variance V_f^2 of the forward contract as follows.

Theorem 2.3. *Suppose* Assumption 2.1 *holds. Then the expected return of forward is* $R_f = f_T - s_0 - c$ *and its variance is* $V_f^2 = 0$ *where* $f_T \equiv \ln(\Theta' F_T)$, *and* $c \equiv \Theta' C / \Theta' S_0$.

Now we derive the expected return R_p and its variance V_p^2 of currency put option as follows.

Theorem 2.4. *Suppose* Assumption 2.1 *holds. Then,*

(a) *the expected return of currency put option is given as:*

$$R_p = x_0 \Phi(z_0) + \sigma\sqrt{T}\phi(z_0) - p$$

and

(b) *its variance of currency put option is:*

$$V_p^2 = x_0^2 \Phi(z_0) + T\sigma^2 E(z_T^2 | z_T \ge z_0)[1 - \Phi(z_0)] - (x_0 \Phi(z_0) + \sigma\sqrt{T}\phi(z_0))^2.$$

where $k \equiv \ln(\Theta' K)$, $p \equiv \Theta' P / \Theta' S_0$, $x_0 \equiv k - s_0$, *and* $z_0 = x_0 / (\sigma\sqrt{T})$ *where* $\phi(z)$ *and* $\Phi(z)$ *are the standard normal density and distribution functions respectively and* $F_{a.0}$ *denotes the distribution function of central* $\chi^2_{(q)}$ *distribution with the degree of freedom q and:*

$$E(z_T^2 | z_T \ge z_0) = \frac{1 - F_{3,0}(z_0^2)}{1 - F_{1,0}(z_0^2)} \quad if\ z_0 \ge 0$$

$$= \frac{F_{3,0}(z_0^2)}{F_{1,0}(z_0^2)}[1 - 2\Phi(z_0)] + \frac{1 - F_{3,0}(z_0^2)}{1 - F_{1,0}(z_0^2)}[1 - \Phi(-z_0)] \quad if\ z_0 < 0.$$

In the above Theorem 2.4, it was suggested that a form of V_p^2 represented by a $\chi^2_{(1)}$ distribution for the computation of conditional expectation $E(z_T^2 | z_T \ge z_0)$. Otherwise, there is a need for integration by a formula $E(z_T^2 | z_T \ge z_0) = \int_{z_0}^{\infty} z_T^2 \phi(z_T) dz_T$, which requires an additional burden.

Next, there is a derivation of the covariance among the three hedging tools. Note the covariance of returns between non-hedging (or option) and forward is obviously zero since the forward return is not random. Then the covariance of returns between put option and non-hedging is given as follows.

Theorem 2.5. Suppose Assumption 2.1 *holds. Then the covariance of returns between put option and non-hedging is:*[8]

$$Cov_{pn} = -x_0 \sigma\sqrt{T}\phi(z_0) + T\sigma^2 E(z_T^2 | z_T \ge z_0)[1 - \Phi(z_0)].$$

2.2. FX Buying Case

First, note we have the same expected return $R_n = 0$ and its variance

[8]See Theorem 2.4 for the definitions.

$V_n^2 = T\sigma^2$ of the non-hedging, as given in Proposition 2.2 for buying a foreign exchange case.

Second, we derive the expected return R_f and its variance V_f^2 of forward contract as follows.

Theorem 2.6. *Suppose* Assumption 2.1 *holds. Then the expected return of forward*[9] *is* $R_f = s_0 - f_T - c$ *and its variance is* $V_f^2 = 0$.

Now we *derive* the expected return R_c and its variance V_c^2 of currency call option as follows.

Theorem 2.7. Suppose Assumption 2.1 *holds. Then,*

(a) *the expected* return *of currency call option is given as*:

$$R_c = \sigma\sqrt{T}\phi(z_0) - x_0\left[1-\Phi(z_0)\right] - p$$

and

(b) *its variance of currency call option is*:

$$V_c^2 = x_0^2\left[1-\Phi(z_0)\right] + T\sigma^2 E\left(z_T^2 \middle| z_T < z_0\right)\Phi(z_0) - \left(x_0\left[1-\Phi(z_0)\right] - \sigma\sqrt{T}\phi(z_0)\right)^2$$

where

$$E\left(z_T^2 \middle| z_T < z_0\right) = \frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)} \text{ if } z_0 < 0$$

$$= \frac{F_{3,0}\left(z_0^2\right)}{F_{1,0}\left(z_0^2\right)}\left[1-2\Phi(-z_0)\right] + \frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)}\Phi(-z_0) \text{ if } z_0 \ge 0.$$

The covariance of returns between call option and the non-hedging is given as follows.

Theorem 2.8. *Suppose* Assumption 2.1 *holds. Then the covariance of returns between put option and non-hedging is*:

$$Cov_{cn} = x_0\sigma\sqrt{T}\phi(z_0) + T\sigma^2 E\left(z_T^2 \middle| z_T < z_0\right)\Phi(z_0).$$

3. Efficient Hedging Frontier Construction

Based upon above derivation of expected return (R) and return variance (V^2) structure, now we can derive the efficient hedging frontier. It is exactly matched with the portfolio possibilities curve in a standard portfolio theory (e.g., Elton, *et al.* (2007) [4]).

For this purpose, first, there is consideration of a portfolio composed of non-hedging and put in the option (for FX selling) that are all risky. Let the weight of non-hedging be as w and 1-w for the option where w is a real number. Then, from the above derivation in Section 2, its expected return is defined as follows.[10]

$$R(w) = wR_n + (1-w)R_p = (1-w)R_p \tag{3.1}$$

[9]Buying the foreign exchange means outflow of domestic currency. So, a negative of the forward amount is taken.

[10]In case of call option, R_p, V_p^2 and Cov_{pn} are replaced by R_c, V_c^2 and Cov_{cn} respectively.

because $R_n = 0$ for the non-hedging, and its variance is given as:

$$V^2(w) = w^2 V_n^2 + (1-w)^2 V_p^2 + 2w(1-w) Cov_{np}.$$

Therefore note $R(0) = R_p$, $R(1) = R_n = 0$, $V^2(0) = V_p^2$ and $V^2(1) = V_n^2$.

In this case, the return of forward has zero variance with the expected return, say, R_f. Thus, it is regarded as a riskless asset in the standard portfolio theory. Now the hedging allocation line (a line of *R* and *V*)[11] connecting the riskless forward contract and a combination of *non-hedging and put option* is defined as follows.

$$R = R_f + \left(\frac{R(w) - R_f}{V(w)}\right) V, \tag{3.2}$$

where *R* denotes the return and *V* denotes the standard deviation of return (as a risk); $\left[R(w) - R_f\right] / V(w)$ is a constant slope for a given *w* where $V(w) = \sqrt{V^2(w)}$.

Then the efficient hedging allocation line[12] is given by solving following problem:

$$\max_w \frac{R(w) - R_f}{V(w)} \tag{3.3}$$

that is maximizing the slope of Equation (3.2) with the argument *w*. The problem (3.3) may be solved without restriction, according to Elton, *et al.* ([4]: pp. 100-103), as follows.

$$w^* = \frac{m_1}{m_1 + m_2} = \frac{-V_p^2 R_f - Cov_{np}\left(R_p - R_f\right)}{R_f\left(Cov_{np} - V_p^2\right) + \left(V_n^2 - Cov_{np}\right)\left(R_p - R_f\right)} \tag{3.4}$$

where $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} V_n^2 & Cov_{np} \\ Cov_{np} & V_p^2 \end{pmatrix}^{-1} \begin{pmatrix} -R_f \\ R_p - R_f \end{pmatrix}$ assuming $\begin{vmatrix} V_n^2 & Cov_{np} \\ Cov_{np} & V_p^2 \end{vmatrix} \neq 0$.

If $w^* \notin [0,1]$, then the maximization problem (3.3) should be solved under the restriction $w \in [0,1]$ using a typical Kuhn-Tucker condition.

Finally, the efficient hedging frontier is given by:

$$R = R_f + \left(\frac{R(w^*) - R_f}{V(w^*)}\right) V \text{ of the left of } \left[R(w^*), V(w^*)\right] \text{ if } w^* \in [0,1] \tag{3.5}$$

$$= \left[R(w), V(w)\right] \text{ of the right of } \left[R(w^*), V(w^*)\right] \quad \text{otherwise.}$$

For the given efficient frontier in (3.5), the optimal hedging (cf., separation theorem) is conducted as follows. First, the hedging ratio between non-hedging and option are set as $(w^*, 1-w^*)$. See **Figure 1**. Second, ρ is set for the forward and $1-\rho$ is set for the first combination of non-hedging and option. So if $\rho = 1$, then the forward becomes the unique hedging tool.

Finally, $\left[\rho, w^*(1-\rho), (1-w^*)(1-\rho)\right]$ becomes the optimal hedging ratio of the forward, non-hedging, and put option. Note the expected utility maximization

[11]It is called as the capital allocation line in the portfolio theory.

[12]It is called as the capital market line in the portfolio theory.

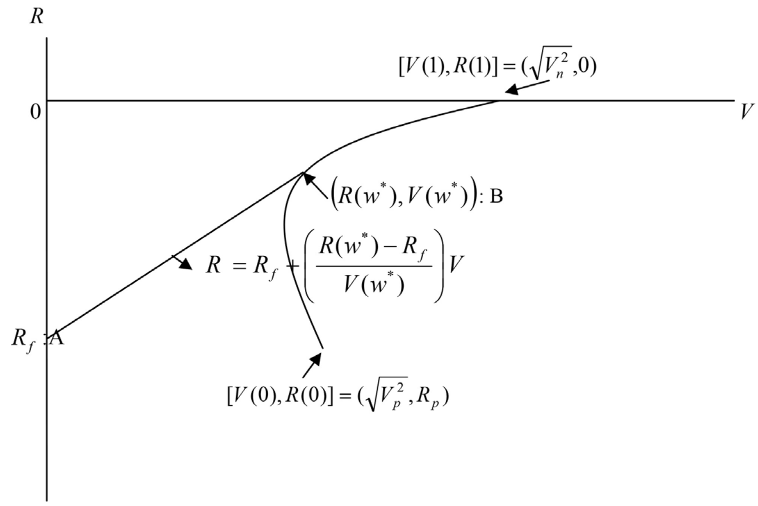

Figure 1. Efficient hedging frontier. (A: Forward only solution, B: Non-Forward solution).

may be a rule to determine an optimal ρ. The following section suggests an optimal hedging solution through determining an optimal ρ under the Leontief utility function.

4. Optimal Hedging under Leontief Utility Function

A Leontief utility (or profit for a firm) function is considered $U=\min(R,\alpha+\beta V)$ as a criterion for hedging performance evaluation where $\beta<0$. Note, for the maximization of a Leontief utility function under the efficient hedging frontier in Figure 1, a pair (V, R) should satisfy a line:

$$R=\alpha+\beta V. \tag{4.1}$$

To show it, let us derive an indifference curve. For this, suppose $R_0=\alpha+\beta V_0$ (as in Figure 2). Then a utility of (V_0,R) has the same utility with (V_0,R_0) for $R_0\le R$ because a utility of (V_0,R) is $\min(R,\alpha+\beta V_0)=\min(R,R_0)=R_0$ while the utility of (V_0,R_0) is $\min(R_0,\alpha+\beta V_0)=R_0$ from $R_0=\alpha+\beta V_0$. Similarly, a utility of (V,R_0) has the same utility with (V_0,R_0) for $V\le V_0$ because a utility of (V,R_0) is $\min(R_0,\alpha+\beta V)=R_0$ using $\alpha+\beta V\ge\alpha+\beta V_0=R_0$ while the utility of (V_0,R_0) is $\min(R_0,\alpha+\beta V_0)=R_0$ from $R_0=\alpha+\beta V_0$. So the North-West direction indicates the increase of utility in a space of (V, R).

Later, the above Equation (4.1) will be called a utility maximizing locus (UML). The UML might be interpreted as that which denotes how V is transformed into R with the same utility. It also denotes a cost of the standard deviation (volatility) for a hedging portfolio. See Figure 2 where the cost for the volatility V_0 is evaluated as $R_0=\alpha+\beta V_0$ in terms of return.

Note, the above Leontief utility function and conformable UML represent an extreme risk averseness. It is related to the marginal rate of substitution of the volatility to a return at the utility maximizing point along UML, which is $+\infty$, *i.e.*, the marginal increase of V requires an infinite return *increase* (as compensation for augmented risk) for the same utility, whereas, a marginal decrease of V does

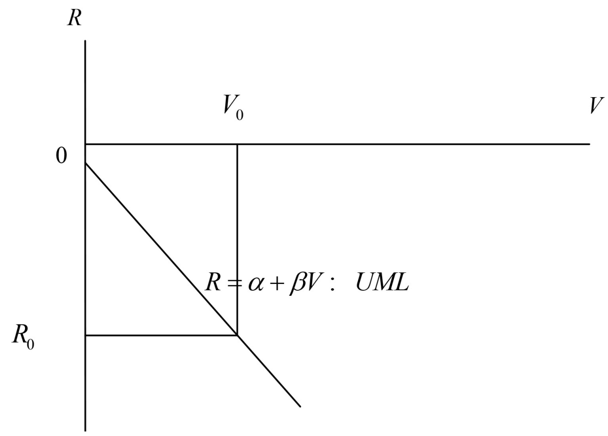

Figure 2. Indifference curve under Leontief utility function.

not require any return to be at the same utility level. This assumption is not so unrealistic because this model is not designed for the speculator but for the hedger/firms in the real world of business who are concerned with the volatility of fund flow.

Now to estimate α and β by an ordinary least square regression, we re-write Equation (4.1) as:

$$E(R) = \alpha + \beta\sqrt{E\left[z - E(z)\right]^2}$$

or approximately

$$R_{Ti} = \alpha + \beta\left|z_{Ti} - \bar{z}\right| + \varepsilon_{Ti} \quad \text{for} \quad i = 1, 2, \cdots, n \tag{4.2}$$

where $z_{Ti} \equiv s_{Ti} - s_{T(i-1)}$ is a change rate of FX asset during a maturity from 0 to *T*, $\bar{z}$ is a sample average of z_{Ti}, and ε_{Ti} is assumed as a mean zero error term that is not correlated with z_{Ti}.

Now note the intersection of UML (4.1) and the efficient hedging frontier (3.5), which is given as follows.

$$\tilde{V} = \frac{R_f - \alpha}{\beta - \frac{R(w^*) - R_f}{V(w^*)}} \quad \text{and} \quad \tilde{R} = \alpha + \beta\tilde{V},$$

after solving two Equations (3.5) and (4.1) with two unknowns R and V when $0 \le w^* \le 1$. The above solution point $(\tilde{V}, \tilde{R})$ helps to find the optimal weight for the riskless forward contract as

$$\rho^* = 1 - \frac{\tilde{V}}{V(w^*)} \text{ [13]} \tag{4.3}$$

when $0 \le \frac{\tilde{V}}{V(w^*)} \le 1$. See **Figure 3**.

Consequently

[13]It is also equivalently written as $\rho = 1 - \frac{\tilde{R} - R_f}{R(w^*) - \tilde{R}}$ from the property of the proportional triangular.

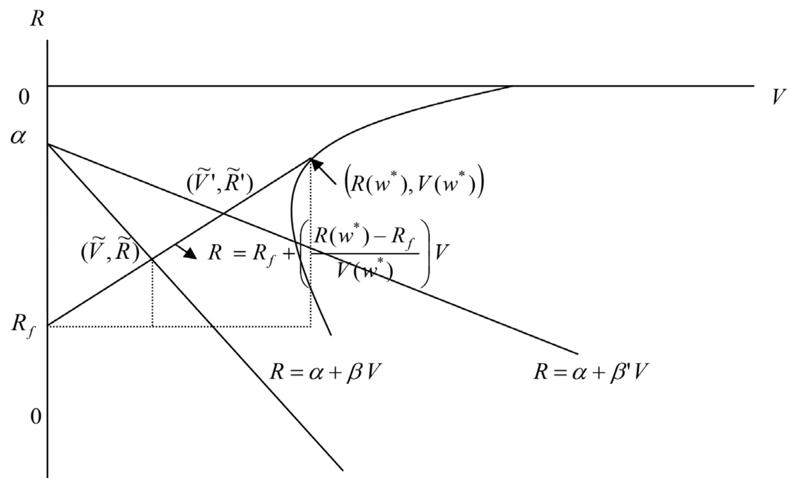

Figure 3. Derivation of optimal weight for forward.

$$\left[\rho^*, w^*\left(1-\rho^*\right), \left(1-w^*\right)\left(1-\rho^*\right)\right] \tag{4.4}$$

becomes the optimal hedging ratio of forward, non-hedging, and put option using (4.3) for the vector Θ. So, for instance, the weight ρ^* of Θ needs to be distributed to the forward.

Note, if the slope coefficient β as a marginal cost of volatility V is decreased to $\beta'(<\beta)$, then the new optimal weight for the forward contract (riskless) is decreased as $\rho^*{}' = 1 - \frac{\tilde{V}'}{V\left(w^*\right)} < \rho^*$. So more risk can be admitted because the marginal cost of volatility is decreased. See **Figure 3** to see this change.

However, if $\tilde{V}$ is larger than $V\left(w^*\right)$ because β is sufficiently small, then the weight for the forward contract (remind $R_n = 0$) may become zero.[14] In this case, w^* is not any further an optimal weight between the leaving open position and the option. Rather, we have to choose it from the intersection of UML and the locus of $\left[V\left(w\right), R\left(w\right)\right]$ which depends on the weight parameter *w*. The new solutions $\left[\bar{V}, \bar{R}\right]$ for the optimization are computed as follows.[15]

Theorem **4.1:** *Suppose a pair* (*V, R*) *satisfies a line* (4.1). *Then*

$\bar{R} = \frac{-b \pm \sqrt{b^2 - ac}}{a}$ and $\bar{V} = \frac{\bar{R} - \alpha}{\beta}$ *assuming* $b^2 - ac \geq 0$ *where*

$a = \frac{R_p^2}{\beta^2} - V_n^2 - V_p^2 + 2Cov_{pn}$, $b = -\alpha \frac{R_p^2}{\beta^2} + R_p V_n^2 - Cov_{pn} R_p$, $c = \frac{\alpha^2 R_p^2}{\beta^2} - R_p^2 V_n^2$.

In Theorem 4.1, we may have two different solutions that need to be selected to maximize the utility. So, we need to select one $\bar{R}$ among them maximizing the utility and define a conformable optimal expected return as $\bar{R}^* \equiv \arg\max_{\bar{R}} U = \max_{\bar{R}} \min\left(\bar{R}, \alpha + \beta\bar{V}\right)$. See following **Figure 4**.

Finally, the optimal hedging ratio of the forward, non-hedging, and put option becomes $\left[0, \bar{w}, \left(1-\bar{w}\right)\right]$ where

[14]It is when the marginal cost of V is small and thus a riskless forward contract is not chosen.

[15]Remind that any forward is not used in this case.

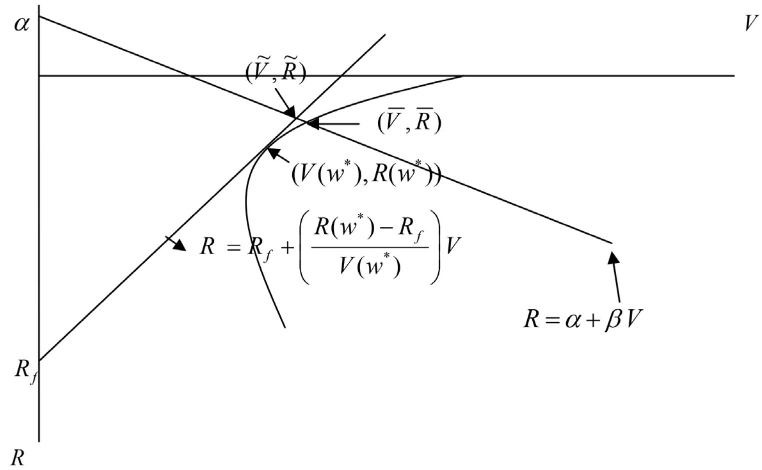

Figure 4. Optimal hedging without forward contract.

$$\bar{w} = -\frac{\bar{R}^* - R_p}{R_p} \tag{4.5}$$

from solving (3.1) for the weight w.

Finally, if $\rho \geq 1$, then a weighting vector (1, 0, 0) that is just selling the forward becomes the optimal hedging ratio.

5. Application Procedures

In application, suppose, at time 0, an investor hopes to sell one unit of foreign exchange at a future time T. Then following steps need to be carried out for hedging.

1) Select three vehicles of hedging as: forward contracts, leaving the position open (Selling foreign exchange case) and European currency put option.

2) Compute mean, variance, and covariance of each tool using the formula in Section 2.

3) Compute a weighting coefficient w^* as in (3.4) or $\bar{w}$ as in (4.5) if $\tilde{V} < \bar{V}$ or leaving the position open against the put option.

4) Decide α and β using OLS regression as in (4.2).

5) Compute an optimal weighting coefficient for the forward against for the portfolio of option and leaving the position open ρ as in (4.3).

6) Finally compute the optimal hedging ratio of the forward, non-hedging, and option. $\left[\rho^*, w^*\left(1-\rho^*\right), \left(1-w^*\right)\left(1-\rho^*\right)\right]$ as in (4.4).

Consequently, we summarize the optimal weighting vectors of forward, option, and non-hedging for optimal hedging, as shown in **Table 1**.

Then we apply the developed method for the exchange rate of the euro against the US dollar. The data frequency and period are presented on a monthly basis from January 1999 to March 2015. All data have been taken from FRED of FRB St. Louis.

Thus we assume, at time 0, *i.e.*, June 1, 2015, 1.1235 dollar price of one euro with $\sigma = 0.024$ \$/€, a hedger hopes to sell one unit of foreign exchange at a

Table 1. Optimal hedging weighting vector.

	$w^* < 0$	$0 \le w^* < 1$	$1 \le w^*$
$\rho^* < 0$	(0,0,1)	$[0, \overline{w}, (1-\overline{w})]$	(0,1,0)
$0 \le \rho^* < 1$	$[\rho^*, 0, (1-\rho^*)]$	$[\rho^*, w^*(1-\rho^*), (1-w^*)(1-\rho^*)]$	$[\rho^*, 1-\rho^*, 0]$
$1 \le \rho^*$	(1,0,0)	(1,0,0)	(1,0,0)

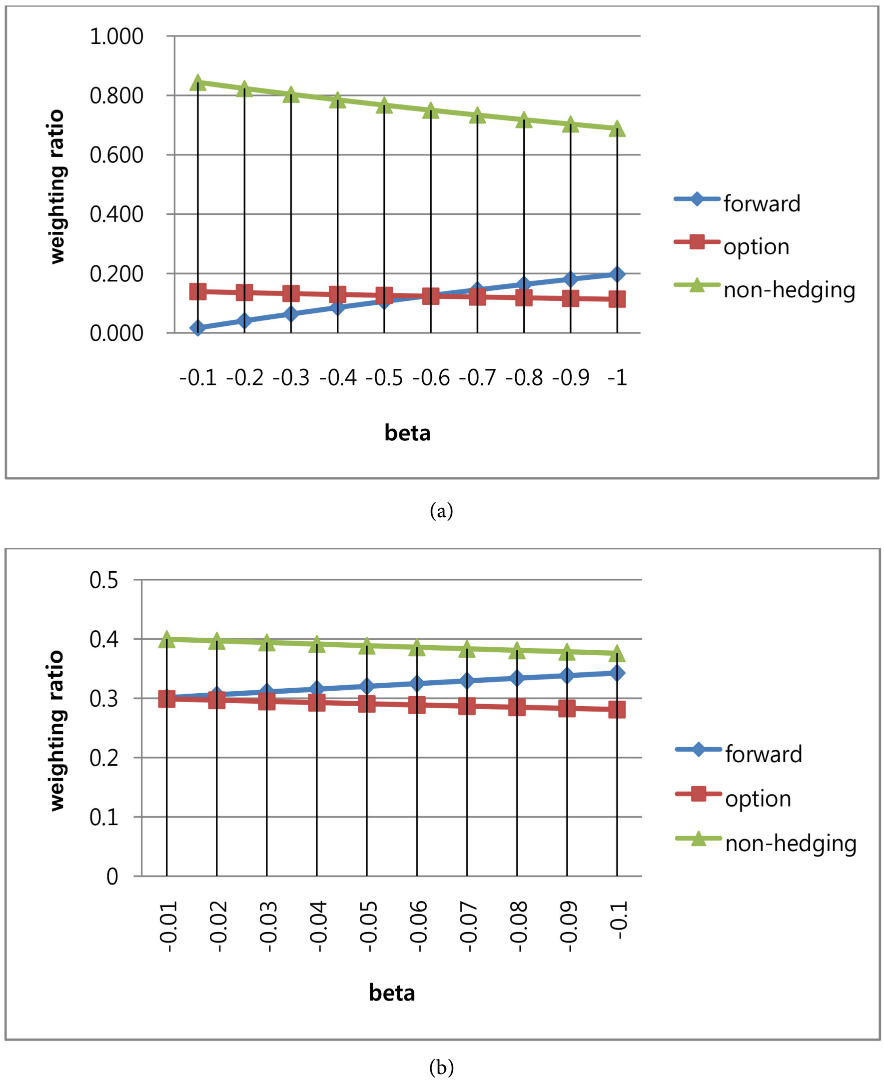

Figure 5. Optimal weighting ratio change as β decrease. (a) Selling FX case; (b) Buying FX case.

future time $T = 6$ months. Further we suppose that there are three hedging tools, *i.e.*, European currency put option, forward contracts, and leaving the position open. Assume a forward contract rate $F = 1.1$ \$/€, selling cost for the forward contract $C = 0.1$ \$, a striking price $K = 1.15$ \$/€, and its premium $P = 0.03$ \$/€ for European put option with the maturity $T = 6$, respectively. Note $z_0 = k - s_0 > 0$ in this case.

We assume $\alpha = 0.01$. See **Figure 5** for the optimal weighting ratio in (4.4) change as β decreases[16]. Note, if β as a marginal cost of volatility V is decreased, then the optimal weight for the forward contract (riskless) is decreased, as expected in the above theoretical explication (see Section 3).

6. Conclusions

This paper introduced the optimal foreign exchange risk hedging solution by exploiting a standard portfolio theory, thus extending Kim (2013) [8] in its following features. First, the case of the selling/buying of multiple foreign currencies is also considered. Second, the cost of handling forward contracts is included. Third, as a criterion of hedging performance evaluation, we consider the Leontief utility function, which represents the risk averseness of a hedger. Fourth, steps are introduced about what is needed to proceed with hedging. There is a computation of the weighting ratios of the optimal combinations of three conventional hedging vehicles, *i.e.*, call/put currency options, forward contracts, and leaving the position open. The closed form solution of mathematical optimization may achieve a lower level of foreign exchange risk for a specified level of expected return. There is also a suggestion provided about a procedure that may be conducted in the business fields by means of Excel.

The structure may be extended to cover the futures and American options and it will be a future research topic for us. However, I hypothesize that a similar logic may be readily applied to these extensions applying developed method in this paper. Furthermore, a development of a convenient computer program for FX risk hedging users, based on above results, would be a useful project.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Beneda, N. (2004) Optimal Hedging and Foreign Exchange Risk. Credit and Financial Management Review, October.

[2] Bodie, Z., Kane, A. and Marcus, A. (2002) Investments. McGraw Hill, New York.

[3] Diebold, F.X. and Nason, J.A. (1990) Nonparametric Exchange Rate Prediction? *Journal of International Economics*, **28**, 315-332. https://doi.org/10.1016/0022-1996(90)90006-8

[4] Elton, E., Gruber, M., Brown, S. and Goetzmann, W. (2007) Modern Portfolio Theory and Investment Analysis. Wiley, New Jersey.

[5] Garman, M and Kohlhagen, S. (1983) Foreign Currency Option Values. *Journal of International Money and Finance*, **2**, 231-237. https://doi.org/10.1016/S0261-5606(83)80001-1

[6] Greene, W. (2003) Econometric Analysis. Pearson Education, London.

[16]EXCEL code for optimal hedging ratio computation is available at https://blog.naver.com/yunyeongkim

[7] Hsiao, C.M. (2017) Enterprise Risk Management with Foreign Exchange Exposures: Evidence from Taiwan Tourism Industry. *Asian Economic and Financial Review*, **7**, 882-906. https://doi.org/10.18488/journal.aefr.2017.79.882.906

[8] Khoury, S. and Chan, K. (1988) Hedging Foreign Exchange Risk: Selecting the Optimal Tool. *Midland Corporate Finance Journal*, **5**, 40-52.

[9] Kim, Y. (2013) Optimal Foreign Exchange Risk Hedging: A Mean Variance Portfolio Approach. *Theoretical Economics Letters*, **3**, 1-6. https://doi.org/10.4236/tel.2013.31001

[10] Marchand, E. (1996) Computing the Moments of a Truncated Noncentral Chi-Square Distribution. *Journal of Statistical Computation and Simulation*, **55**, 23-29. https://doi.org/10.1080/00949659608811746

[11] Sercu, P and Uppal, R. (1995) International Financial Markets and the Firm, South-Western. South-Western College Pub, USA.

Appendix: Proofs of Theorems

Proof of Theorem 2.2: Note the return of non-hedging is approximately the value of following[17]:

$$\frac{\Theta'\left(S_T - S_0\right)}{\Theta' S_0} \cong s_T - s_0 \tag{1}$$

assuming $\Theta'\left(S_T - S_0\right)$ is small. Then, under Assumption 2.1, the claimed results hold as:

$$E\left(s_T - s_0 \middle| \Omega\right) = 0 \quad \text{and} \quad E\left[\left(s_T - s_0\right)^2 \middle| \Omega\right] = T\sigma^2 . \tag{2}$$

Proof of Theorem 2.3: Note the expected return for forward is the value of following:

$$\frac{\Theta'\left(F_T - S_0 - C\right)}{\Theta' S_0} \cong f_T - s_0 - c \tag{3}$$

assuming $\Theta'\left(F_T - S_0 - C\right)$ is small. Its variance is obviously zero since the return is not random. □

Proof of Theorem 2.4: (a) Note the inflow of selling weighted put option at time T is given as $\Theta'\left[\max\left(S_T, K\right) - P\right]$. Thus its return is given as following:

$$\begin{aligned} &\frac{\Theta'\left[\max\left(S_T, K\right) - P - S_0\right]}{\Theta' S_0} \\ &= \max\left(\frac{\Theta'\left[S_T - S_0\right]}{\Theta' S_0}, \frac{\Theta'\left[K - S_0\right]}{\Theta' S_0}\right) - \frac{\Theta' P}{\Theta' S_0} \\ &\cong \max\left(s_T - s_0, k - s_0\right) - p \equiv \max\left(x_T, x_0\right) - p \end{aligned} \tag{4}$$

assuming $S_T - S_0$ and $K - S_0$ are small.

Now the expected return conditional on Ω in (4) is computed as:

$$E\left[\max\left(x_T, x_0\right) \middle| \Omega\right] - p = x_0 \Phi\left(z_0\right) + \sigma\sqrt{T}\phi\left(z_0\right) - p$$

from (4) where $x_T \equiv s_T - s_0$, since[18]

$$\begin{aligned} E\left[\max\left(x_T, x_0\right) \middle| \Omega\right] &= E\left[\max\left(x_T, x_0\right) \middle| x_T < x_0, \Omega\right] \Pr\left(x_T < x_0\right) \\ &\quad + E\left[\max\left(x_T, x_0\right) \middle| x_T \geq x_0, \Omega\right] \Pr\left(x_T \geq x_0\right) \\ &= x_0 \Pr\left(x_T < x_0\right) + E\left(x_T \middle| x_T \geq x_0, \Omega\right) \Pr\left(x_T \geq x_0\right) \\ &= x_0 \Phi\left(z_0\right) + \sigma\sqrt{T} \frac{\phi\left(z_0\right)}{1 - \Phi\left(z_0\right)} \left[1 - \Phi\left(z_0\right)\right] \\ &= x_0 \Phi\left(z_0\right) + \sigma\sqrt{T}\phi\left(z_0\right) \end{aligned} \tag{5}$$

from the definition of conditional expectation, where $x_T \sim N\left(0, T\sigma^2\right)$ from Assumption 2.1 and

[17]It is negative for buying of foreign currency (also for the forward contract) because it means the outflow of domestic currency.

[18]Note $E(x) = \int_A x \frac{f(x)}{\Pr[A]} \mathrm{d}x \Pr[A] + \int_B x \frac{f(x)}{\Pr[B]} \mathrm{d}x \Pr[B] = E(x \mid A) + E(x \mid B)$ where $x \in A \cup B$.

$$E\left(x_T \middle| x_T \geq x_0, \Omega\right) = \sigma\sqrt{T}\frac{\phi(z_0)}{1-\Phi(z_0)} \tag{6}$$

for the third equality (5) from Greene ([6]: p. 759), and

$$\Pr(x_T \geq x_0) = \Pr\left(\frac{x_T}{\sigma\sqrt{T}} \geq \frac{x_0}{\sigma\sqrt{T}}\right) = \Pr(z_T \geq z_0) \equiv 1-\Phi(z_0) \tag{7}$$

where $z_T = x_T/\sigma\sqrt{T}$.

(b) The return's variance of (4) is defined as:

$$\begin{aligned} &E\left(\max(x_T, x_0) - E\left[\max(x_T, x_0)|\Omega\right]|\Omega\right)^2 \\ &= E\left(\left[\max(x_T, x_0)\right]^2|\Omega\right) - \left(E\left[\max(x_T, x_0)|\Omega\right]\right)^2 \end{aligned} \tag{8}$$

Note the second term of right hand side in (8) is derived from (5) directly. Then the first term of right hand side in (8) is arranged as:

$$\begin{aligned} E\left(\left[\max(x_T, x_0)\right]^2|\Omega\right) &= E\left(\left[\max(x_T, x_0)\right]^2 \middle| x_T < x_0, \Omega\right)\Pr(x_T < x_0) \\ &\quad + E\left(\left[\max(x_T, x_0)\right]^2 \middle| x_T \geq x_0, \Omega\right)\Pr(x_T \geq x_0) \\ &= x_0^2 \Pr(x_T < x_0) + E\left(x_T^2 \middle| x_T \geq x_0, \Omega\right)\Pr(x_T \geq x_0) \\ &= x_0^2\Phi(z_0) + E\left(x_T^2 \middle| x_T \geq x_0, \Omega\right)\left[1-\Phi(z_0)\right] \\ &= x_0^2\Phi(z_0) + T\sigma^2 E\left(z_T^2 \middle| z_T \geq z_0\right)\left[1-\Phi(z_0)\right] \end{aligned} \tag{9}$$

where

$$\begin{aligned} E\left(z_T^2 \middle| z_T \geq z_0\right) &= \frac{1-F_{3,0}(z_0^2)}{1-F_{1,0}(z_0^2)} \text{ if } z_0 \geq 0 \\ &= \frac{F_{3,0}(z_0^2)}{F_{1,0}(z_0^2)}\left[1-2\Phi(z_0)\right] + \frac{1-F_{3,0}(z_0^2)}{1-F_{1,0}(z_0^2)}\left[1-\Phi(-z_0)\right] \text{ if } z_0 < 0 \end{aligned}$$

because, for the second term in last equation in (9), we may show that

$$E\left(x_T^2 \middle| x_T \geq x_0\right) = T\sigma^2 E\left(z_T^2 \middle| z_T \geq z_0\right) \tag{10}$$

from

$$\begin{aligned} E\left(x_T^2 \middle| x_T \geq x_0\right) &= \int_{x_T \geq x_0} x_T^2 \frac{g(x_T)}{G(x_T \geq x_0)} \mathrm{d}x_T \\ &= T\sigma^2 \int_{z_T \geq z_0} z_T^2 \frac{g(\sigma\sqrt{T}z_T)}{G(z_T \geq z_0)} \sigma\sqrt{T}\mathrm{d}z_T \\ &= T\sigma^2 E\left(z_T^2 \middle| z_T \geq z_0\right) \end{aligned}$$

since $\frac{g(\sigma\sqrt{T}z_T)}{G(z_T \geq z_0)}\sigma\sqrt{T}$ is the truncated density function of variable z_T where

$$1 = \int_{x_T \geq x_0} \frac{g(x_T)}{G(x_T \geq x_0)} \mathrm{d}x_T = \int_{z_T \geq z_0} \frac{g(\sigma\sqrt{T}z_T)}{G(z_T \geq z_0)} \sigma\sqrt{T}\mathrm{d}z_T$$

from the change of variable formula where g and G denote the density and distribution functions of x_T respectively, and $\sigma\sqrt{T}\mathrm{d}z_T = \mathrm{d}x_T$ since $z_T = x_T/\sigma\sqrt{T}$

by definition. Note z_T has a standard normal, z_T^2 has a central $\chi^2_{(1)}$ distribution respectively.

and;

Case 1: if $z_0 \ge 0$, then

$$E\left(z_T^2 \middle| z_T \ge z_0\right) = E\left(z_T^2 \middle| z_T^2 \ge z_0^2\right) = \frac{1 - E\left(z_T^2 \middle| z_T^2 < z_0^2\right) F_{1,0}\left(z_0^2\right)}{1 - F_{1,0}\left(z_0^2\right)} \tag{11}$$

because

$$\begin{aligned} E\left(z_T^2 \middle| z_T^2 \ge z_0^2\right) &= E\left(z_T^2 \middle| z_T \ge z_0 \text{ or } z_T < -z_0\right) \\ &= E\left(z_T^2 \middle| z_T \ge z_0\right) \frac{\Pr\left[z_T > z_0\right]}{\Pr\left[z_T^2 \ge z_0^2\right]} + E\left(z_T^2 \middle| z_T < -z_0\right) \frac{\Pr\left[z_T < -z_0\right]}{\Pr\left[z_T^2 \ge z_0^2\right]} \\ &= E\left(z_T^2 \middle| z_T \ge z_0\right) \end{aligned} \tag{12}$$

from $\dfrac{\Pr\left[z_T \ge z_0\right]}{\Pr\left[z_T^2 \ge z_0^2\right]} = \dfrac{\Pr\left[z_T < -z_0\right]}{\Pr\left[z_T^2 \ge z_0^2\right]} = \dfrac{1}{2}$ and

$$E\left(z_T^2 \middle| z_T \ge z_0\right) = \int_{z_0}^{\infty} z_T^2 \phi\left(z_T\right) \mathrm{d}z_T = \int_{-\infty}^{-z_0} z_T^2 \phi\left(z_T\right) \mathrm{d}z_T = E\left(z_T^2 \middle| z_T < -z_0\right) \tag{13}$$

using the symmetry of normal distribution; and

$$E\left(z_T^2 \middle| z_T^2 \ge z_0^2\right) = \frac{1 - E\left(z_T^2 \middle| z_T^2 < z_0^2\right) F_{1,0}\left(z_0^2\right)}{1 - F_{1,0}\left(z_0^2\right)} \tag{14}$$

solving following equation for $E\left(z_T^2 \middle| z_T^2 \ge z_0^2\right)$

$$1 = E\left(z_T^2\right) = E\left(z_T^2 \middle| z_T^2 < z_0^2\right) F_{1,0}\left(z_0^2\right) + E\left(z_T^2 \middle| z_T^2 \ge z_0^2\right)\left[1 - F_{1,0}\left(z_0^2\right)\right]$$

for the final equality of (11). Further note

$$E\left(z_T^2 \middle| z_T^2 < z_0^2\right) = 2 \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{F_{3,0}\left(z_0^2\right) - F_{3,0}\left(0\right)}{F_{1,0}\left(z_0^2\right) - F_{1,0}\left(0\right)} = \frac{F_{3,0}\left(z_0^2\right)}{F_{1,0}\left(z_0^2\right)} \tag{15}$$

from Marchand ([10]: p. 26 and Remark 4), where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, where $h(0,1,0) = F_{1,0}\left(z_0^2\right) - F_{1,0}(0)$ and $h(0,3,0) = F_{3,0}\left(z_0^2\right) - F_{3,0}(0)$ in Marchand ([10]: p. 26 and Remark 4) where $F_{1,0}(0) = F_{3,0}(0) = 0$ with $p = 1$, $\alpha = 1$ and $\lambda = 0$ that is a non-centrality parameter.

Plugging (15) into (11) results in

$$E\left(z_T^2 \middle| z_T \ge z_0\right) = \frac{1 - \dfrac{F_{3,0}\left(z_0^2\right)}{F_{1,0}\left(z_0^2\right)} F_{1,0}\left(z_0^2\right)}{1 - F_{1,0}\left(z_0^2\right)} = \frac{1 - F_{3,0}\left(z_0^2\right)}{1 - F_{1,0}\left(z_0^2\right)}. \tag{16}$$

Case 2: $z_0 < 0$

$$\begin{aligned}
&E\left(z_T^2 \middle| z_T \geq z_0\right) \\
&= E\left(z_T^2 \middle| z_T \geq z_0, z_0 \leq z_T < -z_0\right)\Pr\left[z_0 \leq z_T < -z_0\right] \\
&\quad + E\left(z_T^2 \middle| z_T \geq z_0, -z_0 < z_T\right)\Pr\left[-z_0 < z_T\right] \\
&= E\left(z_T^2 \middle| z_T^2 < z_0^2\right)\left[1-2\Phi\left(z_0\right)\right] + E\left(z_T^2 \middle| -z_0 < z_T\right)\left[1-\Phi\left(-z_0\right)\right] \\
&= E\left(z_T^2 \middle| z_T^2 < z_0^2\right)\left[1-2\Phi\left(z_0\right)\right] + E\left(z_T^2 \middle| z_0^2 < z_T^2\right)\left[1-\Phi\left(-z_0\right)\right] \\
&= E\left(z_T^2 \middle| z_T^2 < z_0^2\right)\left[1-2\Phi\left(z_0\right)\right] + \frac{1-E\left(z_T^2 \middle| z_T^2 < z_0^2\right)F_{1,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)}\left[1-\Phi\left(-z_0\right)\right] \qquad (17) \\
&= \frac{F_{3,0}\left(z_0^2\right)}{F_{1,0}\left(z_0^2\right)}\left[1-2\Phi\left(z_0\right)\right] + \frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)}\left[1-\Phi\left(-z_0\right)\right]
\end{aligned}$$

from (12) for the third equality, from (14) for the fourth equality and from (16) for the final equality.

Consequently we get,

$$\begin{aligned}
V_p^2 &= E\left(\left[\max\left(x_T, x_0\right)\right]^2 \middle| \Omega\right) - \left(E\left[\max\left(x_T, x_0\right) \middle| \Omega\right]\right)^2 \\
&= x_0^2\Phi\left(z_0\right) + T\sigma^2 E\left(z_T^2 \middle| z_T \geq z_0\right)\left[1-\Phi\left(z_0\right)\right] - \left(x_0\Phi\left(z_0\right) + \sigma\sqrt{T}\phi\left(z_0\right)\right)^2
\end{aligned}$$

from (5) and (9). □

Proof of Theorem 2.5: Note the covariance between non-hedging and put option conditional on Ω is defined as:

$$\begin{aligned}
&E\left[\left(\max\left(x_T, x_0\right) - p - E\left[\max\left(x_T, x_0\right) - p \middle| \Omega\right]\right)x_T \middle| \Omega\right] \\
&= E\left[\left(\max\left(x_T, x_0\right) - E\left[\max\left(x_T, x_0\right) \middle| \Omega\right]\right)x_T \middle| \Omega\right] \\
&= E\left[\max\left(x_T, x_0\right)x_T \middle| \Omega\right] - E\left[\max\left(x_T, x_0\right) \middle| \Omega\right]E\left(x_T \middle| \Omega\right) \\
&= E\left[\max\left(x_T, x_0\right)x_T \middle| \Omega\right]
\end{aligned}$$

since $E\left[\max\left(x_T, x_0\right) \middle| \Omega\right]$ is constant conditional on Ω for the second equality and the fourth equality holds from $E\left(x_T \middle| \Omega\right) = 0$.

Now the claimed result is derived since

$$\begin{aligned}
&E\left[\max\left(x_T, x_0\right)x_T \middle| \Omega\right] \\
&= E\left[\max\left(x_T, x_0\right)x_T \middle| x_T < x_0, \Omega\right]\Pr\left(x_T < x_0\right) \\
&\quad + E\left[\max\left(x_T, x_0\right)x_T \middle| x_T \geq x_0, \Omega\right]\Pr\left(x_T \geq x_0\right) \\
&= x_0 E\left(x_T \middle| x_T < x_0, \Omega\right)\Pr\left(x_T < x_0\right) + E\left(x_T^2 \middle| x_T \geq x_0, \Omega\right)\Pr\left(x_T \geq x_0\right) \\
&= x_0 E\left(x_T \middle| x_T < x_0, \Omega\right)\Phi\left(z_0\right) + E\left(x_T^2 \middle| x_T \geq x_0, \Omega\right)\left[1-\Phi\left(z_0\right)\right] \\
&= -x_0\sigma\sqrt{T}\phi\left(z_0\right) + T\sigma^2 E\left(z_T^2 \middle| z_T \geq z_0\right)\left[1-\Phi\left(z_0\right)\right].
\end{aligned}$$

from (10) and

$$E\left(x_T \middle| x_T < x_0, \Omega\right) = -\sigma\sqrt{T}\frac{\phi\left(z_0\right)}{\Phi\left(z_0\right)} \qquad (18)$$

from Greene ([6]: p. 759) for the last two equations. □

Proof of Theorem 2.6: Note the expected return for forward is the value of following:

$$\frac{\Theta'\left(S_0 - F_T - C\right)}{\Theta' S_0} \cong s_0 - f_T - c \tag{19}$$

assuming $F_T - S_0$ is small. Its variance is obviously zero since the return is not random. □

Proof of Theorem 2.7: (a) Note the outflow of buying call option at time T is given as $\min\left(S_T, K\right) + P$. Thus its return normalized by S_0 is given as the negative value of following:

$$\begin{aligned} & -\frac{\Theta'\left[\min\left(S_T, K\right) + P - S_0\right]}{\Theta' S_0} \\ & = -\min\left(\frac{\Theta'\left[S_T - S_0\right]}{\Theta' S_0}, \frac{\Theta'\left[K - S_0\right]}{\Theta' S_0}\right) - \frac{\Theta' P}{\Theta' S_0} \\ & \cong -\min\left(s_T - s_0, k - s_0\right) - p \equiv -\min\left(x_T, x_0\right) - p \end{aligned} \tag{20}$$

assuming $S_T - S_0$ and $K - S_0$ are small.

Now the expected return conditional on Ω is value of following:

$$-E\left[\min\left(x_T, x_0\right) \middle| \Omega\right] - p = \sigma\sqrt{T}\phi\left(z_0\right) - x_0\left[1 - \Phi\left(z_0\right)\right] - p \tag{21}$$

from (17) where $x_T \equiv s_T - s_0$, since

$$\begin{aligned} E\left[\min\left(x_T, x_0\right) \middle| \Omega\right] &= E\left[\min\left(x_T, x_0\right) \middle| x_T < x_0, \Omega\right] \Pr\left(x_T < x_0\right) \\ & \quad + E\left[\min\left(x_T, x_0\right) \middle| x_T \geq x_0, \Omega\right] \Pr\left(x_T \geq x_0\right) \\ & = E\left(x_T \middle| x_T < x_0, \Omega\right) \Pr\left(x_T < x_0\right) + x_0 \Pr\left(x_T \geq x_0\right) \\ & = -\sigma\sqrt{T}\frac{\phi\left(z_0\right)}{\Phi\left(z_0\right)}\Phi\left(z_0\right) + x_0\left[1 - \Phi\left(z_0\right)\right] \\ & = -\sigma\sqrt{T}\phi\left(z_0\right) + x_0\left[1 - \Phi\left(z_0\right)\right] \end{aligned} \tag{22}$$

from the definition of conditional expectation, where $x_T \sim N\left(0, T\sigma^2\right)$ from Assumption 2.1 and (18)

and

$$\Pr\left(x_T < x_0\right) = \Pr\left(\frac{x_T}{\sigma\sqrt{T}} < \frac{x_0}{\sigma\sqrt{T}}\right) = \Pr\left(z_T < z_0\right) \equiv \Phi\left(z_0\right) \tag{23}$$

where $z_0 = x_0 / \sigma\sqrt{T}$ and $z_T = x_T / \sigma\sqrt{T}$.

(b) The return's variance of call option conditional on Ω is given as:

$$\begin{aligned} & E\left(\min\left(x_T, x_0\right) - E\left[\min\left(x_T, x_0\right) \middle| \Omega\right] \middle| \Omega\right)^2 \\ & = E\left(\left[\min\left(x_T, x_0\right)\right]^2 \middle| \Omega\right) - \left(E\left[\min\left(x_T, x_0\right) \middle| \Omega\right]\right)^2 \end{aligned} \tag{24}$$

Note the second term of right hand side in (24) is derived from (21) directly. Then the first term of right hand side in (24) is arranged as:

$$\begin{aligned} E\left(\left[\min\left(x_T,x_0\right)\right]^2|\Omega\right) &= E\left(\left[\min\left(x_T,x_0\right)\right]^2|x_T<x_0,\Omega\right)\Pr\left(x_T<x_0\right) \\ &\quad + E\left(\left[\min\left(x_T,x_0\right)\right]^2|x_T\ge x_0,\Omega\right)\Pr\left(x_T\ge x_0\right) \\ &= E\left(x_T^2|x_T<x_0,\Omega\right)\Pr\left(x_T<x_0\right)+x_0^2\Pr\left(x_T\ge x_0\right) \\ &= E\left(x_T^2|x_T<x_0,\Omega\right)\Phi\left(z_0\right)+x_0^2\left[1-\Phi\left(z_0\right)\right] \\ &= T\sigma^2E\left(z_T^2|z_T<z_0\right)\Phi\left(z_0\right)+x_0^2\left[1-\Phi\left(z_0\right)\right] \end{aligned} \quad (25)$$

where

$$\begin{aligned} E\left(z_T^2|z_T<z_0\right) &= \frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)} \text{ if } z_0<0 \\ &= \frac{F_{3,0}\left(z_0^2\right)}{F_{1,0}\left(z_0^2\right)}\left[1-2\Phi\left(-z_0\right)\right]+\frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)}\Phi\left(-z_0\right) \text{ if } z_0\ge 0 \end{aligned}$$

from $E\left(x_T^2|x_T<x_0\right)=T\sigma^2E\left(z_T^2|z_T<z_0\right)$ as similarly in (10) and

Case 1: $z_0<0$

$$E\left(z_T^2|z_T<z_0\right)=E\left(\left(-z_T\right)^2|-z_T\ge -z_0\right)=E\left(z_T^2|z_T^2\ge z_0^2\right)=\frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}\left(z_0^2\right)} \quad (26)$$

From symmetry and z_T has a standard normal distribution, from (12) for the second equality, from (11) and (16) for the final equality.

Case 2: $z_0\ge 0$

$$\begin{aligned} E\left(z_T^2|z_T<z_0\right) &= E\left(z_T^2|z_T<z_0,-z_0<z_T<z_0\right)\Pr\left[-z_0<z_T<z_0\right] \\ &\quad + E\left(z_T^2|z_T<z_0,z_T<-z_0\right)\Pr\left[z_T<-z_0\right] \\ &= E\left(z_T^2|z_T^2<z_0^2\right)\left[1-2\Phi\left(-z_0\right)\right]+E\left(z_T^2|z_T<-z_0\right)\Phi\left(-z_0\right) \\ &= \frac{F_{3,0}\left(z_0^2\right)}{F_{1,0}\left(z_0^2\right)}\left[1-2\Phi\left(-z_0\right)\right]+\frac{1-F_{3,0}\left(z_0^2\right)}{1-F_{1,0}(z_0^2)}\Phi\left(-z_0\right) \end{aligned} \quad (27)$$

from (15) and (16) for the final equality. Consequently, we get,

$$\begin{aligned} V_c^2 &= E\left(\left[\min\left(x_T,x_0\right)\right]^2|\Omega\right)-\left(E\left[\min\left(x_T,x_0\right)|\Omega\right]\right)^2 \\ &= T\sigma^2E\left(z_T^2|z_T<z_0\right)\Phi\left(z_0\right)+x_0^2\left[1-\Phi\left(z_0\right)\right]-\left(-\sigma\sqrt{T}\phi\left(z_0\right)+x_0\left[1-\Phi\left(z_0\right)\right]\right)^2 \end{aligned}$$

from (25) and (22). □

Proof of Theorem 2.8: Note the covariance between non-hedging and call option conditional on Ω is defined as:

$$\begin{aligned} & E\left[\left(-\min\left(x_T,x_0\right)-p-E\left[-\min\left(x_T,x_0\right)-p|\Omega\right]\right)\left(-x_T\right)|\Omega\right] \\ &= E\left[\left(\min\left(x_T,x_0\right)-E\left[\min\left(x_T,x_0\right)|\Omega\right]\right)x_T|\Omega\right] \\ &= E\left[\min\left(x_T,x_0\right)x_T|\Omega\right]-E\left[\min\left(x_T,x_0\right)|\Omega\right]E\left(x_T|\Omega\right) \\ &= E\left[\min\left(x_T,x_0\right)x_T|\Omega\right] \end{aligned}$$

since the fourth equality holds from $E\left(x_T|\Omega\right)=0$.

Now the claimed result is derived since

$$
\begin{aligned}
&E\left[\min\left(x_T,x_0\right)x_T\left|\Omega\right.\right] \\
&=E\left[\min\left(x_T,x_0\right)x_T\left|x_T<x_0,\Omega\right.\right]\Pr\left(x_T<x_0\right) \\
&\quad+E\left[\min\left(x_T,x_0\right)x_T\left|x_T\ge x_0,\Omega\right.\right]\Pr\left(x_T\ge x_0\right) \\
&=E\left(x_T^2\left|x_T<x_0,\Omega\right.\right)\Pr\left(x_T<x_0\right)+x_0E\left(x_T\left|x_T\ge x_0,\Omega\right.\right)\Pr\left(x_T\ge x_0\right) \\
&=E\left(x_T^2\left|x_T<x_0,\Omega\right.\right)\Phi\left(z_0\right)+x_0E\left(x_T\left|x_T\ge x_0,\Omega\right.\right)\left[1-\Phi\left(z_0\right)\right] \\
&=T\sigma^2E\left(z_T^2\left|z_T<z_0\right.\right)\Phi\left(z_0\right)+x_0\sigma\sqrt{T}\frac{\phi\left(z_0\right)}{1-\Phi\left(z_0\right)}\left[1-\Phi\left(z_0\right)\right] \\
&=T\sigma^2E\left(z_T^2\left|z_T<z_0\right.\right)\Phi\left(z_0\right)+x_0\sigma\sqrt{T}\phi\left(z_0\right)
\end{aligned}
$$

from (6) and from $E\left(x_T^2\left|x_T<x_0\right.\right)=T\sigma^2E\left(z_T^2\left|z_T<z_0\right.\right)$ as similarly in (10) for the last two equations. □

Proof of Theorem 4.1: To get such a solution point, we solve following three equations:

$$R=\left(1-w\right)R_p \tag{28}$$

$$V^2=w^2V_n^2+\left(1-w\right)^2V_p^2+2w\left(1-w\right)Cov_{pn} \tag{29}$$

and

$$V=\frac{R-\alpha}{\beta} \tag{30}$$

from the UML $R=\alpha+\beta\ V$. Note

$$w\equiv\frac{R-R_p}{-R_p}\ \text{ and }\ 1-w\equiv\frac{R}{R_p} \tag{31}$$

from (28). Therefore

$$\left(\frac{R-\alpha}{\beta}\right)^2=\left(\frac{R-R_p}{-R_p}\right)^2V_n^2+\left(\frac{R}{R_p}\right)^2V_p^2+2Cov_{pn}\left(\frac{R-R_p}{-R_p}\right)\left(\frac{R}{R_p}\right) \tag{32}$$

by plugging (30) and (31) into (29). Then we solve (32) as

$$\frac{R_p^{\ 2}}{\beta^2}\left(R-\alpha\right)^2=\left(R-R_p\right)^2V_n^2+R^2V_p^2-2Cov_{pn}\left(R-R_p\right)R.$$

or

$$
\begin{aligned}
&\left[\frac{R_p^2}{\beta^2}-V_n^2-V_p^2+2Cov_{pn}\right]R^2+2\left[-\alpha\frac{R_p^2}{\beta^2}+R_pV_n^2-Cov_{pn}R_p\right]R \\
&+\frac{\alpha^2R_p^2}{\beta^2}-R_p^2V_n^2=0
\end{aligned} \tag{33}
$$

which is the second order polynomial equation of unknown *R*.

Now solving (33) results in two roots

$$\bar{R}_1=\frac{-b+\sqrt{b^2-ac}}{a}\ \text{ and }\ \bar{R}_2=\frac{-b-\sqrt{b^2-ac}}{a}$$

assuming $b^2 - ac \geq 0$ where

$$a = \frac{R_p^2}{\beta^2} - V_n^2 - V_p^2 + 2Cov_{pn}, \quad b = -\alpha\frac{R_p^2}{\beta^2} + R_p V_n^2 - Cov_{pn} R_p, \quad c = \frac{\alpha^2 R_p^2}{\beta^2} - R_p^2 V_n^2.$$

Then we select $\overline{V}_i = \dfrac{\overline{R}_i - \alpha}{\beta}$ for $i = 1, 2$. □