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On Semi π-Regular Local Ring

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Abstract

A ring R is said to be a right (left) semi π-regular local ring if and only if for all a in R, either a or $(1-a)$ is a right (left) semi π-regular element. The purpose of this paper is to give some characterization and properties of semi π-regular local rings, and to study the relation between semi π-regular local rings and local rings. From the main results of this work: 1) Let R be a semi π-regular reduced ring. Then the idempotent associated element is unique. 2) Let R be a ring. Then R is a right semi π-regular local ring if and only if either $r(a^n)$ or $r((1-a)^n)$ is direct summand for all $a \in R$ and $n \in Z^+$. If R is a local ring with $r(a^n) \subseteq r(a)$ for all $a \in R$ and $n \in Z^+$, then R is a right semi π-regular local ring.

Subject Areas

Algebra

Keywords

Local, Ring, Semi π-Regular

1. Introduction

Throughout this paper, R will be an associative ring with identity. For $a \in R$, $r(a)$, $(l(a))$ denote the right (left) annihilator of *a*. A ring R is reduced if R contains, no non-zero nilpotent element.

A ring R is said to be Von Neumann regular (or just regular) if and only if for each *a* in *R*, there exists *b* in R such that $a = aba$ [1]. Following [2], a ring R is said to be right semi-regular if and only if for each *a* in *R*, there exists *b* in R such that $a = ab$ and $r(a) = r(b)$.

By extending the notion of a right semi π–regular ring to a right semi-regular ring is defined as follows:

A ring R is said to be right semi π-regular if and only if for each a in R, there exist positive integers n and b in R such that $a^n = a^n b$ and $r(a^n) = r(b)$ [3].

Following [4], a ring R is said to be π-regular if and only if for each a in R, there exist positive integers n and b in R such that $a^n = a^n b a^n$. A ring R is called a local ring, if it has exactly one maximal ideal [5].

A ring R is said to be a local semi-regular ring, if for all a in R, either a or $(1-a)$ is a semi-regular element [6].

We extend the notion of the local semi-regular ring to the semi π-regular local ring defined as follows:

A ring R is said to be a semi π-regular local ring, if for all a in R, either a or $(1-a)$ is a semi π-regular element.

Clearly that every π-regular ring is a semi π-regular local ring.

2. A Study of Some Characterization of Semi π-Regular Local Ring

In this section we give the definition of a semi π-regular local ring with some of its characterization and basic properties.

2.1. Definition

A ring R is said to be right (left) semi π-regular local ring if and only if for all a in R, either a or $(1-a)$ is right (left) semi π-regular element for every a in R.

Examples:

Let $(Z_2,+,\cdot)$ be a ring and let $G = \{g : g^2 = 1\}$ is cyclic group, then $Z_2 G = \{0,1,g,1+g\}$ is π-regular ring. Thus R is semi π-regular local ring.

Let R be the set of all matrix in Z_2 which is defined as:

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a,b,d \in Z_2 \right\}.$$

It easy to show that R is semi π-regular local ring.

2.2. Proposition

Let R be a right semi π-regular local ring. Then the associated elements are idempotents.

Proof:

Let $a \in R$, since R is right semi π-regular local ring. Then either a or $(1-a)$ is right semi π-regular element, that there exists b in R such that $a^n = a^n b$ and $r(a^n) = r(b)$, so $a^n(1-b) = 0$, gives $(1-b) \in r(a^n) = r(b)$. Thus $b(1-b) = 0$, which implies $b = b^2$. Now, if $(1-a)$ is right semi π-regular element, then there exists c in R such that $(1-a)^n = (1-a)^n c^c$ and $r((1-a)^n) = r(c)$. So $(1-a)^n(1-c) = 0$, thus $(1-c) \in r((1-a)^n) = r(c)$. Hence $c(1-c) = 0$ and therefore $c = c^2$.

In general the associated element is not unique. But the following proposition give the necessary condition to prove the associated element is unique.

2.3. Proposition

Let R be a right semi π-regular local reduced ring. Then the idempotent associated element is unique.

Proof:

Let $a \in R$, since R is right semi π-regular local ring. Then either a or $(1-a)$ is right semi π-regular element in R. If a is right semi π-regular element, then there exists $b \in R$ such that $a^n = a^n b$ and $r(a^n) = r(b)$. Assume that, there is an element $\bar{b}$ in R such that $a^n = a^n \bar{b}$ and $r(a^n) = r(\bar{b})$, which implies that $a^n(b-\bar{b}) = 0$, hence $(b-\bar{b}) \in r(a^n) = r(b) = r(\bar{b})$ and $\bar{b}(b-\bar{b}) = 0$, that is $b(b-\bar{b}) = 0$ and then $\bar{b}b = \bar{b}^2$, $b^2 = b\bar{b}$, which implies $\bar{b}b = \bar{b}$, $b = b\bar{b}$.

Since R is reduced ring, then $r(b) = l(b) = l(\bar{b})$. Hence $(b-\bar{b}) \in l(b) = l(\bar{b})$ and then $(b-\bar{b})b = 0$ and $(b-\bar{b})\bar{b} = 0$ which implies $b^2 = b\bar{b}$ and $b\bar{b} = \bar{b}^2$. Hence $b = \bar{b}b$ and $b\bar{b} = \bar{b}$, and therefore $b = \bar{b}b = b\bar{b} = \bar{b}$. Now, if $(1-a)$ is right semi π-regular element, then there exists an element $c \in R$ such that $(1-a)^n = (1-a)^n c$ and $r((1-a)^n) = r(c)$. Now, we assume that the associated element c is not unique.

Then, there exists $\bar{c} \in R$ such that $r((1-a)^n) = r(\bar{c})$, $(1-a)^n = (1-a)^n \bar{c}$, then $(1-a)^n c = (1-a)^n \bar{c}$ which implies that $(1-a)^n (c-\bar{c}) = 0$, that is $(c-\bar{c}) \in r((1-a)^n) = r(c) = r(\bar{c})$. Hence $c(c-\bar{c}) = 0$ and $\bar{c}(c-\bar{c}) = 0$, implies that $c^2 = c\bar{c}$ and $\bar{c}c = \bar{c}^2$, that is $c = c\bar{c}$ and $\bar{c}c = \bar{c}$. Since R is reduced ring, then $l(\bar{c}) = r(c) = l(c)$ and then $(c-\bar{c})c = 0$, $(c-\bar{c})\bar{c} = 0$, that is $c^2 = \bar{c}c$ and $c\bar{c} = \bar{c}^2$. Thus $c = \bar{c}c$ and $c\bar{c} = \bar{c}$. Therefore $c = \bar{c}c = c\bar{c} = \bar{c}$.

The following theorem give the condition to a semi π-regular local ring to be π-regular ring.

2.4. Theorem

Let R be a right semi π-regular local ring. Then any element $a \in R$ is π-regular if $Ra^n = Rb$ for any associated element b in R.

Proof:

Let $a \in R$ and R be a right semi π-regular local ring. Then either a or $(1-a)$ is right semi π-regular element in R. If a is right semi π-regular element in R, then there exists $b \in R$ such that $a^n = a^n b$ and $r(a^n) = r(b)$.

Now, assume that $Ra^n = Rb$. Then $ra^n = b$ and $ra^n \in Ra^n$, $b \in Rb$. Since b is idempotent element, then $b + (1-b) = 1$ and $ra^n + (1-b) = 1$, it follows that $a^n r^n a^n + a^n(1-b) = a^n$.

Thus $a^n ra^n = a^n$. Therefore a is π-regular element in R.

Now, if $(1-a)$ is right semi π-regular element, then there exists an element $c \in R$ such that: $(1-a)^n = (1-a)^n c$ and $r((1-a)^n) = r(c)$.

If $R(1-a)^n = Rc$, assume that $s(1-a^n) = c$, where $s(1-a) \in R(1-a)$, $c \in R$. Since c is idempotent element, then $c + (1-c) = 1$ and $S(1-a)^n + (1-c) = 1$, it follows that $(1-a)^n S(1-a)^n + (1-a)^n(1-c) = (1-a)^n$, that is $(1-a)^n S(1-a)^n + (1-a)^n - (1-a)^n c = (1-a)^n$.

Thus $(1-a)^n S(1-a)^n = (1-a)^n$. Therefore $(1-a)$ is π-regular element in *R*.

2.5. Proposition

The epimorphism image of right semi π-regular local ring is right semi π-regular local ring.

Proof:

Let $f : R \to \overline{R}$ be epimorphism homomorphism function from the ring π in to the ring $\overline{R}$, where R is right semi π-regular local ring and let $\overline{e}, y, \overline{1}$ be element s in $\overline{R}$. Then there exists elements $e, x, 1$ in R such that

$$f(e) = \overline{e}, f(x) = y, f(1) = \overline{1}.$$

Now, since R is right semi π-regular local ring, then either x or $(1-x)$ is right semi π-regular element, that is $x^n = x^n e$ and $r(x^n) = r(e)$. Then

$$y^n = (f(x))^n = f(x^n) = f(x^n e) = f(x^n) f(e) = y^n \overline{e}.$$

Now, to prove $r(y^n) = r(\overline{e})$. If $a \in r(y^n)$, then $y^n a = 0$, that is $(f(x))^n a = 0$, then $f(x^n) a = 0$, and $f^{-1} f(x^n) f^{-1}(a) = 0$, hence $x^n f^{-1}(a) = 0$.

Thus $f^{-1}(a) \in r(x^n) = r(e)$, that is $ef^{-1}(a) = 0$. Then $f(e) a = 0$, thus $\overline{e} a = 0$. Hence $a \in r(\overline{e})$. Therefore,

$$r(y^n) \subseteq r(\overline{e}) \tag{1}$$

Now, let $b \in r(\overline{e})$. Then $\overline{e} b = 0$, it follows that $y \overline{e} b = 0$ and then $y^n \overline{e} b = 0$.

Thus $y^n b = 0$ and hence $b \in r(y^n)$. Therefore

$$r(\overline{e}) \subseteq r(y^n) \tag{2}$$

from (1) and (2), we obtain $r(\overline{e}) = r(y^n)$.

Now, if $(1-x)$ is right semi π-regular element in *R*, then $(1-x)^n = (1-x)^n e$ and $r(1-x)^n = r(e)$.

Now, $f(1-x)^n = (f(1-x))^n = (f(1) + f(-x))^n = (f(1) - f(x))^n = (\overline{1} - y)^n$. Thus $(\overline{1} - y)^n = f(1-x)^n = f((1-x)^n e) = f(1-x)^n f(e) = (\overline{1} - y)^n \overline{e}$.

Now, to prove $r(\overline{1} - y)^n = r(\overline{e})$.

Let $c \in r(\overline{1} - y)^n$. Then $(\overline{1} - y)^n c = 0$. That is $(f(1) - f(x))^n c = 0$, then $(f(1-x))^n c = 0$ and $f(1-x)^n c = 0$. Then $(1-x)^n f^{-1}(c) = 0$ and hence $f^{-1}(c) \in r(1-x)^n = r(e)$, that is $ef^{-1}(c) = 0$, it follows that $f(e) c = 0$.

Hence $\overline{e} c = 0$, thus $c \in r(\overline{e})$. Therefore

$$r(\overline{1} - y)^n \subseteq r(\overline{e}) \tag{3}$$

Now, let $d \in r(\overline{e})$, implies to $\overline{e} d = 0$, hence $(\overline{1} - y)^n \overline{e} d = 0$, thus $(\overline{1} - y)^n d = 0$. Hence $d \in r(\overline{1} - y)^n$. Therefore

$$r(\overline{e}) \subseteq r(\overline{1} - y)^n \tag{4}$$

from (3) and (4) we obtain $r(\overline{e}) = r(\overline{1} - y)^n$, that is either y or $(\overline{1} - y)$ is right semi π-regular element in $\bar{R}$. Therefore $\bar{R}$ is right semi π-regular local ring.

2.6. Theorem

Let R be a ring. Then R is right semi π-regular local ring if and only if either $r(a^n)$ or $r((1-a)^n)$ is direct summand for all $a \in R$ and $n \in z^+$.

Proof:

Let $a \in R$ and $r(a^n)$ is direct summand. Then there exists an ideal $I \subset R$, such that $R = r(a^n) \oplus I$. Thus, there is $d \in r(a^n)$ and $b \in I$, such that $d + b = 1$ and hence $a^n d + a^n b = a^n$ and therefore $a^n b = a^n$. Now, to prove $r(a^n) = r(b)$, let $x \in r(a^n)$. Then $a^n x = 0$, that is $a^n bx = 0$ and $bx \in r(a^n)$. But $bx \in I$ and $r(a^n) \cap I = 0$. Then $bx = 0$ and $x \in r(b)$, hence

$$r(a^n) \subseteq r(b) \tag{5}$$

and by the same way we can prove

$$r(b) \subseteq r(a^n) \tag{6}$$

from (5) and (6) we obtain $r(a^n) = r(b)$. Therefore *a* is right semi π-regular element. Now, if $((1-a)^n) \in R$ and $r((1-a)^n)$ is direct summand.

Then, there exists an ideal $I \subset R$ such that, $R = r((1-a)^n) \oplus J$ and there exists $c \in J$ and $f \in r((1-a)^n)$, such that $1 = f + c$. Thus

$$(1-a)^n = (1-a)^n f + (1-a)^n c.$$

Therefore $(1-a)^n = (1-a)^n c$. Now, to prove $r((1-a)^n) = r(c)$.

Let $w \in r((1-a)^n)$. Then $(1-a)^n w = 0$ and hence $(1-a)^n cw = 0$

Thus, $cw \in r((1-a)^n)$. But $cw \in J$ and $I \cap r((1-a)^n) = 0$, then $cw = 0$ and therefore $w \in r(c)$, hence

$$r((1-a)^n) \subseteq r(c) \tag{7}$$

Now, let $z \in r(c)$. Then $cz = 0$ and hence $(1-a)^n cz = 0$, Thus $(1-a)^n z = 0$, therefore $z \in r((1-a)^n)$ and we have

$$r(c) \subseteq r((1-a)^n) \tag{8}$$

form (7) and (8) we obtain $r(c) = r((1-a)^n)$. Therefore $(1-a)^n$ is right semi π-regular element. That is R is right semi π-regular local ring.

Now, let R be aright semi π-regular local ring. Then either *a* or $(1-a)$ is right semi π-regular element in *R*. If *a* is right semi π-regular element, then there exists $b \in R$ and $n \in Z^+$ such that $a^n = a^n b$ and $r(a^n) = r(b)$.

Hence, $a^n(1-b) = 0$, that is $(1-b) \in r(a^n)$, then $1 = b + (1-b)$ and thus $R = bR + (1-b)R$. Therefore $R = bR + r(a^n)$.

Now, to prove $bR \cap r(a^n) = 0$, suppose that $x \in bR \cap r(a^n)$, then $x \in bR$ and $x \in r(a^n)$. Hence $x = br$ for some $r \in R$ and $ax = 0$, since $x \in r(a^n) = r(b)$, then $bx = 0$ and $b \cdot br = 0$, that is $br = 0$ [proposition 2.2]. Thus $x = 0$ and therefore $bR \cap r(a^n) = 0$, that is $r(a^n)$ is direct summand of *R*.

Now, if $(1-a)^n$ is right semi π-regular element, then there exists $c \in R$ such that $(1-a)^n = (1-a)^n c$ and $r\left((1-a)^n\right) = r(c)$. Since $(1-a)^n (1-c) = 0$, we have $(1-c) \in r\left((1-a)^n\right)$, and since $1 = c + (1-c)$.

Hence, $R = cR + (1-c)R$. Thus, $R = cR + r\left((1-a)^n\right)$.

Now, to prove $r\left((1-a)^n\right) \cap cR = 0$. Let $y \in r\left((1-a)^n\right) \cap cR$.

Then $y \in r\left((1-a)^n\right)$ and $y \in cR$, hence $(1-a)^n y = 0$ and $y = cr$ for some $r \in R$. Since $y \in r\left((1-a)^n\right) = r(c)$ then $cy = 0$ and $c \cdot cr = 0$.

Hence $cr = 0$ [proposition 2.2] and thus $y = cr$ and then $y = 0$.

That is $r(1-a)^n \cap cR = 0$. Therefore $r\left((1-a)^n\right)$ is direct summand of *R*.

Now, to give the relation between semi π-regular local ring and local ring.

2.7. Theorem

If *R* is local ring with $r(a^n) \subseteq r(a)$ for all $a \in R$ and $n \in z^+$, then *R* is right semi π-regular local ring.

Proof:

Let *R* be local ring. Then either *a* or $(1-a)$ is invertible element in *R* [6].

If *a* is invertible, then there exists an element *b* in *R* such that $ab = 1$, hence $aba = a$ and then $a^n ba = a^n$. Let $e = ba$. Then $a^n e = a^n$. To prove $(a^n) = r(e)$. Let $x \in r(a^n) \subseteq r(a)$. Then $x = 0$, it follows that $bax = 0$ and then $ex = 0$, that is $x \in r(e)$. Hence

$$r(a^n) \subseteq r(e) \tag{9}$$

Now, let $y \in r(e)$. Then $ey = 0$ and hence $a^n ey = 0$ that is $a^n y = 0$, thus $y \in r(a^n)$. Therefore

$$r(e) \subset r(a^n) \tag{10}$$

from (9) and (10) we obtain $r(a^n) = r(e)$. Hence *a* is right semi π-regular element in *R*. Now, if $(1-a)$ is invertible element in *R*, then there exists an element *c* in *R* such that $(1-a)c = 1$. That is $(1-a)c(1-a) = (1-a)$, it follows that $(1-a)^n c(1-a) = (1-a)^n$. let $d = c(1-a)$. Then $(1-a)^n d = (1-a)^n$. To prove $r\left((1-a)^n\right) = r(d)$, let $x \in r\left((1-a)^n \subseteq r(1-a)\right)$, then $(1-a)x = 0$ that is $c(1-a)x = 0$ and hence $x = 0$, and then $x \in r(a)$. Thus

$$r\left((1-a)^n\right) \subseteq r(a) \tag{11}$$

Now, let $y \in r(d)$, that is $dy = 0$ and hence $(1-a)^n dy = 0$, it follows that $(1-a)^n y = 0$, that is $y \in r(1-a)^n$. Hence

$$r(a) \subseteq r(1-a)^n \tag{12}$$

form (11) and (12) we have $r(1-a)^n = r(d)$. Thus $(1-a)$ is right semi π-regular element. Therefore *R* is right semi π-regular ring.

3. The Conclusion

From the study on characterization and properties of semi π-regular local rings, we obtain the following results:

1) Let R be a right semi π-regular local ring. Then the associated elements are idempotents.

2) Let R be a right semi π-regular local ring. Then the idempotent associated element is unique.

3) Let R be a right semi π-regular local ring. Then any element $a \in R$ is π-regular if $Ra^n = Rb$ for any associated element b in R.

4) The epimorphism image of right semi π-regular local ring is right semi π-regular local ring.

5) Let R be a ring. Then R is a right semi π-regular local ring if and only if either $r\left(a^n\right)$ or $r\left(\left(1-a\right)^n\right)$ is direct summand for all $a \in R$ and $n \in Z^+$.

If R is a local ring with $r\left(a^n\right) \subseteq r\left(a\right)$ for all $a \in R$ and $n \in Z^+$, then R is a right semi π-regular local ring.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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