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Comparative Study of the Adomian Decomposition Method and Alternating Direction Implicit (ADI) for the Resolution of the Problems of Advection-Diffusion-Reaction

André Bitsindou*, Joseph Bonazebi-Yindoula, Gabriel Bissanga

Laboratory of Numerical Analysis, Computer Science and Applications, University Marien NGOUABI, Brazzaville, Congo
Email: *bonayindoula@yahoo.fr, andre2bitsindou@outlook.fr, bissanga1@yahoo.fr

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Abstract

In this paper, we use the Adomian decomposition method (ADM), the finite differences method and the Alternating Direction Implicit method to estimate the advantages and the weakness of the above methods. For it, we make a numerical simulation of the different solutions constructed with these methods and compare the error investigated case.

Keywords

Adomian Decomposition Method, Finite Differences Method, Advection-Diffusion-Reaction Equations

1. Introduction

The advection-diffusion-reaction equation is a combination of the advection, diffusion and reaction equation. It describes physical phenomena, where the energy of the particles or other physical sizes is transferred in a physical system because of three processes: advection, diffusion and reaction. According to this definition, it follows that the equation of advection-diffusion contains a parabolic part (diffusion) and hyperbolic part (advection).

In the case of constant coefficient of diffusion and constant rate of flow, the equation in 1D can be written in the following form:

$$\frac{\partial u(x,t)}{\partial t}+\underbrace{\varepsilon\frac{\partial u(x,t)}{\partial x}}_{\text{Advection}}=\underbrace{\gamma\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)}_{\text{Diffusion}}+\underbrace{\beta u^m}_{\text{reaction}}+\underbrace{f(x,t)}_{\text{source}} \tag{1}$$

where: ε is the speed of transport, γ the coefficient of diffusion, β the chemical coefficient of reaction, $f(x,t)$ the source function and $m = 1,2,3,\cdots$.

The equations of advection-diffusion-reaction are used to describe the problems of transport (the transport of pollutants, flows in the conduits, the modeling of the air pollution, etc.) [1] [2] [3].

2. The Adomian Decomposition Method

Suppose that we need to solve the following equation:

$$Fu = f \tag{2}$$

in a real Hilbert space *H*, where $F : H \to H$ is a linear or a nonlinear operator, $f \in H$ and u is the unknown function. The principle of the ADM is based on the decomposition of the operator F in the following form [2] [4] [5] [6]

$$F = L + R + N \tag{3}$$

where $L + R$ is a linear part, N nonlinear operator.

We suppose that L is an invertible operator in the sense of Adomian with L^{-1} as inverse.

Using that decomposition, Equation (2) is equivalent to

$$u = \theta + L^{-1} f - L^{-1} Ru - L^{-1} Nu \tag{4}$$

where θ verifies $L\theta = 0$. (4) is called the Adomian's fundamental equation or Adomian's canonical form. We look for the solution of (2) in the following series expansion form $u = \sum_{n=0}^{+\infty} u_n$ and we consider that $Nu = \sum_{n=0}^{+\infty} A_n$ where A_n are special polynomials of variables $u_0, u_1, \cdots, u_n$ called Adomian polynomials and defined by [2],

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left[N \left(\sum_{i=0}^{+\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0,1,2,\cdots \tag{5}$$

where λ is a parameter used by "convenience". Thus Equation (4) can be rewritten as follows:

$$\sum_{n=0}^{+\infty} u_n = \theta + L^{-1} f - L^{-1} R \left(\sum_{n=0}^{+\infty} u_n \right) - L^{-1} \left(\sum_{n=0}^{+\infty} A_n \right) \tag{6}$$

We suppose that the series

$$\sum_{n=0}^{+\infty} u_n \text{ and } \sum_{n=0}^{+\infty} A_n$$

are convergent, and we obtain the following Adomian algorithm:

$$\begin{cases} u_0 = \theta + L^{-1} f \\ u_1 = -L^{-1}(Ru_0) - L^{-1} A_0 \\ \qquad \vdots \\ u_{n+1} = -L^{-1}(Ru_n) - L^{-1} A_n, \quad n \geq 0 \end{cases} \tag{7}$$

In practice it is often difficult to calculate all the terms of an Adomian series solution, so we approach the series solution by the truncated series: $u = \sum_{i=0}^{n} u_i$, where the choice of n depends on error requirements. If this series converges, the solution of (2) is:

$$u = \lim_{n\to+\infty} \sum_{i=0}^{n} u_i \tag{8}$$

3. Applications

3.1. Problem 1

Let's consider the following advection-diffusion: [7] [8] [9]

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \varepsilon \frac{\partial u(x,t)}{\partial x} = \lambda \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le 1, t \ge 0 \\ u(x,0) = g(x), \ 0 \le x \le 1 \\ \frac{\partial u(0,t)}{\partial x} = h(t), \quad t \ge 0 \\ u(0,t) = 0, \quad t \ge 0 \\ u(1,t) = 0, \quad t \ge 0 \end{cases} \tag{9}$$

with

$$g(x) = \mathrm{e}^{\frac{11}{50}x} \sin \pi x, \ h(t) = \pi \mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t}, \varepsilon = \frac{11}{50}, \lambda = \frac{1}{2} \tag{10}$$

3.1.1. Resolution by the Adomian Decomposition Method

The equation of state of the problem is:

$$\frac{\partial u(x,t)}{\partial t} + \frac{11}{50}\frac{\partial u(x,t)}{\partial x} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{11}$$

From (11), we have

$$u(x,t) = \mathrm{e}^{\frac{11x}{50}} \sin \pi x + \frac{1}{2}\int_0^t \frac{\partial^2 u(x,s)}{\partial x^2} \mathrm{d}s - \frac{11}{50}\int_0^t \frac{\partial u(x,s)}{\partial x} \mathrm{d}s \tag{12}$$

and

$$u(x,t) = u(0,t) + x\frac{\partial u(0,t)}{\partial x} + \int_0^x \left(\int_0^s \left(2\frac{\partial u(z,t)}{\partial t} + \frac{11}{25}\frac{\partial u(z,t)}{\partial x} \right) \mathrm{d}z \right) \mathrm{d}s \tag{13}$$

(13) is equivalent to

$$u(x,t) = x\pi \mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} + \int_0^x \left(\int_0^s \left(2\frac{\partial u(z,t)}{\partial t} + \frac{11}{25}\frac{\partial u(z,t)}{\partial x} \right) \mathrm{d}z \right) \mathrm{d}s \tag{14}$$

(12) and (14) give the following canonical form

$$\begin{aligned} u(x,t) = {} & \mathrm{e}^{\frac{11}{50}x-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \sin \pi x + \frac{1}{2}\mathrm{e}^{\frac{11}{50}x} \sin \pi x + \frac{1}{2} x\pi \mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \\ & - \mathrm{e}^{\frac{11}{50}x-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \sin \pi x + \frac{1}{4}\int_0^t \frac{\partial^2 u(x,s)}{\partial x^2} \mathrm{d}s - \frac{11}{100}\int_0^t \frac{\partial u(x,s)}{\partial x} \mathrm{d}s \\ & + \int_0^x \left(\int_0^s \frac{\partial u(z,t)}{\partial t} \mathrm{d}z \right) \mathrm{d}s + \frac{11}{50}\int_0^x \left(\int_0^s \frac{\partial u(z,t)}{\partial x} \mathrm{d}z \right) \mathrm{d}s \end{aligned} \tag{15}$$

From (16) one obtains the following Adomian algorithm:

$$\begin{cases} u_0(x,t) = \mathrm{e}^{\frac{11}{50}x - \left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \sin \pi x \\ u_1(x,t) = \frac{1}{2}\mathrm{e}^{\frac{11}{50}x} \sin \pi x + \frac{1}{2} x\pi \mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} - \mathrm{e}^{\frac{11}{50}x - \left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \sin \pi x \\ \qquad + \frac{1}{4}\int_0^t \frac{\partial^2 u_0(x,s)}{\partial x^2}\mathrm{d}s - \frac{11}{100}\int_0^t \frac{\partial u_0(x,s)}{\partial x}\mathrm{d}s \\ \qquad + \int_0^x \left(\int_0^s \frac{\partial u_0(z,t)}{\partial t}\mathrm{d}z\right)\mathrm{d}s + \frac{11}{50}\int_0^x \left(\int_0^s \frac{\partial u_0(z,t)}{\partial x}\mathrm{d}z\right)\mathrm{d}s \\ u_{n+1}(x,t) = \frac{1}{4}\int_0^t \frac{\partial^2 u_n(x,s)}{\partial x^2}\mathrm{d}s - \frac{11}{100}\int_0^t \frac{\partial u_n(x,s)}{\partial x}\mathrm{d}s + \int_0^x \left(\int_0^s \frac{\partial u_n(z,t)}{\partial t}\mathrm{d}z\right)\mathrm{d}s \\ \qquad + \frac{11}{50}\int_0^x \left(\int_0^s \frac{\partial u_n(z,t)}{\partial x}\mathrm{d}z\right)\mathrm{d}s, \forall n \geq 1 \end{cases} \tag{16}$$

Calculation of $u_1(x,t)$

$$\begin{aligned} u_1(x,t) = &\frac{1}{2}\mathrm{e}^{\frac{11}{50}x} \sin \pi x + \frac{1}{2} x\pi \mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} - \mathrm{e}^{\frac{11}{50}x - \left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \sin \pi x \\ &+ \frac{1}{4}\left(\int_0^t \frac{\partial^2 u_0(x,s)}{\partial x^2}\mathrm{d}s\right) - \frac{11}{100}\left(\int_0^t \frac{\partial u_0(x,s)}{\partial x}\mathrm{d}s\right) \\ &+ \int_0^x \left(\int_0^s \frac{\partial u_0(z,t)}{\partial t}\mathrm{d}z\right)\mathrm{d}s + \frac{11}{50}\int_0^x \left(\int_0^s \frac{\partial u_0(z,t)}{\partial x}\mathrm{d}z\right)\mathrm{d}s \end{aligned}$$

$$\begin{aligned} u_1(x,t) = &\frac{1}{2}\mathrm{e}^{\frac{11}{50}x} \sin \pi x + \frac{1}{2} x\pi \mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} - \mathrm{e}^{\frac{11}{50}x - \left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t} \sin \pi x \\ &- \frac{242\exp\left(\frac{11}{50}x - \frac{121}{5000}t - \frac{1}{2}\pi^2 t\right)}{10000\pi^2 + 484}\sin \pi x \\ &- \frac{2200\pi\exp\left(\frac{11}{50}x - \frac{121}{5000}t - \frac{1}{2}\pi^2 t\right)}{10000\pi^2 + 484}\cos \pi x \\ &+ \frac{5000\pi^2\exp\left(\frac{11}{50}x - \frac{121}{5000}t - \frac{1}{2}\pi^2 t\right)}{10000\pi^2 + 484}\sin \pi x \\ &+ \frac{242\mathrm{e}^{\frac{11}{50}x}}{10000\pi^2 + 484}\sin \pi x + \frac{2200\pi\mathrm{e}^{\frac{11}{50}x}}{10000\pi^2 + 484}\cos \pi x \\ &- \frac{5000\pi^2\mathrm{e}^{\frac{11}{50}x}}{10000\pi^2 + 484}\sin \pi x + \frac{11}{100}\left(\frac{1100}{2500\pi^2 + 121}\right)\mathrm{e}^{\frac{11}{50}x - \frac{121}{5000}t - \frac{1}{2}\pi^2 t} \sin \pi x \\ &+ \frac{11}{100}\left(\frac{5000\pi}{2500\pi^2 + 121}\right)\mathrm{e}^{\frac{11}{50}x - \frac{121}{5000}t - \frac{1}{2}\pi^2 t} \cos \pi x \\ &- \frac{11}{100}\left(\frac{1100}{2500\pi^2 + 121}\right)\mathrm{e}^{\frac{11}{50}x} \sin \pi x - \frac{11}{100}\left(\frac{5000\pi}{2500\pi^2 + 121}\right)\mathrm{e}^{\frac{11}{50}x} \cos \pi x \\ &- \frac{1100\pi\mathrm{e}^{-\frac{121}{5000}t - \frac{1}{2}\pi^2 t}}{242 + 5000\pi^2} - \frac{2500\pi^3 x\mathrm{e}^{-\frac{121}{5000}t - \frac{1}{2}\pi^2 t}}{242 + 5000\pi^2} - \frac{121\mathrm{e}^{\frac{11}{50}x - \frac{121}{5000}t - \frac{1}{2}\pi^2 t}}{242 + 5000\pi^2}(\sin \pi x) \end{aligned}$$

$$
\begin{aligned}
&-\frac{121\pi x\mathrm{e}^{-\frac{121}{5000}t-\frac{1}{2}\pi^2 t}}{242+5000\pi^2}+\frac{2500\pi^2\mathrm{e}^{\frac{11}{50}x+-\frac{121}{5000}t-\frac{1}{2}\pi^2 t}}{242+5000\pi^2}(\sin\pi x)\\
&+\frac{1100\pi\mathrm{e}^{\frac{11}{50}x+-\frac{121}{5000}t-\frac{1}{2}\pi^2 t}}{242+5000\pi^2}(\cos\pi x)+\frac{550\pi\mathrm{e}^{-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t}}{2500\pi^2+121}\\
&+\frac{121\exp\left(\frac{11}{50}x-\frac{121}{5000}t-\frac{1}{2}\pi^2 t\right)}{2500\pi^2+121}(\sin\pi x)\\
&-\frac{550\pi\exp\left(\frac{11}{50}x-\frac{121}{5000}t-\frac{1}{2}\pi^2 t\right)}{2500\pi^2+121}(\cos\pi x)
\end{aligned}
\tag{17}
$$

(16) gives us:

$$
\begin{aligned}
u_1(x,t)=&\left(-1+\frac{2500\pi^2}{2500\pi^2+121}+\frac{121}{2500\pi^2+121}-\frac{121}{2500\pi^2+121}\right.\\
&\left.+\frac{121}{2500\pi^2+121}\right)\mathrm{e}^{\frac{11}{50}x-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t}(\sin\pi x)+\left(\frac{1100\pi}{242+5000\pi^2}\right.\\
&\left.-\frac{1100\pi}{5000\pi^2+242}+\frac{550\pi}{2500\pi^2+121}-\frac{550\pi}{2500\pi^2+121}\right)\mathrm{e}^{\frac{11}{50}x-\frac{121}{5000}t-\frac{1}{2}\pi^2 t}(\cos\pi x)\\
&+\left(\frac{1}{2}\pi-\frac{1}{2}\pi\right)x\mathrm{e}^{-\frac{121}{5000}t-\frac{1}{2}\pi^2 t}+\left(\frac{1}{2}+\frac{121}{5000\pi^2+242}-\frac{2500\pi^2}{5000\pi^2+242}\right.\\
&\left.-\frac{242}{5000\pi^2+242}\right)\mathrm{e}^{\frac{11}{50}x}\sin\pi x+\left(\frac{550\pi}{2500\pi^2+121}-\frac{550\pi}{2500\pi^2+121}\right)\mathrm{e}^{\frac{11}{50}x}\cos\pi x\\
&+\left(\frac{550\pi}{2500\pi^2+121}-\frac{550\pi}{2500\pi^2+121}\right)\mathrm{e}^{-\frac{121}{5000}t-\frac{1}{2}\pi^2 t}=0
\end{aligned}
\tag{18}
$$

therefore

$$
\begin{cases}
u_0(x,t)=\mathrm{e}^{\frac{11}{50}x-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t}\sin\pi x\\
u_n(x,t)=0,\forall n\geq 1
\end{cases}
\tag{19}
$$

The exact solution of the problem (9) by the Adomian decomposition method is:

$$
u(x,t)=\mathrm{e}^{\frac{11}{50}x-\left(\frac{121}{5000}+\frac{1}{2}\pi^2\right)t}\sin\pi x
\tag{20}
$$

and we remark that $u(1,t)=0$.

3.1.2. Resolution by ADI Method

Grid of the field

$$
x_i=ih,\ h=\frac{1-0}{n}=\frac{1}{n}\ \text{with } i\in\{0,1,\cdots,n\}
\tag{21}
$$

$$
t_k=k\tau,\ k>0\ \text{ where } \tau=\frac{T}{\Delta_t}
\tag{22}
$$

let's note

$$u(ih,k\tau)=u_i^k \tag{23}$$

3.1.3. Semi-Discretization in Relation to the Space

Let's consider the following equation:

$$\frac{\partial u(x,t)}{\partial t}-\gamma\frac{\partial^2 u(x,t)}{\partial x^2}+\varepsilon\frac{\partial u(x,t)}{\partial x}=0 \tag{24}$$

Let's put

$$-\gamma\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}+\varepsilon\frac{\mathrm{d}u}{\mathrm{d}x}=f \tag{25}$$

where f is a function that depends on x

$$u(x_i+h)=u(x_i)+hu'(x_i)+\frac{h^2}{2}u''(x_i)+\frac{h^3}{6}u^{(3)}(x_i)+\frac{h^4}{24}u^{(4)}(x_i)+0(h^4) \tag{26}$$

and

$$u(x_i-h)=u(x_i)-hu'(x_i)+\frac{h^2}{2}u''(x_i)-\frac{h^3}{6}u^{(3)}(x_i)+\frac{h^4}{24}u^{(4)}(x_i)+0(h^4) \tag{27}$$

(26) and (27) give

$$\begin{cases} u(x_i+h)+u(x_i-h)-2u(x_i)=+h^2u''(x_i)+\dfrac{h^4}{12}u^{(4)}(x_i)+0(h^4) \\ u(x_i+h)-u(x_i-h)=2hu'(x_i)+\dfrac{h^3}{3}u^{(3)}(x_i)+0(h^4) \end{cases} \tag{28}$$

and

$$u(x_i-h)=u(x_i)-hu'(x_i)+\frac{h^2}{2}u''(x_i)-\frac{h^3}{6}u^{(3)}(x_i)+\frac{h^4}{24}u^{(4)}(x_i)+0(h^4) \tag{29}$$

$$\begin{cases} u(x_i+h)+u(x_i-h)-2u(x_i)=+h^2u''(x_i)+\dfrac{h^4}{12}u^{(4)}(x_i)+0(h^4) \\ u(x_i+h)-u(x_i-h)=2hu'(x_i)+\dfrac{h^3}{3}u^{(3)}(x_i)+0(h^4) \end{cases} \tag{30}$$

(28) is equivalent to

$$\begin{cases} \dfrac{u(x_i+h)-2u(x_i)+u(x_i-h)}{h^2}=u''(x_i)+\dfrac{h^2}{12}u^{(4)}(x_i)+0(h^4) \\ \dfrac{u(x_i+h)-u(x_i-h)}{2h}=u'(x_i)+\dfrac{h^2}{6}u^{(3)}(x_i)+0(h^4) \end{cases} \tag{31}$$

with

$$\begin{cases} \delta_x^2 u_i=\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2}+\dfrac{h^2}{12}\dfrac{\mathrm{d}^4 u}{\mathrm{d}x^4}+0(h^4) \\ \delta_x u_i=\dfrac{\mathrm{d}u}{\mathrm{d}x}+\dfrac{h^2}{6}\dfrac{\mathrm{d}^3 u}{\mathrm{d}x^3}+0(h^4) \end{cases} \tag{32}$$

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = \delta_x^2 u_i - \dfrac{h^2}{12}\dfrac{\mathrm{d}^4u}{\mathrm{d}x^4} + 0\left(h^4\right) \\ \dfrac{\mathrm{d}u}{\mathrm{d}x} = \delta_x u_i - \dfrac{h^2}{6}\dfrac{\mathrm{d}^3u}{\mathrm{d}x^3} + 0\left(h^4\right) \end{cases} \tag{33}$$

The discretisation of the Equation (24) is:

$$-\gamma\delta_x^2 u_i + \varepsilon\delta_x u_i - \frac{h^2}{12}\left(2\varepsilon\frac{\mathrm{d}^3u}{\mathrm{d}x^3} - \gamma\frac{\mathrm{d}^4u}{\mathrm{d}x^4}\right) = f_i + 0\left(h^4\right) \tag{34}$$

Calculation of $\dfrac{\mathrm{d}^3u}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^4u}{\mathrm{d}x^4}$.

From (32), we have

$$\begin{cases} -\gamma\dfrac{\mathrm{d}^3u}{\mathrm{d}x^3} + \varepsilon\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = \dfrac{\mathrm{d}f}{\mathrm{d}x} \\ -\gamma\dfrac{\mathrm{d}^4u}{\mathrm{d}x^4} + \varepsilon\dfrac{\mathrm{d}^3u}{\mathrm{d}x^3} = \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2} \end{cases} \tag{35}$$

who gives us

$$\begin{cases} \dfrac{\mathrm{d}^3u}{\mathrm{d}x^3} = \dfrac{1}{\gamma}\left(\varepsilon\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \dfrac{\mathrm{d}f}{\mathrm{d}x}\right) = \dfrac{\varepsilon}{\gamma}\delta_x^2 u_i - \dfrac{1}{\gamma}\delta_x f_i + 0\left(h^4\right) \\ \dfrac{\mathrm{d}^4u}{\mathrm{d}x^4} = \dfrac{\varepsilon^2}{\gamma^2}\delta_x^2 u_i - \dfrac{\varepsilon}{\gamma^2}\delta_x f_i - \dfrac{1}{\gamma}\left(\delta_x^2 f_i\right) + 0\left(h^4\right) \end{cases} \tag{36}$$

from (36) we obtain:

$$\begin{aligned} &-\gamma\delta_x^2 u_i + \varepsilon\delta_x u_i - \frac{h^2}{12}\left[2\varepsilon\left(\frac{\varepsilon}{\gamma}\delta_x^2 u_i - \frac{1}{\gamma}\delta_x f_i\right) - \gamma\left(\frac{\varepsilon^2}{\gamma^2}\delta_x^2 u_i - \frac{\varepsilon}{\gamma^2}\delta_x f_i - \frac{1}{\gamma}\left(\delta_x^2 f_i\right)\right)\right] \\ &= f_i + 0\left(h^4\right) \end{aligned} \tag{37}$$

that is equivalent to

$$\left[-\left(\gamma + \frac{h^2}{12}\frac{\varepsilon^2}{\gamma}\right)\delta_x^2 + \varepsilon\delta_x\right]u_i = \left[1 + \frac{h^2}{12}\left(\delta_x^2 - \frac{\varepsilon}{\gamma}\delta_x\right)\right]f_i + 0\left(h^4\right) \tag{38}$$

Let's note

$$\begin{cases} A_x = -\left(\gamma + \dfrac{h^2}{12}\dfrac{\varepsilon^2}{\gamma}\right)\delta_x^2 + \varepsilon\delta_x \\ L_x = 1 + \dfrac{h^2}{12}\left(\delta_x^2 - \dfrac{\varepsilon}{\gamma}\delta_x\right) \end{cases} \tag{39}$$

and we obtain

$$A_x u_i = L_x f_i + 0\left(h^4\right) \tag{40}$$

that is equivalent to

$$L_x^{-1} A_x u_i = f_i + 0\left(h^4\right) \tag{41}$$

Finally the semi-discretisation of Equation (24) is:

$$L_x^{-1}A_x u_i^k = -\frac{\partial u_i^k}{\partial t} + 0\left(h^4\right) \tag{42}$$

Let's note $v_i^k = \dfrac{\partial u_i^k}{\partial t}$

we have

$$L_x^{-1}A_x u_i^k = -v_i^k + 0\left(h^4\right) \tag{43}$$

One obtains the following diagram of the finite differences

$$\begin{aligned}&\left(\frac{1}{12}+\frac{h\varepsilon}{24\gamma}\right)v_{i-1}^k + \frac{5}{6}v_i^k + \left(\frac{1}{12}-\frac{h\varepsilon}{24\gamma}\right)v_{i+1}^k \\ &= \left(\frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h}\right)u_{i-1}^k + \left(-\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma}\right)u_i^k + \left(\frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h}\right)u_{i+1}^k\end{aligned} \tag{44}$$

In the matrix form (44) becomes

$$\begin{cases} AV(t) = BU(t) \\ U(0) = U_0 \end{cases} \tag{45}$$

that is equivalent to:

$$\begin{cases} A\dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = BU(t) \\ U(0) = U_0 \end{cases} \tag{46}$$

where

$$\begin{cases} U(t) = \left[u_1(t), u_2(t), \cdots, u_{n-1}(t)\right]^{\mathrm{T}} \\ U(0) = \left[g_1(x), g_2(x), \cdots, g_{n-1}(x)\right]^{\mathrm{T}} \end{cases} \tag{47}$$

Here A and B are the $n-1$ order tridiagonale matrixes of the following form:

$$A = \begin{pmatrix} \frac{5}{6} & \frac{1}{12}-\frac{h\varepsilon}{24\gamma} & 0 & \cdots & 0 & 0 \\ \frac{1}{12}+\frac{h\varepsilon}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{h\varepsilon}{24\gamma} & 0 & \cdots & 0 \\ 0 & \frac{1}{12}+\frac{h\varepsilon}{24\gamma} & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \frac{1}{12}-\frac{h\varepsilon}{24\gamma} & 0 \\ 0 & \vdots & \ddots & \frac{1}{12}+\frac{h\varepsilon}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{h\varepsilon}{24\gamma} \\ 0 & 0 & \cdots & 0 & \frac{1}{12}+\frac{h\varepsilon}{24\gamma} & \frac{5}{6} \end{pmatrix} \tag{48}$$

and

$$B=\begin{pmatrix} -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0 & \cdots & 0 & 0 \\ \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0 & \cdots & 0 \\ 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0 \\ 0 & \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} \\ 0 & 0 & \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} \end{pmatrix} \quad (49)$$

The numerical solution is represented for $h=\frac{1}{50}$.

In the following, we give the numerical simulation of the approximate solution, the exact solution and the error between these two solutions in three-dimensional space.

On Figure 1(1) and Figure 1(2) we have the respective curves of diffusion of the exact and of the solution approached.

Figure 1(3) gives us the error between the exact and approached solution.

On Figure 1(4) and Figure 1(5) we have the project of Figure 1(1) and Figure 1(2) on the plane. On Figure 1(7) and Figure 1(8) we have the respective consecutive curves at different instants on the plane.

4. Problem 2

Let's consider the following nonlinear diffusion-reaction problem: [10] [11] [12]

$$\begin{cases} \frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}-u^3(x,t),\ 0<x<1:,t>0 \\ u(x,0)=\sin(\pi x),\ t\geq 0 \\ u(0,t)=0 \\ u(1,t)=0 \end{cases} \quad (50)$$

4.1. Resolution by Adomian Decomposition Method

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}-u^3(x,t) \quad (51)$$

From (50) we have the following canonical form:

$$u(x,t)=\sin(\pi x)+\int_0^t \frac{\partial^2 u(x,s)}{\partial x^2}\mathrm{d}s-\int_0^t u^3(x,s)\mathrm{d}s \quad (52)$$

let's suppose

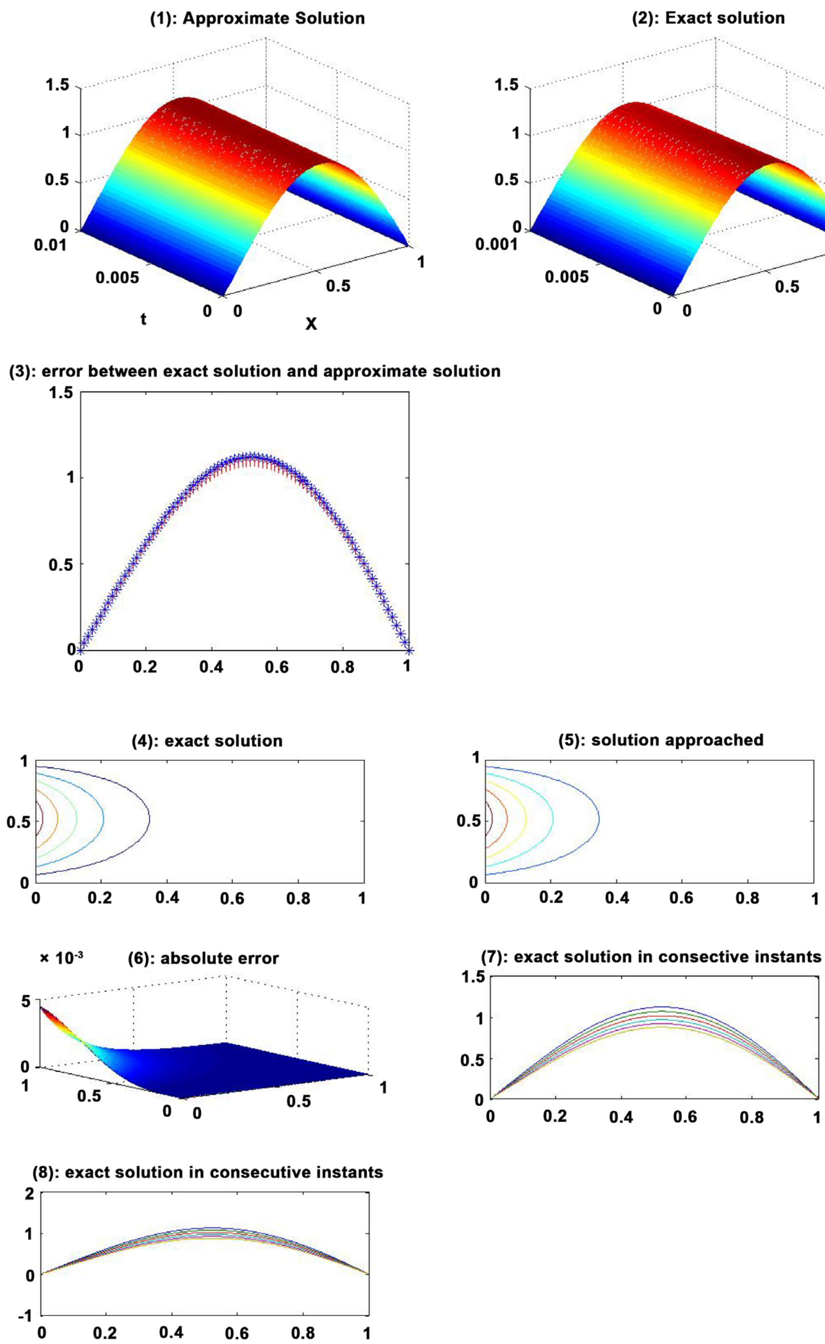

Figure 1. Simulation of problem 1.

$$\begin{cases} u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \\ u^3(x,t) = \sum_{n=0}^{\infty} A_n(x,t) \end{cases} \tag{53}$$

(51) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin(\pi x) + \sum_{n=0}^{\infty} \left(\int_0^t \frac{\partial^2 u_n(x,s)}{\partial x^2} \mathrm{d}s - \int_0^t A_n(x,t) \mathrm{d}s \right) \tag{54}$$

One obtains the following Adomian algorithm:

$$\begin{cases} u_0(x,t)=\sin(\pi x) \\ u_{n+1}(x,t)=\int_0^t \dfrac{\partial^2 u_n(x,s)}{\partial x^2}\mathrm{d}s-\int_0^t A_n(x,t)\mathrm{d}s, \forall n\geq 0 \end{cases} \tag{55}$$

where A_n are given by

$$\begin{cases} A_0(x,t)=u_0^3(x,t)=\sin^3(\pi x) \\ A_1(x,t)=3u_1(x,t)\left(u_0(x,t)\right)^2=\left(-3\pi^2\sin^3\pi x-3\sin^5\pi x\right)t \\ A_2(x,t)=3u_0u_1^2+3u_0^2u_2 \\ \qquad =6\left(\pi^4\sin^3\pi x+12\pi^2\sin^5\pi x+6\sin^7\pi x+3\pi^4\sin^2\pi x\right. \\ \qquad \quad \left.+18\pi^2\sin^4\pi x-18\pi^2\sin^2\pi x\cos^2\pi x+9\sin^6\pi x\right)\dfrac{t^2}{2} \\ \qquad \vdots \end{cases} \tag{56}$$

We get

$$\begin{cases} u_0(x,t)=\sin(\pi x) \\ u_1(x,t)=\left(-\pi^2\sin\pi x-\sin^3\pi x\right)t \\ u_2(x,t)=\left(\pi^4\sin\pi x+3\pi^2\sin^3\pi x-6\pi^2\cos^2\pi x\sin\pi x\right. \\ \qquad \qquad \left.+3\pi^2\sin^3\pi x+3\sin^3\pi x\right)\left(\dfrac{1}{2}t^2\right) \\ u_3(x,t)=\left(78\pi^2\cos^2\pi x\sin^3\pi x+78\pi^4\cos^2\pi x\sin\pi x-15\sin^7\pi x\right. \\ \qquad \qquad \left.-45\pi^2\sin^3\pi x-39\pi^4\sin^3\pi x-\pi^6\sin\pi x\right)\left(\dfrac{1}{6}t^3\right) \end{cases} \tag{57}$$

Thus the approximate solution of 50 is:

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots \\ &= \sin(\pi x)+\left(-\pi^2\sin\pi x-\sin^3\pi x\right)t+\left(\pi^4\sin\pi x+3\pi^2\sin^3\pi x\right. \\ &\quad \left.-6\pi^2\cos^2\pi x\sin\pi x+3\pi^2\sin^3\pi x+3\sin^3\pi x\right)\left(\frac{1}{2}t^2\right) \\ &\quad +\left(78\pi^2\cos^2\pi x\sin^3\pi x+78\pi^4\cos^2\pi x\sin\pi x-15\sin^7\pi x\right. \\ &\quad \left.-45\pi^2\sin^3\pi x-39\pi^4\sin^3\pi x-\pi^6\sin\pi x\right)\left(\frac{1}{6}t^3\right)+\cdots \end{aligned} \tag{58}$$

4.2. Resolution by the Finite Difference Method

Discretisation of the space

$$x_i=ih \text{ or } h=\frac{1}{N+1}, i=0,1,\cdots,N+1 \tag{59}$$

Let's note

$$u(x_i,t)=u_i \text{ and } \underline{u}=\left(u_1,u_2,\cdots,u_N\right)^{\mathrm{T}} \tag{60}$$

The discretised problem is:

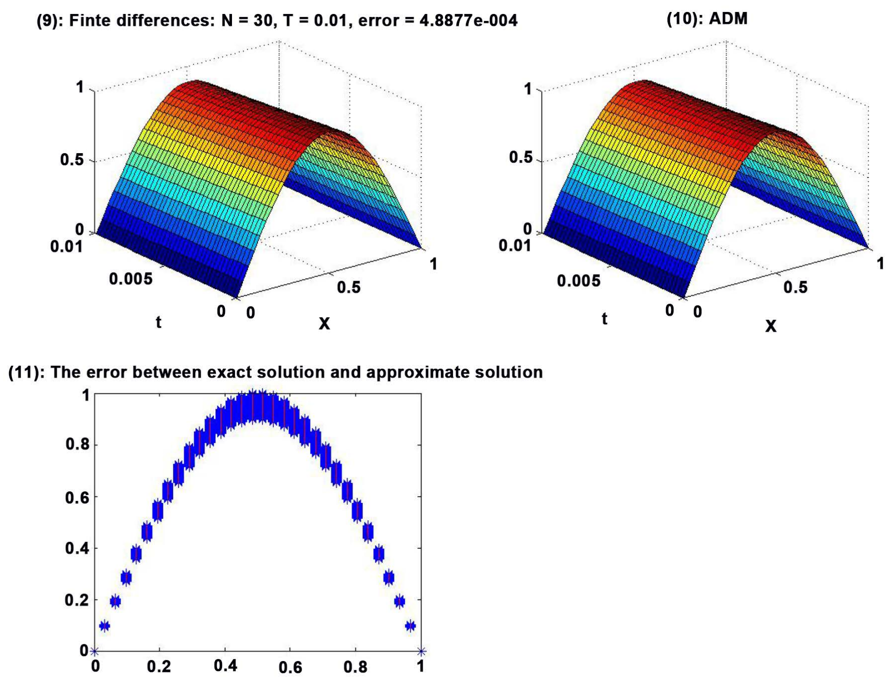

Figure 2. Simulation of problem 2.

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = A\underline{u} - \underline{u^3} \\ \underline{u^0} = \sin(\pi \underline{x}) \end{cases} \tag{61}$$

where A is the matrix of differentiation of the partial derivative of order two defined by:

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 & \\ & & & & 1 & -2 \end{pmatrix} \tag{62}$$

The method of Euler give us the following diagram of finite differences:

$$\begin{cases} \underline{u^{n+1}} = \underline{u^n} + \mathrm{d}t\left(A\underline{u^n} - \underline{\left(u^3\right)^n}\right) \\ \underline{u^0} = \sin(\pi \underline{x}) \end{cases} \tag{63}$$

4.2.1. Numerical Simulation

We choose

$$0 \le \frac{\Delta_t}{\Delta_x} \le \frac{1}{2}; N = 100, h = \frac{1}{N+1}, T_{\max} = 10^{-3}, \Delta_t = rh^2, r = \frac{1}{4} = \frac{\Delta_t}{\Delta_x} \tag{64}$$

We obtain **Figure 2**.

Here we have the numerical simulation of exact and ADM solution on three-dimensional space and the error between these solutions on plane point by point.

5. Conclusion

In this paper, two examples have been investigated. In the first example, we got the exact solution, using the ADM and the comparison has been done with the numerical solution obtained by ADI method. We find that the solution by the ADI method approaches the exact solution quite well, and the error is consisted between 0 and 0.005. In the second example, using the ADM, we got the approached solution; we remark that, the error between the solution gotten by the ADM and the one gotten by the finite differences method is very minimal.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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