

Optimal Investment-Reinsurance Strategies for Insurers with Mean-Reversion and Mispricing under Variance Premium Principle

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Abstract

This paper considers a robust optimal reinsurance-investment problem for an insurer with mispricing and model ambiguity. The surplus process is described by a classical Cramér-Lundberg model and the financial market contains a market index, a risk-free asset and a pair of mispriced stocks, where the expected return rate of the stocks and the mispricing follow mean reverting processes which take into account liquidity constraints. In particular, both the insurance and reinsurance premium are assumed to be calculated via the variance premium principle. By employing the dynamic programming approach, we derive the explicit optimal robust reinsurance-investment strategy and the optimal value function.

Keywords

Proportional Reinsurance, Robust Control, Optimal Investment Strategy, Utility Function, Mispricing

1. Introduction

Reinsurance and investment are the main tools for insurers to manage the risk and profit. Insurers can transfer the risk to reinsurers by purchasing reinsurance; meanwhile, they would invest their surplus into financial market to pursue extra profit. Recently, the problem of optimal reinsurance and investment has attracted great interest. For example, Schmidli [1], Bai and Guo [2] and Chen *et al.* [3] investigated the optimal investment-reinsurance strategies for insurers to minimize the ruin probability; Bäuerle [4], Bai and Zhang [5], Zeng and Li [6] studied the optimal reinsurance and investment strategies for insurers with mean-variance criteria; Maximizing the expected utility from terminal wealth

was investigated by many literature, see among Yang and Zhang [7], Wang [8], Xu *et al.* [9] and so on.

On the other hand, it is a common belief that there is no agreement on which model, or real world probability, should be used in practice. So model uncertainty exists widely in the field of portfolio selection. Thus, some scholars have advocated and investigated the effect of model uncertainty on portfolio selection. Robust decision making in the portfolio context is introduced by Maenhout [10]. A number of other papers are built on Maenhout [10] to address the implications of ambiguity on portfolio choice. Liu *et al.* [11] discussed the robust consumption and portfolio choice for time-varying investment opportunities. Yi *et al.* [12] focus on an optimal portfolio selection problem under SV model with model uncertainty. For more details, we refer the reader to Li *et al.* [13], Sun *et al.* [14] and references therein.

Theoretically, real markets are full of friction. Hence, there exists mispricing between a pair of assets. We can find vivid examples of mispricing in certain Chinese companies (such as Bank of China) traded on both Chinese stock exchanges as share A and Hong Kong stock exchanges as shares. Yi *et al.* [15] first consider a dynamic portfolio problem with mispricing and model ambiguity. Gu *et al.* [16] discuss optimal proportional reinsurance-investment problem for an insurer with mispricing and model ambiguity. In this paper, we seek the optimal proportional reinsurance-investment problem with mispricing under observed mean-reverting stochastic risk premium.

Under the criteria of maximizing the expected utility of terminal wealth, most of the literature mentioned above are based on expected value premium principle due to its simplicity and popularity in practice. Sun *et al.* [14] consider the optimal investment-reinsurance strategies under variance premium principle. Zeng *et al.* [17] and Gu *et al.* [18] investigate the optimal proportional reinsurance-investment problem for an insurer with mispricing, model ambiguity with mean-reversion under expected value premium principle. Motivated by these papers, both the insurance and reinsurance premium payments are calculated by using the variance premium principle in this paper.

The rest of the paper is organized as follows. In Section 2, we provide the financial market and the insurance model. In Section 3, the optimal robust proportional reinsurance-investment problem is established. Optimal proportional reinsurance-investment strategies and their corresponding optimal value functions are given in Section 4.

2. Economy and Assumption

In this section, we formulate a continuous-time financial model where the insurers can trade in the financial market and in the insurance market with no taxes or fees. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, 0 \leq t \leq T\}, P)$ be a probability space, in which Ω is the state space and \mathcal{F} is a σ -algebra on Ω . $T > 0$ is a fixed constant, representing the time horizon, $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration, which describes the flow

of information over time, the σ -algebra $\{\mathcal{F}_t\}$ describes the information available up to time t , and $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual condition (it contains all P-null sets and is right continuous). We denote P as a reference measure and suppose that all stochastic processes given in the following are assumed to be adapted on this space.

2.1. Surplus Process

We assume that the insurer's surplus is given by the classical Cramér-Lunderberg risk model (without reinsurance and investment), the insurer's surplus R is given by

$$dR(t) = cdt - d \sum_{i=1}^{N_t} Y_i + \sigma_1 dB(t), \tag{1}$$

where c is the premium rate, the claim arrival process $\{N_t\}_{t > 0}$ is a Poisson process with constant intensity $\lambda > 0$ and the random variables $Y_i, i = 1, 2, \dots$ are i.i.d claim sizes independent of N_t . We let $F(y)$ denote the claim size distribution with finite first-order moment m_1 and second-order moment m_2 . The stochastic process $\{B(t), 0 \leq t \leq T\}$ is a standard Brownian motion independent of N , representing the diffusion risk of the surplus process. The premium rate c is assumed to be calculated via the expected value principle, *i.e.*,

$$c = E_p \left[\sum_{i=1}^{N(t)} Y_i \right] + \theta_0 Var \left[\sum_{i=1}^{N(t)} Y_i \right] = \lambda (m_1 + \theta_0 m_2), \tag{2}$$

where θ_0 is the insurer's safety loading.

In this paper, the insurer can purchase proportional reinsurance or acquire new business to adjust the exposure to insurance risk. The proportional reinsurance/new business level is associated with the risk exposure $q(t)$ at time t . When $q(t) \in [0, 1]$, it means the insurer purchases proportional reinsurance. In this case, for each claim, the insurer only pays its $q(t)Y$, while the reinsurer pays the rest $(1 - q(t))Y$ for each claim. When $q(t) \in [1, \infty]$, it means the insurer acquires new business from the other insurers as a reinsurer. Then, the aggregate reinsurance premium under the variance principle takes the form

$$E_p \left[\sum_{i=1}^{N(t)} (Y_i - qY_i) \right] + \theta_1 Var \left[\sum_{i=1}^{N(t)} (Y_i - qY_i) \right] = \lambda \left[(1 - q)m_1 + \theta_1 (1 - q)^2 m_2 \right] t, \tag{3}$$

with $\theta_1 > 0$. Thus, the surplus process of the insurer after taking into account reinsurance is governed by

$$\begin{aligned} dR(t) &= \left[q(t)m_1 + \left[\theta_0 - \theta_1 (1 - q(t))^2 \right] m_2 \right] \lambda dt - q(t) d \sum_{i=1}^{N_t} Y_i \\ &= \left[q(t)m_1 + \left[\theta_0 - \theta_1 (1 - q(t))^2 \right] m_2 \right] \lambda dt - q(t) \int_{R^+} y(t) N(dt, dy), \end{aligned} \tag{4}$$

where $N(\cdot, \cdot)$ defined on $\Omega \times [0, T] \times [0, \infty)$ is a Poisson random measures used to represent the compound Poisson process $\sum_{i=1}^{N(t)} Y_i$ as

$$\sum_{i=1}^{N(t)} Y_i = \int_0^t \int_{R^+} y N(dt, dy). \tag{5}$$

If we denote $\nu(dt, dy) = \lambda dt dF(y)$, then

$$E \left[\sum_{i=1}^{N(t)} Y_i \right] = \int_0^t \int_{R^+} y \nu(dt, dy), \tag{6}$$

and $\nu(dt, dy)$ is the compensator of the random measure $N(\cdot, \cdot)$. Thus, the compensated measures $\tilde{N}(\cdot, \cdot) = N(\cdot, \cdot) - \nu(\cdot, \cdot)$ is related to the compound Poisson process $\sum_{i=1}^{N(t)} Y_i$ as follows

$$\int_0^t \int_{R^+} y \tilde{N}(dt, dy) = \sum_{i=1}^{N(t)} Y_i - E \left[\sum_{i=1}^{N(t)} Y_i \right], \quad \forall t \in [0, T]. \tag{7}$$

2.2. Financial Market

We assume that financial market consists of one risk-free asset, and a pair of stocks with mispricing (two price processes for the same asset on two distinct stock markets). We assume for the moment that the price of risk-free asset P_0 is given by

$$dP_0(t) = rP_0(t)dt, \quad P_0(0) = p_0, \tag{8}$$

where constant $r > 0$ is the risk-free interest rate. The price process of the market index is expressed as

$$\frac{dP_m(t)}{P_m(t)} = (r + \mu_m)dt + \sigma_m dZ_m(t), \tag{9}$$

where the market risk premium μ_m , the market volatility σ_m are positive constants, and $Z_m(t)$ is a standard Brownian motion on (Ω, \mathcal{F}, P) . The two mispriced price processes are modeled as a pair of stocks P_1 and P_2 which are coupled via the pricing error

$$X(t) := \ln \frac{P_1(t)}{P_2(t)}. \tag{10}$$

We assume the vector (P_1, P_2) solves the system of stochastic differential equations

$$\frac{dP_1(t)}{P_1(t)} = (r + a(t))dt + \sigma dZ(t) + b dZ_1(t) - \lambda_1 X(t)dt, \quad P_1(0) = p_{10}, \tag{11}$$

$$\frac{dP_2(t)}{P_2(t)} = (r + a(t))dt + \sigma dZ(t) + b dZ_2(t) + \lambda_2 X(t)dt, \quad P_2(0) = p_{20}, \tag{12}$$

in which $\lambda_1, \lambda_2, \sigma, b$ are constant parameters. And we assume the premium for the common risk part $\sigma dZ(t)$ is the observed mean-reverting process $a(t)$ whose dynamics are given by

$$da(t) = \kappa(\theta - a(t))dt + \sigma_a dZ_a(t), \quad a(0) = a_0, \tag{13}$$

where κ is the long-run mean of the risk premium, and θ is the degree of

mean reversion (or mean-reversion rate). The term $\lambda_i X(t)dt$ shows the effect of mispricing on the i th stock's price. The term $\sigma dZ(t) + bdZ_i(t)$ describe the idiosyncratic risk of stock i , $\sigma dZ(t)$ describes the common risk, while the individual risk $bdZ_i(t)$ is generated by the asset i , $i = 1, 2$. The term $\lambda_i X(t)dt$ shows the effect of mispricing on i th stock's price via the pricing error $X(t)$ defined above. We assume that all standard Brownian motions $Z(t)$, $Z_1(t)$, $Z_2(t)$ and $Z_a(t)$ are independent of each other and all are independent of $N(dt, dy)$. Based on (11) and (12), ordinary stochastic calculus implies that the dynamics of the pricing error $X(t)$ is given by the following equation

$$\begin{aligned} dX(t) &= -(\lambda_1 + \lambda_2)X(t)dt + bdZ_1(t) - bdZ_2(t), \quad X(0) = x_0. \\ dX(t) &= -(\lambda_1 + \lambda_2)X(t)dt + bdZ_1(t) - bdZ_2(t), \quad X(0) = x_0. \end{aligned} \tag{14}$$

Thus Equation (14) shows that X is also a MR process. Note that ρ is the correlation coefficient between the Brownian motions $Z(t)$ and $B(t)$, and ρ_0 is the correlation coefficient between $Z_m(t)$ and $Z_a(t)$. Moreover, there exist Brownian motions $\hat{B}(t)$ and $Z_0(t)$ satisfying the following equations, where. Moreover, we assume all standard Brownian motions $\{Z_m(t)\}$, $\{B(t)\}$, $\{\hat{B}(t)\}$, $\{Z_1(t)\}$, $\{Z_2(t)\}$ and $\{Z_0(t)\}$ are independent each other and all are independent of $N(dy, dt)$. To capture the features of mispricing, we assume $\lambda_1 + \lambda_2 > 0$.

In addition to reinsurance, the insurer can invest in our financial market. Thus we denote by $u(t) = (\ell_m(t), \ell_1(t), \ell_2(t))$ the corresponding investment strategy, where $\ell_m(t)$ shows the amount of the wealth invested in the market index, $\ell_1(t)$, and $\ell_2(t)$ denote the amounts invested in the two stocks, respectively, hence the remainder, $R(t) - \ell_m(t) - \ell_1(t) - \ell_2(t)$ is invested in the risk-free asset.

2.3. Wealth Process

We denote the whole reinsurance-investment strategy by

$\ell := \{\ell(t), 0 \leq t \leq T\} = \{q(t), u(t), t \in [0, T]\}$. As a result of adopting the strategy ℓ , the insurer's corresponding reserve $\{W^\ell(t)\}_{0 \leq t \leq T}$ satisfies the following stochastic dynamics:

$$\begin{aligned} & dW^\ell(t) \\ &= \left[W^\ell(t)r + \ell_m(t)\mu_m + a(t)(\ell_1(t) + \ell_2(t)) + X(t)(\lambda_2\ell_2(t) - \lambda_1\ell_1(t)) \right] dt \\ &+ \sigma_1 dB(t) + \sigma(\ell_1(t) + \ell_2(t))(\rho dB(t) + \hat{\rho}d\hat{B}(t)) + b\ell_1(t)dZ_1(t) \tag{15} \\ &+ b\ell_2(t)dZ_2(t) + \ell_m(t)\sigma_m dZ_m(t) + \left[q(t)m_1 + \left[\theta_0 - \theta_1(1 - q(t))^2 \right] m_2 \right] \lambda dt \\ &- q(t) \int_{R^+} y(t) N(dt, dy), \end{aligned}$$

where $W^\ell(0) = w_0$ and w_0 is the insurer's initial wealth.

3. Robust Problem with Mispricing

To derive explicit results, we need to make some assumptions regarding the am-

biguity-averse insurer's utility. Suppose that the insurer has exponential utility function, *i.e.*

$$U(x) = -\frac{1}{\gamma} \exp\{-\gamma x\}, \tag{16}$$

where $\gamma > 0$ is a constant representing the coefficient of absolute risk aversion. The insurer is assumed to be ambiguity-neutral with objective function

$$\max_{\ell \in \Pi_0} E_0^P [U(W^\ell(T))] = \max_{\ell \in \Pi_0} E_0^P \left[-\frac{1}{\gamma} \exp\{-\gamma W^\ell(T)\} \right]. \tag{17}$$

where Π_0 is the set of admissible strategies for an ambiguity-neutral insurer in a given market and $E_t^P[\cdot] = E^P[\cdot | \mathcal{F}_t]$ represents the conditional expectation under the probability measure P .

Definition 3.1. A strategy $\ell = \{q(t), \ell_m(t), \ell_1(t), \ell_2(t)\}_{t \in [0, T]}$ is said to be admissible, if

- 1) $\forall t \in [0, T], q(t) \in [0, \infty)$;
- 2) ℓ is predictable with respect to \mathcal{F}_t and $E^{Q^*} \left[\int_0^T \|v(t)\|^2 dt \right] < \infty$, where $\|v(t)\|^2 = q(t)^2 + \ell_m^2(t) + \ell_1^2(t) + \ell_2^2(t)$;
- 3) $\forall (t, w, x, a) \in [0, T] \times R \times R \times R$, Equation (5) has a unique solution $\{W^\ell(t)\}_{t \in [0, T]}$ with $E_{t, w, x, a}^{Q^*} [U(W^\ell(t))] < \infty$, where Q^* is the chosen model to describe the worst case and $E_{t, w}^{Q^*}[\cdot] = E^{Q^*}[\cdot | W^\ell(t) = w]$.

Denote by ℓ the set of all admissible strategies.

In order to incorporate the ambiguity, we assume that the insurer does not have full confidence in the reference probability P . She uses the probability P as her reference probability for the wealth process, but takes some alternative models at the same time. Every alternative model is characterized by another stochastic process and the associated probability measure Q , which is equivalent to the reference measure P . We denote this class of probability measures by \mathcal{Q} : $\mathcal{Q} := \{Q : Q \sim P\}$.

The change of measure from P to Q can be defined by its Randon-Nikodym derivative, *i.e.*, there exists a progressively measurable process $\epsilon = (\varphi(s), \phi(t))$, such that every Q in \mathcal{Q} satisfies

$$\frac{dQ}{dP} = \xi(T) \tag{18}$$

where

$$\begin{aligned} \xi(t) = \exp \left\{ -\int_0^t \varphi(s) dZ(s) - \frac{1}{2} \int_0^t \|\varphi(s)\|^2 ds - \int_0^t h(s) dB(s) - \frac{1}{2} \int_0^t h^2(s) ds \right. \\ \left. + \int_0^t \int_0^\infty \ln \phi(s) N(ds, dy) + \int_0^t \int_0^\infty (1 - \phi(s)) \nu(dy, dt) \right\}, \end{aligned} \tag{19}$$

where $\varphi(t) = (h_m(t), \hat{h}(t), h_1(t), h_2(t), h_0(t))$, $\|\varphi(t)\|^2 = h_m^2(t) + \hat{h}^2(t) + h_1^2(t) + h_2^2(t) + h_0^2(t)$ and

$dZ(t) = (dZ_m(t), d\hat{B}(t), dZ_1(t), dZ_2(t), dZ_0(t))$. According to Girsanov's theorem (Øksendal [19]), under the alternative measure Q , the stochastic process $\{Z_0^Q(t)\}, \{Z_1^Q(t)\}, \{Z_2^Q(t)\}, \{Z_m^Q(t)\}, \{B^Q(t)\}$ and $\{B^Q(t)\}$ are standard Brownian motions, where

$$\begin{aligned} dZ_0^Q(t) &= dZ_0(t) + h_0(t)dt, & dZ_1^Q(t) &= dZ_1(t) + h_1(t)dt, \\ dZ_2^Q(t) &= dZ_2(t) + h_2(t)dt, & dZ_m^Q(t) &= dZ_m(t) + h_m(t)dt, \\ dB^Q(t) &= dB(t) + h(t)dt, & d\hat{B}^Q(t) &= d\hat{B}(t) + \hat{h}(t)dt. \end{aligned} \tag{20}$$

Moreover, the intensity of the Poisson process becomes $\lambda\phi(t)$, that is,

$$\tilde{N}^Q(dt, dy) = N(dt, dy) - \lambda^Q dF(y)dt = N(dt, dy) - \lambda\phi(t)dF(y)dt, \tag{21}$$

is a martingale. For tractability and ease of interpretation, the distribution of the claim Y is assumed to be known, and is restricted to be identical under P and Q . Thus, the dynamics of the wealth process under Q is

$$\begin{aligned} dW^\ell(t) &= \left[W^\ell(t)r + \ell_m(t)\mu_m + a(t)(\ell_1(t) + \ell_2(t)) + X(t)(\lambda_2\ell_2(t) - \lambda_1\ell_1(t)) \right. \\ &\quad \left. - \ell_m(t)\sigma_m h_m(t) - \sigma(\ell_1(t) + \ell_2(t))(\rho h(t) + \hat{\rho}\hat{h}(t)) - b(\ell_1(t)h_1(t) \right. \\ &\quad \left. + \ell_2(t)h_2(t)) \right] dt - \sigma_1 h(t)dt + \ell_m \sigma_m dZ_m^Q(t) + \ell_1(t) b dZ_1^Q(t) \\ &\quad + \ell_2(t) b dZ_2^Q(t) + \sigma_1 dB^Q(t) + \sigma(\ell_1(t) + \ell_2(t))(\rho dB(t) + \hat{\rho}d\hat{B}(t)) \\ &\quad + \left[q(t)m_1 + \left[\theta_0 - \theta_1(1 - q(t))^2 \right] m_2 \right] \lambda dt \\ &\quad - q(t) \int_{R^+} \tilde{N}^Q(dt, dy) + \lambda\phi(t)dF(y)dt, \end{aligned} \tag{22}$$

Meanwhile, the dynamics of the mispricing error under Q can be given by

$$dX(t) = \left[-(\lambda_1 + \lambda_2)X(t) - bh_1(t) + bh_2(t) \right] dt + b dZ_1^Q(t) - b dZ_2^Q(t), \tag{23}$$

and (13) can be modified to

$$\begin{aligned} da(t) &= \left[\kappa(\theta - a(t)) - h_m(t)\rho_0\sigma_a - h_0(t)\hat{\rho}_0\sigma_a \right] dt \\ &\quad + \sigma_a \rho_0 dZ_m^Q(t) + \hat{\rho}_0 \sigma_a dZ_0^Q(t). \end{aligned} \tag{24}$$

Following from Maenhout [10], Gu *et al.* [16] and Zeng *et al.* [20], we show that the increase in relative entropy from t to $t + dt$ equals

$$\begin{aligned} &\frac{1}{2} \left(h_m^2(t) + h^2(t) + \hat{h}^2(t) + h_1^2(t) + h_2^2(t) + h_0^2(t) \right) dt \\ &+ \lambda \left(\phi(t) \ln \phi(t) - \phi(t) + 1 \right) dt. \end{aligned} \tag{25}$$

Denote that

$$J(t, w, x, a) = \sup_{t \in \mathbb{T}} \inf_{Q \in \mathcal{Q}} E_{t, w, x, a}^Q \left\{ -\frac{1}{\gamma} e^{-\gamma w(t)} + \int_t^T \left(\frac{R_1(s)}{\psi_1(s)} + \frac{R_2(s)}{\psi_2(s)} + \frac{R_3(s)}{\psi_3(s)} \right) ds \right\}, \tag{26}$$

where $R_1(t) = \varphi^2(s) = \frac{1}{2} \left(h_m^2(t) + \hat{h}^2(t) + h_1^2(t) + h_2^2(t) + h_0^2(t) \right)$, $R_2(t) = \frac{1}{2} h^2(t)$, $R_3(t) = \lambda \left(\phi(t) \ln \phi(t) - \phi(t) + 1 \right)$. For convenience, similar to Maenhout [10] and Gu *et al.* [16], we assume that $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are non-negative and state-dependent functions which are inversely proportional to the value

function:

$$\psi_1(s) = -\frac{\alpha_1}{\gamma J(s, w, x, a)}, \psi_2(s) = -\frac{\alpha_2}{\gamma J(s, w, x, a)}, \psi_3(s) = -\frac{\alpha_3}{\gamma J(s, w, x, a)}, \quad (27)$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\alpha_3 \geq 0$ represent the insurer's ambiguity aversion levels to the diffusion modeling and jump modeling risk.

Next we aim to derive the explicit solution to the HJB Equation (22) with preference parameter (27).

4. Optimal Robust Investment and Reinsurance Strategy

The purpose of this section is to find the optimal investment strategy $(\pi_m(t), \pi_1(t), \pi_2(t))$ and the optimal proportional reinsurance strategy $q(t)$ under the worst-case scenario. According to the principle of dynamic programming, the robust Hamilton-Jacobi-Bellmann (HJB) equation established by Anderson *et al.* [21] to express the value function (26), can be derived as

$$\begin{aligned} & \sup_{\ell \in \Pi(\varphi, \phi)} \inf \left\{ J_t + J_w \left[wr + a(t)(\ell_1(t) + \ell_2(t)) - \sigma(\rho h(t) + \hat{\rho} \hat{h}(t))(\ell_1(t) + \ell_2(t)) \right. \right. \\ & \quad - \sigma_1 h(t) + x(\lambda_2 \ell_2(t) - \lambda_1 \ell_1(t)) - b(\ell_1(t) h_1(t) + \ell_2(t) h_2(t)) + \ell_m(t) \mu_m \\ & \quad \left. \left. - \ell_m(t) \sigma_m h_m(t) \right] + \frac{1}{2} J_{ww} (\sigma^2 (\ell_1(t) + \ell_2(t))^2 + b^2 \ell_1^2(t) + b^2 \ell_2^2(t) + \ell_m^2(t) \sigma_m^2 \right. \\ & \quad \left. + \sigma_1^2 + 2\sigma_1 \sigma \rho (\ell_1(t) + \ell_2(t))) + b^2 J_{xx} + \frac{1}{2} \sigma_a^2 J_{aa} + J_{wx} b^2 (\ell_1(t) - \ell_2(t)) \right. \\ & \quad \left. + J_{wa} \sigma_a \rho_0 \ell_m(t) \sigma_m + J_a (\kappa(\theta - a(t)) - \sigma_a \rho_0 h_m(t) - \sigma_a \hat{\rho} h_0(t)) \right. \\ & \quad \left. + J_w \left[q(t) m_1 + [\theta_0 - \theta_1 (1 - q(t))]^2 m_2 \right] \lambda + J_x (b h_2(t) - b h_1(t) - (\lambda_1 + \lambda_2) x) \right. \\ & \quad \left. - \frac{\gamma J \lambda}{\alpha_3} (\phi(t) \ln \phi(t) - \phi(t) + 1) - \frac{\gamma J}{\alpha_1} \frac{1}{2} \|\varphi(t)\|^2 - \frac{\gamma J}{\alpha_2} \frac{1}{2} h(t)^2 \right. \\ & \quad \left. + \phi(t) \lambda E^Q [J(t, w - q(t) Y, x, a) - J(t, w, x, a)] \right\} = 0, \end{aligned} \quad (28)$$

with the boundary condition $J(T, w, x, a) = -\frac{1}{\gamma} \exp\{-\gamma w\}$, and $J_t, J_w, J_a, J_x, J_{ww}, J_{wx}, J_{xx}$ and J_{aa} represent the value function's partial derivative w.r.t the corresponding variables.

In order to obtain the solution J of (28), we conjecture that $J(t, w, x, a)$ has the ansatz form

$$\begin{aligned} J(t, w, x, a) = & -\frac{1}{\gamma} \exp \left\{ -\gamma \left[P(t) w + \frac{1}{2} A_2(t) a^2 + A_1(t) a + A_0(t) \right. \right. \\ & \left. \left. + \frac{1}{2} B_1(t) x^2 + B_2(t) x + B_3(t) ax \right] \right\} \end{aligned} \quad (29)$$

with the boundary condition $P(T) = 1$,

$A_0(T) = A_1(T) = A_2(T) = B_1(T) = B_2(T) = 0$. A direct calculation yields partial derivatives

$$\begin{aligned}
 J_t &= -\gamma \left(P'(t)w + \frac{1}{2} A_2'(t)a^2 + A_1'(t)a + A_0'(t) + \frac{1}{2} B_1'(t)x^2 + B_2'(t)x + B_3'(t)ax \right) J, \\
 J_a &= -\gamma (A_2(t)a + A_1(t) + B_2(t)x)J, \quad J_w = -\gamma P(t)J, \quad J_{ww} = \gamma^2 P^2(t)J, \\
 J_x &= -\gamma (B_1(t)x + B_2(t) + B_3(t)a)J, \quad J_{wx} = \gamma^2 P(t)(B_1(x) + B_2(t) + B_3(t)a)J, \\
 J_{xx} &= \left[-\gamma B_1(t) + \gamma^2 (B_1(t)x + B_2(t) + B_3(t)a)^2 \right] J, \\
 J_{aa} &= \gamma \left((A_2(t)a + A_1(t) + B_2x)^2 - A_2(t) \right) J, \\
 J_{wa} &= \gamma^2 P(t)(A_2(t)a + A_1(t) + B_2(t)x)J, \\
 \frac{J_w}{J_{ww}} &= -\frac{1}{\gamma P(t)}, \quad \frac{J_{wx}}{J_{ww}} = \frac{B_1(x) + B_2(t) + B_3(t)a}{P(t)} \\
 J(t, w - qY, x, a) - J(t, w, x, a) &= J(t, w, x, a) \left(\exp\{\gamma P(t)q(t)Y\} - 1 \right), \tag{30}
 \end{aligned}$$

where $J = J(t, w, x, a)$.

Substituting (30) back into Equation (28) and according to the first-order conditions for $(h_m(t), h(t), h_1(t), h_2(t), h_0(t), \phi)$, we can obtain the minimum point $(h_m^*(t), h^*(t), h_1^*(t), h_2^*(t), h_0^*(t), \phi^*)$ given by

$$\begin{aligned}
 h_m^*(t) &= -\frac{\alpha_1}{\gamma J} (J_w \pi_m \sigma_m + J_a \sigma_a \rho_0) \\
 &= \alpha_1 (\pi_m(t) \sigma_m P(t) + \sigma_a \rho_0 (A_2(t)a + A_1(t) + B_3(t))); \\
 \hat{h}^*(t) &= -\frac{\alpha_1}{\gamma J} (\pi_1(t) + \pi_2(t)) \sigma \hat{\rho} J_w = \alpha_1 \sigma \hat{\rho} P(t) (\pi_1(t) + \pi_2(t)); \\
 h_0^*(t) &= -\frac{\alpha_1}{\gamma J} \sigma_a J_a \hat{\rho}_0 = \alpha_1 \sigma_a \hat{\rho}_0 (A_2(t)a + A_1(t) - B_3(t)a); \\
 h_1^*(t) &= -\frac{\alpha_1}{\gamma J} (b\pi_1(t)J_w + bJ_x) = \alpha_1 b (\pi_1(t)P(t) + B_1(t)x + A_2(t) + B_3(t)a); \\
 h_2^*(t) &= -\frac{\alpha_1}{\gamma J} (b\pi_2(t)J_w - bJ_x) = \alpha_1 b (\pi_2(t)P(t) - B_1(t)x - A_2(t) - B_3(t)a), \\
 \phi^*(t) &= \exp \left\{ \frac{\alpha_3}{\gamma} E^P \left[e^{P(t)q(t)\gamma Y} - 1 \right] \right\}. \tag{31}
 \end{aligned}$$

We substitute (31) into the HJB Equation (28), and differentiating w.r.t. $q(t)$ implies

$$\begin{aligned}
 &\sup_{\ell \in \Pi} \left\{ J_t + J_w \left[rw + a(\ell_1(t) + \ell_2(t)) + x(\lambda_2 \ell_2(t) - \lambda_1 \ell_1(t)) + \ell_m(t) \mu_m \right] \right. \\
 &\quad + J_a \kappa (\theta - a) + J_w \left[q(t) m_1 + (\theta_0 - \theta_1 (1 - q(t))^2) m_2 \right] \lambda + J_{wx} b^2 w (\ell_1(t) - \ell_2(t)) \\
 &\quad + b^2 J_{xx} + \frac{1}{2} \sigma_a^2 J_{aa} + \frac{1}{2} J_{ww} w^2 \left(\ell_m^2(t) \sigma_m^2 + \sigma^2 (\ell_1(t) + \ell_2(t))^2 + b^2 \ell_1^2(t) + b^2 \ell_2^2(t) \right) \\
 &\quad \left. + \sigma_1^2 + 2\sigma_1 \sigma \rho (\ell_1(t) + \ell_2(t)) \right) - J_x (\lambda_1 + \lambda_2) x + \frac{\gamma J}{\alpha_3} (\phi^*(t) - 1) \lambda
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \frac{\alpha_1}{\gamma J} w^2 \sigma^2 (\ell_1(t) + \ell_2(t))^2 J_w^2 + J_{wa} \sigma_a \rho_0 \ell_m \sigma_m + \frac{\alpha_1}{2\gamma J} (w b \ell_1(t) J_w + b J_x)^2 \\
 & + \frac{\alpha_1}{2\gamma J} (b \ell_2(t) J_w - b J_x)^2 + \frac{\alpha_1}{2\gamma J} \sigma_a^2 J_a^2 \hat{\rho}_0^2 \frac{\alpha_1 \sigma^2 \hat{\rho}^2 J_w^2 (\ell_1(t) + \ell_2(t))^2}{2\gamma J} \\
 & + \frac{\alpha_1}{2\gamma J} (J_w \ell_m(t) \sigma_m + J_a \sigma_a \rho_0)^2 + \frac{\alpha_2 J_w^2 (\sigma_1 + (\ell_1(t) + \ell_2(t)) \sigma \rho)^2}{2\gamma J} \Big\} = 0.
 \end{aligned} \tag{32}$$

Differentiating Equation (32) w.r.t. $q(t)$, we get the optimal reinsurance strategy $q^*(t)$ satisfies

$$m_1 + 2\theta_1(1 - q(t))m_2 = \exp\left\{\frac{\alpha_3}{\gamma} E^P\left[e^{P(t)q^*(t)\gamma Y} - 1\right]\right\} E^P\left[Y e^{P(t)q^*(t)\gamma Y}\right]. \tag{33}$$

Using the result of Gu *et al.* [16], we get $q^*(t) > 0$. If this were not true, then $e^{P(t)q^*(t)\gamma Y} < 1$ and

$\exp\left\{\frac{\alpha_3}{\gamma} E^P\left[e^{P(t)q^*(t)\gamma Y} - 1\right]\right\} E^P\left[Y e^{P(t)q^*(t)\gamma Y}\right] < E^P[Y] < m_1 + 2\theta_1(1 - q(t))m_2$. This would cause a contradiction with (33).

According to the first-order condition for $\ell_m(t)$, $\ell_1(t)$ and $\ell_2(t)$, we have

$$\begin{aligned}
 & \ell_m^*(t) \\
 & = e^{-r(T-t)} \left[\frac{\mu_m}{\sigma_m(\alpha_1 + \gamma)} - \frac{\sigma_a \rho_0}{\sigma_m} (A_2(t)a + A_1(t) + B_3x) \right],
 \end{aligned} \tag{34}$$

$$\begin{aligned}
 & \ell_1^*(t) \\
 & = \frac{1}{P(t)} \left[\left(\frac{1}{K} a \right) + \left(\frac{\lambda_1 - \lambda_2}{2K} \right) x - (B_1(t)x + B_2(t) + B_3(t)a) \right] - \frac{(\alpha_2 + \gamma) \sigma_1 \sigma \rho}{K},
 \end{aligned} \tag{35}$$

$$\begin{aligned}
 & \ell_2^*(t) \\
 & = \frac{1}{P(t)} \left[\left(\frac{1}{K} a \right) + \left(\frac{\lambda_1 - \lambda_2}{2K} \right) x + (B_1(t)x + B_2(t) + B_3(t)a) \right] - \frac{(\alpha_2 + \gamma) \sigma_1 \sigma \rho}{K},
 \end{aligned} \tag{36}$$

where $K = 2\sigma^2(\gamma + \alpha_1 \hat{\rho}^2 + \alpha_2 \rho^2) + b^2(\alpha_1 + \gamma)$.

Inserting $q^*(t)$, $\ell_m^*(t)$, $\ell_1^*(t)$, $\ell_2^*(t)$ into (32) and letting the coefficients of w , a , x , ax , a^2 and x^2 be zero, we get

$$\begin{aligned}
 & -\gamma P'(t) - \gamma r P(t) = 0, \\
 & -\frac{1}{2} B_1'(t) + \frac{1}{2} \sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2 B_3^2(t) - \frac{1}{4} \left(\frac{(\lambda_1 - \lambda_2)^2}{K} + \frac{(\lambda_1 + \lambda_2)^2}{\alpha_1 + \gamma} b^2 \right) = 0, \\
 & -B_2'(t) + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_1(t) B_3(t) + \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) B_3(t) \\
 & + \frac{\alpha_2 \sigma_1 \sigma \rho P(t) (\lambda_2 - \lambda_1)}{K} = 0, \\
 & -\frac{1}{2} A_2'(t) + \frac{1}{2} \sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2 A_2^2(t) + \kappa A_2(t) - \frac{1}{K} = 0,
 \end{aligned}$$

$$\begin{aligned}
 & -A_1'(t) + \sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2 A_1^2(t) A_2(t) + \kappa A_2(t) + \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) A_2(t) \\
 & + \frac{2(\alpha_2 + \gamma) \sigma_1 \sigma P(t)}{K} - \frac{2b^2 (\alpha_1 + \gamma) \gamma \sigma_1 \sigma \rho P(t)}{K^2} = 0. \\
 & -B_3'(t) + \kappa B_3(t) + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_1^2(t) - \frac{\lambda_2 - \lambda_1}{K} = 0, \\
 & -A_0'(t) + \frac{1}{2} (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_1^2(t) - b^2 B_1(t) + \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) A_1(t) - \frac{1}{2} \sigma_a^2 A_2(t) \\
 & - \frac{\mu_m^2}{2\sigma_m^2 (\alpha_1 + \gamma)} + \frac{(\alpha_2 + \gamma) (\gamma - \alpha_2) \sigma_1^2 \sigma^2 \rho^2 P^2(t)}{K} + \frac{1}{2} \sigma_1^2 (\alpha_2 + \gamma) P^2(t) \tag{37} \\
 & - \left[q^*(t) m_1 + \left(\theta_0 - \theta_1 (1 - q^*(t))^2 \right) m_2 \right] \lambda P(t) + \frac{\lambda}{\alpha_3} (\phi^*(t) - 1) = 0.
 \end{aligned}$$

Taking into account the boundary conditions $P(T) = 1$, $A_0(T) = A(T) = A_2(T) = B_1(T) = B_2(T) = 0$, we know that

$$P(t) = e^{r(T-t)}, \tag{38}$$

$$B_1(t) = \sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2 \int_t^T B_3(s) ds - \frac{T-t}{2} \left(\frac{(\lambda_2 - \lambda_1)^2}{K} + \frac{(\lambda_1 + \lambda_2)^2}{(\alpha_1 + \gamma) b^2} \right), \tag{39}$$

$$\begin{aligned}
 B_2(t) = & \sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2 \int_t^T A_1(s) B_3(s) ds - \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) \int_t^T B_3(s) ds \\
 & - \frac{\alpha_2 \sigma_1 \sigma \rho P(t) (\lambda_2 - \lambda_1)}{K} (T-t), \tag{40}
 \end{aligned}$$

$$B_3(t) = e^{-\int_t^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(s) ds)} \times \int_t^T \frac{\lambda_2 - \lambda_1}{K} e^{\int_s^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(u) du)}, \tag{41}$$

$$\begin{aligned}
 A_0(t) = & -\frac{\sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2}{2} \int_t^T A_1^2(s) ds + b^2 \int_t^T B_1(s) ds + \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) \int_t^T A_1(s) ds \\
 & + \frac{\sigma_a^2}{2} \int_t^T A_2(s) ds + \int_t^T \left(\frac{\mu_m^2}{2\sigma_m^2 (\alpha_1 + \gamma)} - \frac{(\alpha_2 + \gamma) (\gamma - \alpha_2) \sigma_1^2 \sigma^2 \rho^2 P^2(s)}{K} \right. \\
 & \left. - \frac{1}{2} \sigma_1^2 (\alpha_2 + \gamma) P^2(s) \right) ds + \int_t^T \left[q^*(s) m_1 + \left(\theta_0 - \theta_1 (1 - q^*(s))^2 \right) m_2 \right] \lambda e^{r(T-s)} ds \\
 & - \int_t^T \frac{\lambda}{\alpha_3} (\phi^*(s) - 1) ds, \tag{42}
 \end{aligned}$$

$$A_2(t) = \frac{e^{-\epsilon_1(T-t)} - e^{-\epsilon_2(T-t)}}{\left(\frac{e^{-\epsilon_2(T-t)}}{\epsilon_2} - \frac{e^{-\epsilon_1(T-t)}}{\epsilon_1} \right) (\alpha_1 + \gamma) \hat{\rho}_0^2 \sigma_a^2}, \tag{43}$$

$$\begin{aligned}
 A_1(t) = & -e^{-\int_t^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(s) ds)} \int_t^T \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) A_2(s) \times e^{\int_s^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(u) du)} ds \\
 & - e^{-\int_t^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(s) ds)} \int_t^T \frac{2\sigma_1 \sigma \rho P(s)}{K} (\alpha_2 + \gamma) \times e^{\int_s^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(u) du)} ds \\
 & - e^{-\int_t^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(s) ds)} \int_t^T \frac{2\sigma_1 \sigma \rho P(s) b^2 (\alpha_1 + \gamma)}{K^2} \times e^{\int_s^T (\kappa + (\alpha_1 + \gamma) \sigma_a^2 \hat{\rho}_0^2 A_2(u) du)} ds,
 \end{aligned}$$

where $\epsilon_1 = \kappa + \sqrt{\kappa^2 + \frac{2\sigma_a^2(\alpha_1 + \gamma)\hat{\rho}_0^2}{K}}$, $\epsilon_2 = \epsilon - \sqrt{\kappa^2 + \frac{2\sigma_a^2(\alpha_1 + \gamma)\hat{\rho}_0^2}{K}}$. Using the verification of Yi *et al.* [15], we have the following theorem.

Theorem 4.1. For the optimization problem (26), the value function is given by

$$J(t, w, x, a) = -\frac{1}{\gamma} \exp \left\{ -\gamma \left[P(t)w + \frac{1}{2} A_2(t)a^2 + A_1(t)a + A_0(t) + \frac{1}{2} B_1(t)x^2 + B_2(t)x + B_3(t)ax \right] \right\} \tag{44}$$

where $P(t)$, $A_1(t)$, $A_2(t)$, $A_0(t)$, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are given by (38)-(43), and the corresponding optimal reinsurance-investment strategy is given by

$$\ell = (q^*(t), \ell_m(t), \ell_1(t), \ell_2(t)),$$

where $q^*(t)$ is given by Equation (33), and $\ell_m(t)$, $\ell_1(t)$, $\ell_2(t)$ are given by Equations (34)-(36).

If all the ambiguity-aversion coefficients equal 0, i.e. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in our model, the optimization problem degenerates into optimization problem (17) without ambiguity aversion. Similarly to Theorem 4.1, we have the following Proposition.

Proposition 4.1. For optimization problem (17), the value function is given by

$$J^0(t, w, x, a) = -\frac{1}{\gamma} \exp \left\{ -\gamma \left[e^{r(T-t)}w + \frac{1}{2} A_{20}(t)a^2 + A_{10}(t)a + A_{00}(t) + \frac{1}{2} B_{10}(t)x^2 + B_{20}(t)x + B_{30}(t)ax \right] \right\} \tag{45}$$

and the optimal strategy $\ell_0(t) = \ell^*(t)|_{\alpha_1=\alpha_2=0}$, where $P(t)$, $A_{10}(t)$, $A_{20}(t)$, $A_{00}(t)$, $B_{10}(t)$, $B_{20}(t)$ and $B_{30}(t)$ are given by (46)-(52)

$$P(t) = e^{r(T-t)}, \tag{46}$$

$$B_{10}(t) = \sigma_a^2 \gamma \hat{\rho}_0^2 \int_t^T B_3(s) ds - \frac{T-t}{2} \left(\frac{(\lambda_2 - \lambda_1)^2}{(2\sigma^2 + b^2)\gamma} + \frac{(\lambda_1 + \lambda_2)^2}{(\alpha_1 + \gamma)b^2} \right), \tag{47}$$

$$B_{20}(t) = \sigma_a^2 \gamma \hat{\rho}_0^2 \int_t^T A_1(s) B_3(s) ds - \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) \int_t^T B_3(s) ds, \tag{48}$$

$$B_{30}(t) = e^{-\int_t^T (\kappa + \gamma \sigma_a^2 \hat{\rho}_0^2 A_2(s) ds)} \times \int_t^T \frac{\lambda_2 - \lambda_1}{(2\sigma^2 + b^2)\gamma} e^{\int_s^T (\kappa + \gamma \sigma_a^2 \hat{\rho}_0^2 A_2(u) du)} ds, \tag{49}$$

$$\begin{aligned} A_{00}(t) = & -\frac{\sigma_a^2 \gamma \hat{\rho}_0^2}{2} \int_t^T A_1^2(s) ds + b^2 \int_t^T B_1(s) ds + \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) \int_t^T A_1(s) ds \\ & + \frac{\sigma_a^2}{2} \int_t^T A_2(s) ds + \int_t^T \left(\frac{\mu_m^2}{2\sigma_m^2 \gamma} - \frac{\gamma \sigma_1^2 \sigma^2 \rho^2 P^2(s)}{2\sigma^2 + b^2} - \frac{1}{2} \sigma_1^2 \gamma P^2(s) \right) ds \\ & + \int_t^T \left[q^*(s) m_1 + \left(\theta_0 - \theta_1 (1 - q^*(s))^2 \right) m_2 \right] \lambda e^{r(T-s)} ds \\ & + \lambda E^Q \left[e^{P(t)q_0(t)\gamma T} - 1 \right], \end{aligned} \tag{50}$$

$$A_{20}(t) = \frac{e^{-\epsilon_1(T-t)} - e^{-\epsilon_2(T-t)}}{\left(\frac{e^{-\epsilon_2(T-t)}}{\epsilon_2} - \frac{e^{-\epsilon_1(T-t)}}{\epsilon_1}\right) \gamma \hat{\rho}_0^2 \sigma_a^2}, \tag{51}$$

$$\begin{aligned} A_{10}(t) = & -e^{-\int_t^T (\kappa+\gamma) \hat{\rho}_0^2 \sigma_a^2 A_2(s) ds} \int_t^T \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) A_2(s) \times e^{\int_s^T (\kappa+\gamma) \hat{\rho}_0^2 \sigma_a^2 A_2(u) du} ds \\ & - e^{-\int_t^T (\kappa+\gamma) \hat{\rho}_0^2 \sigma_a^2 A_2(s) ds} \int_t^T \frac{2\sigma_1 \sigma \rho P(s)}{(2\sigma^2 + b^2) \gamma} \gamma \times e^{\int_s^T (\kappa+\gamma) \hat{\rho}_0^2 \sigma_a^2 A_2(u) du} ds \\ & - e^{-\int_t^T (\kappa+\gamma) \hat{\rho}_0^2 \sigma_a^2 A_2(s) ds} \int_t^T \frac{2\sigma_1 \sigma \rho P(s) b^2 \gamma}{\left((2\sigma^2 + b^2) \gamma\right)^2} \times e^{\int_s^T (\kappa+\gamma) \hat{\rho}_0^2 \sigma_a^2 A_2(u) du} ds, \end{aligned} \tag{52}$$

where $\epsilon_1 = \kappa + \sqrt{\kappa^2 + \frac{2\sigma_a^2 \hat{\rho}_0^2}{2\sigma^2 + b^2}}$, $\epsilon_2 = \epsilon - \sqrt{\kappa^2 + \frac{2\sigma_a^2 \hat{\rho}_0^2}{2\sigma^2 + b^2}}$.

If the insurer only invests in the financial markets, and does not purchase reinsurance, the optimization problem (26) degenerates to investment only problem. With $q^*(t) = 1$ in Theorem 4.1, we easily derive the following proposition.

Proposition 4.2. *For optimization problem (26), if only investment is discussed, the value function is given by*

$$\begin{aligned} J(t, w, x, a) = & -\frac{1}{\gamma} \exp \left\{ -\gamma \left[e^{r(T-t)} w + \frac{1}{2} A_2(t) a^2 + A_1(t) a + A_{01}(t) \right. \right. \\ & \left. \left. + \frac{1}{2} B_1(t) x^2 + B_2(t) x + B_3(t) a x \right] \right\}, \end{aligned} \tag{53}$$

where $A_1(t)$, $A_2(t)$, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are given by Theorem 4.1, and

$$\begin{aligned} A_{01}(t) = & -\frac{\sigma_a^2 (\alpha_1 + \gamma) \hat{\rho}_0^2}{2} \int_t^T A_1^2(s) ds + b^2 \int_t^T B_1(s) ds + \left(\frac{\mu_m \sigma_a \rho_0}{\sigma_m} - \kappa \theta \right) \int_t^T A_1(s) ds \\ & + \frac{\sigma_a^2}{2} \int_t^T A_2(s) ds + \int_t^T \left(\frac{\mu_m^2}{2\sigma_m^2 (\alpha_1 + \gamma)} - \frac{(\alpha_2 + \gamma)(\gamma - \alpha_2) \sigma_1^2 \sigma^2 \rho^2 e^{2r(T-s)}}{K} \right. \\ & \left. - \frac{1}{2} \sigma_1^2 (\alpha_2 + \gamma) e^{2r(T-s)} \right) ds + \int_t^T [q^*(s) m_1 + \theta_0 m_2] \lambda e^{r(T-s)} ds \\ & - \int_t^T \frac{\lambda}{\alpha_3} \left(e^{\frac{\alpha_3 E^Q}{\gamma} [e^{r(T-s)} \gamma Y - 1]} - 1 \right) ds. \end{aligned} \tag{54}$$

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Competing Interests

The authors declare that they have no conflict of interests.

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