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The Qualitative Analysis for a Class Nonlinear Systems with High Degree

Songlin Chen*, Yajie Du

School of Mathematics & Physics, Anhui University of Technology, Ma'anshan, China
Email: *slchen@ahut.edu.cn

Abstract

In the paper, the problems of identification of focus or center for a class of nonlinear systems are studied. As results, the identifying rule for the type of singular point of the system and the calculating formula for the focus value are obtained by the classical theory of Poincare and Lyapunov. At the end, the numerical simulation for a polynomial system of seven-degree are showed to corroborate the theoretical results of the method.

Keywords

Focus, Center, Focus Quantities, Polynomial System

1. Introduction

A class of plane polynomial nonlinear systems with a singular point are studied in the paper. We know that the singular point maybe a center or a focus, thus the identification for focus or center need to be performed. The problem of identification for focus and center has inseparable relationship to the study of stability of the system. The conventional methods are the methods of successive function and the formal series [1] [2] [3]. This kind of systems arise frequently in many applied science fields such as biomathematics, fluid dynamics, quantum mechanics, chemical reaction and so on [4] [5]. In the paper, the center or focus of a class of nonlinear system with higher degree polynomial are analyzed by using the methods in [6] [7] [8].

The paper is organized in the following style: in Section 2, the identification method is constructed, and the calculating formula for the focus value is obtained. At the end, in Section 3, a seven-degree polynomial system is numerically simulated.

2. Main Results

The following nonlinear system with higher degree polynomial

$$\begin{cases} \dot{x} = -2x^2y - y^3 + x^3y^2 f(x,y) \\ \dot{y} = x^3 + x^2y^3 f(x,y) \end{cases}, \ \partial f(x,y) \geq 1 \tag{1}$$

are considered.

Conventionally, applying the polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

to the System (1) results in

$$\begin{aligned} \frac{dr}{d\theta} &= -r*\cos\theta*\sin\theta + r^3*\cos^2\theta*\sin^2\theta*f(r\cos\theta, r\sin\theta): \\ &= R(r,\theta) \end{aligned} \tag{2}$$

Express initial value problem of the differential Equation (2) as following form

$$\begin{cases} \dfrac{dr}{d\theta} = R(\theta,r) = \displaystyle\sum_{i=1}^{\infty} R_i(\theta)r^i = R_1(\theta)r + R_2(\theta)r^2 + \cdots + R_k(\theta)r^k + \cdots \\ r(0) = c = \text{constant} \end{cases} \tag{3}$$

Moreover, assume the formal series solution for problem (3) is

$$r(\theta,c) = \sum_{i=1}^{\infty} r_i(\theta)c^i = r_1(\theta)c + r_2(\theta)c^2 + \cdots + r_j(\theta)c^j + \cdots \tag{4}$$

Substitute (4) into (3) gives

$$\sum_{i=1}^{\infty} r_i'(\theta)c^i = \sum_{k=1}^{\infty} \left[R_k(\theta)\left(\sum_{i=1}^{\infty} r_i(\theta)c^i \right)^k \right] \tag{5}$$

Expand the right hand of the equality and compare the bilateral coefficients of c^i, the results are stated as follows

$$\begin{cases} \dfrac{dr_1}{d\theta} = R_1(\theta)*r_1(\theta) \\ r_1(0) = 1 \end{cases}$$

$$\begin{cases} \dfrac{dr_k}{d\theta} = R_1(\theta)*r_k(\theta) + Q_k(R_1, R_2, \cdots, R_k; r_1, r_2, \cdots, r_{k-1}) \\ r_k(0) = 0 \end{cases}, \ k \geq 2 \tag{6}$$

thus $r_1(\theta), r_2(\theta), \cdots, r_k(\theta), \cdots$ are solved successively and the series solution of the problem (3) is thus obtained by (4).

Next we'll obtain an equivalent description to the focus value $r(2\pi, c)$.

Denote the first integral of (3) by

$$c = \varphi(\theta, r) = \sum_{i=1}^{\infty} \varphi_i(\theta)r^i$$

then we have the following results.

Theorem *System* (1) *or* (2) *has the following recurrence formula:*

$$1° \begin{cases} \varphi_1'(\theta) = -R_1(\theta) * \varphi_1(\theta) \\ \varphi_n'(\theta) = -\sum_{k=1}^{n} k * R_{n-k+1} * \varphi_k(\theta), n \geq 2 \\ \varphi_1(0) = 1, \varphi_k(0) = 0, \ k \geq 2 \end{cases}$$ (7)

$$2° \quad \varphi_n(2\pi) = -r_n(2\pi) \quad \forall n \geq 2, n \in N$$

Proof Because

$$c = \varphi(\theta, r) = \sum_{i=1}^{\infty} \varphi_i(\theta) r^i = r(0)$$

we have

$$\frac{\partial \varphi}{\partial \theta} + R(\theta, r)\frac{\partial \varphi}{\partial r} = 0$$

substitute $\varphi(\theta, r) = \sum_{i=1}^{\infty} \varphi_i(\theta) r^i$ and $R(\theta, r) = \sum_{i=1}^{\infty} R_i(\theta) r^i$ into the partial differential equation, and expand the equality as follows

$$\begin{aligned} & \left[\varphi_1'(\theta) r + \varphi_2'(\theta) r^2 + \varphi_3'(\theta) r^3 + \varphi_4'(\theta) r^4 + \cdots + \varphi_m'(\theta) r^m + \cdots \right] \\ & + \left[R_1(\theta)\varphi_1(\theta) r + 2R_1(\theta)\varphi_2(\theta) r^2 + 3R_1(\theta)\varphi_3(\theta) r^3 + 4R_1(\theta)\varphi_4(\theta) r^4 \right. \\ & \left. + 2R_1(\theta)\varphi_5(\theta) r^5 + \cdots + mR_1(\theta)\varphi_m(\theta) r^m + \cdots \right] \\ & + \left[R_2(\theta)\varphi_1(\theta) r^2 + 2R_2(\theta)\varphi_2(\theta) r^3 + 3R_2(\theta)\varphi_3(\theta) r^4 + 4R_2(\theta)\varphi_4(\theta) r^5 \right. \\ & \left. + 5R_2(\theta)\varphi_5(\theta) r^6 + \ldots + mR_2(\theta)\varphi_m(\theta) r^{m+1} + \cdots \right] \\ & + \left[R_3(\theta)\varphi_1(\theta) r^3 + 2R_3(\theta)\varphi_2(\theta) r^4 + 3R_3(\theta)\varphi_3(\theta) r^5 + 4R_3(\theta)\varphi_4(\theta) r^6 \right. \\ & \left. + 5R_3(\theta)\varphi_5(\theta) r^7 + \cdots + mR_3(\theta)\varphi_m(\theta) r^{m+2} + \cdots \right] \\ & + \cdots = 0 \end{aligned}$$ (8)

Comparing the bilateral coefficients of r^i gives the conclusion 1° directly.

To prove 2°, with the expressions $c = \varphi(\theta, r) = \sum_{i=1}^{\infty} \varphi_i(\theta) r^i = r(0)$ and (4), we have $r = \sum_{j=1}^{\infty} r_j \left(\sum_{i=1}^{\infty} \varphi_i r^i \right)^j$, that is

$$\begin{aligned} r = & r_1 \left[\varphi_1 r + \varphi_2 r^2 + \cdots \right] + r_2 \left[\varphi_1 r + \varphi_2 r^2 + \cdots \right]^2 + r_3 \left[\varphi_1 r + \varphi_2 r^2 + \cdots \right]^3 \\ & + \cdots + r_m \left[\varphi_1 r + \varphi_2 r^2 + \cdots \right]^m + \cdots \end{aligned}$$ (9)

From the identity it is obvious that

$$\varphi_2(2\pi) = -r_2(2\pi), \quad \varphi_3(2\pi) = -r_3(2\pi)$$

with the help of the method of mathematical induction, $\varphi_n(2\pi) = -r_n(2\pi)$ $\forall n \geq 2, n \in N$ will be true.

According to the classical theory of Poincaré and Lyapunov, it is well known that when $r_k(2\pi, c)$ is non-zero, its subscript must be an odd number. If $r_{2k+1}(2\pi, c) = 0$ for all k, then the original point is a center. If $r_{2k+1}(2\pi, c) \neq 0$ for some specified k, then the original point is a k-order fine focus; meanwhile,

$r_{k+1}(2\pi, c)$ is the focus value of the k-order fine focus, according to the theorem above we known $\varphi_{2k+1}(2\pi)$ can be regarded as an equivalent description for the k-order fine focus.

3. Simulations

Consider the specified systems:

$$\begin{cases} \dot{x} = -2x^2 y - y^3 + x^3 y^2 f(x, y) \\ \dot{y} = x^3 + x^2 y^3 f(x, y) \end{cases}, \partial f(x, y) \geq 1 \tag{10}$$

wherein $f(x, y) = a_1 x + b_1 y + a_2 x^2 + b_2 y^2$. The system can be expressed by polar coordinate (r, θ) such as

$$\frac{dr}{d\theta} = -r * \cos\theta * \sin\theta + r^3 * \cos^2\theta * \sin^2\theta * f(r\cos\theta, r\sin\theta)$$

$$:= R(r, \theta) = -r\sin\theta\cos\theta + r^4 \left(a_1 \cos^3\theta\sin^2\theta + b_1 \cos^2\theta\sin^3\theta\right)$$

$$+ r^5 \left(a_2 \cos^4\theta\sin^2\theta + b_2 \cos^2\theta\sin^4\theta\right)$$

thus the following relationships are obtained

$$R_1(\theta) = -\sin\theta\cos\theta, \, R_2(\theta) = 0, \, R_3(\theta) = 0$$

$$R_4(\theta) = a_1 \cos^3\theta\sin^2\theta + b_1 \cos^2\theta\sin^3\theta$$

$$R_5(\theta) = a_2 \cos^4\theta\sin^2\theta + b_2 \cos^2\theta\sin^4\theta$$

The recurrence formula which quoted from above is stated as follows

$$\begin{cases} \varphi_1'(\theta) = -R_1(\theta) * \varphi_1(\theta) \\ \varphi_n'(\theta) = -\sum_{k=1}^{n} k * R_{n-k+1} * \varphi_k(\theta) \quad n \geq 2 \\ \varphi_1(0) = 1, \varphi_k(0) = 0 \quad k \geq 2 \end{cases}$$

Moreover, for simulating the result, we specify the parameters in system involved in $f(x, y)$ as

$$a_1 = 1, b_1 = 2, a_2 = -3, b_2 = 4 \tag{11}$$

According to **Theorem** in Section 2, it is sufficient to investigate the values of $\varphi_{2k+1}(2\pi)$ which can be obtained approximately by the drawings of $\varphi_3(\theta)$ and $\varphi_5(\theta)$ on interval $[0, 2\pi]$ as follows:

Figure 1, **Figure 2** and the numerical result show that $\varphi_3(2\pi) = 0$ and $\varphi_5(2\pi) = 0.0784437 \neq 0$, thus according to the **Theorem** in Section 2, we know that the singular point of system (10 - 11), (0, 0), is a 2-order fine focus with the focus value being −0.0784437.

To verify the results above, the phase portrait of the specified system (10-11) is depicted directly by **Mathematica** as in **Figure 3**.

Figure 3 shows that the original point (0, 0) is a focus which corroborates in practice the theoretical analysis of the method in the paper.
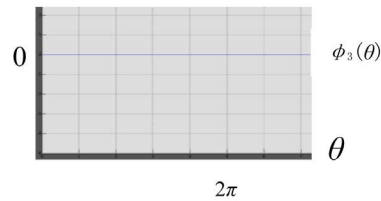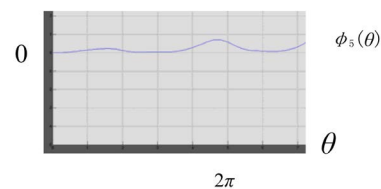
Figure 1. $\varphi_3(\theta)$.
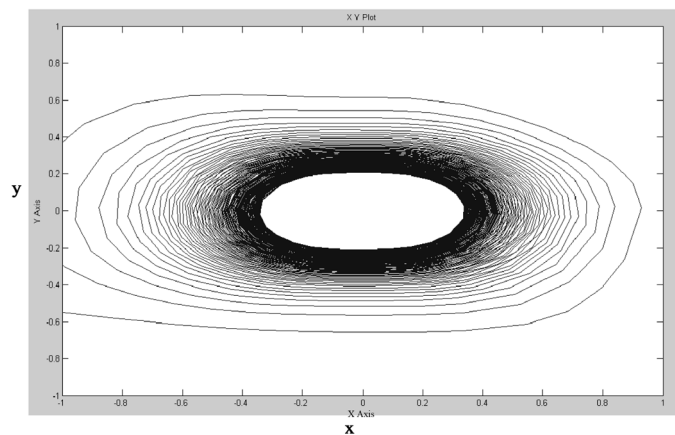

Figure 2. $\varphi_5(\theta)$.

Figure 3. The phase portrait of the specified system (10), (11).

4. Conclusion

The paper considers the problems of identification of focus or center for a class of nonlinear systems. As results, the identifying rule for the type of singular point of the system and the calculating formula for the focus value are obtained by the classical theory of Poincare and Lyapunov. The numerical simulation shows that the theoretical results of the method is valid.

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