

Was Polchinski Wrong? Colombeau Distributional Rindler Space-Time with Distributional Levi-Civita Connection Induced Vacuum Dominance. Unruh Effect Revisited

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Abstract

The vacuum energy density of free scalar quantum field Φ in a Rindler distributional space-time with distributional Levi-Civita connection is considered. It has been widely believed that, except in very extreme situations, the influence of acceleration on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is wrong by showing that in a Rindler distributional background space-time with distributional Levi-Civita connection the vacuum energy of free quantum fields is forced, by the very same background distributional space-time such a Rindler distributional background space-time, to become dominant over any classical energy density component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on a Rindler distributional space-times with distributional Levi-Civita connection. In particular we obtain that the vacuum fluctuations $\langle \Phi^2 \rangle$ have a singular behavior at a Rindler horizon $\delta_r = 0: \langle \Phi^2(\delta) \rangle \asymp \delta^{-4}, \delta \asymp c^2/a, a \rightarrow \infty$. Therefore sufficiently strongly accelerated observer burns up near the Rindler horizon. Thus Polchinski's account doesn't violate the Einstein equivalence principle.

Keywords

Vacuum Energy Density, Rindler Distributional Space-Time, Levi-Civita Connection, Semiclassical Gravity Effect, Einstein Equivalence Principle

1. Introduction

In March 2012, Joseph Polchinski claimed that the following three statements cannot all be true [1]: 1) Hawking radiation is in a pure state, 2) the information carried by the radiation is emitted from the region near the horizon, with low energy effective field theory valid beyond some microscopic distance from the horizon, 3) the infalling observer encounters nothing unusual at the horizon. Joseph Polchinski argues that the most conservative resolution is: the infalling observer burns up at the horizon. In Polchinski's account, quantum effects would turn the event horizon into a seething maelstrom of particles. Anyone who fell into it would hit a wall of fire and be burned to a crisp in an instant. As pointed out by physics community such firewalls would violate a foundational tenet of contemporary physics known as the equivalence principle, it states in part that an observer falling in a gravitational field—even the powerful one inside a black hole—will see exactly the same phenomena as an accelerated observer floating in empty space.

In this paper we argue that Polchinski was not wrong, but Unruh effect revision is needed.

1.1. What Is Colombeau Distributional Semi-Riemannian Geometry?

Recall that the classical Cartan's structural equations show in a compact way the relation between a connection and its curvature, and reveal their geometric interpretation in terms of moving frames. In order to study the mathematical properties of singularities, we need to study the geometry of manifolds endowed on the tangent bundle with a symmetric bilinear form it is allowed to become degenerate (singular).

Remark 1.1.1. But if the fundamental tensor is allowed to be degenerate (singular), there are some obstructions in constructing the geometric objects normally associated to the fundamental tensor. Also, local orthonormal frames and coframes no longer exist, as well as the metric connection and its curvature operator [2].

Remark 1.1.2. "Singular Semi-Riemannian Geometry"—the main branch of contemporary semi-Riemannian geometry in which have been studied a smooth manifolds M furnished with a degenerate (singular) on a smooth submanifold $M' \subsetneq M$ metric tensor of arbitrary signature have been studied [2].

Remark 1.1.3. In order to solve problems of the gravitational singularity in classical general relativity the singular semi-Riemannian geometry based on Colombeau calculus and Colombeau generalized functions was much developed, see [3]-[22].

Remark 1.1.4. Let $\mathbf{G}(M')$ be algebra of Colombeau generalized functions on $M' \subset M$, let $\tilde{\mathbb{R}}$ be the ring of Colombeau generalized numbers [3] [4] [5]. Let $(g_\varepsilon)_\varepsilon$ be Colombeau generalized metric tensor on M and let $\mathbf{Ric}_{M'}(p)$ be generalized Ricci tensor of the metric $(g_\varepsilon(p))_\varepsilon|_{M'}$ [20] [21]. The main properties of such nonclassical manifolds with a degenerate (singular) metric tensor that is $\mathbf{Ric}_{M'}(p) \in \mathbf{G}(M') \setminus C^\infty(M')$, i.e. for all $p \in M' : \mathbf{Ric}_{M'}(p) \in \tilde{\mathbb{R}} \setminus \mathbb{R}$.

Definition 1.1.1. Let $\mathbf{G}(M')$ be algebra of Colombeau generalized functions on $M' \subset M$, and let $(g_\varepsilon(p))_\varepsilon$ be Colombeau generalized metric tensor on M such that $(g_\varepsilon(p))_\varepsilon$ is the Colombeau solution of the Einstein field Equation (1.3.19), (see Remark 1.3.7). We define now the Colombeau distributional scalar curvature $\mathbf{R}_M(p) = [(\mathbf{R}_{\varepsilon,M}(p))_\varepsilon]$ (or distributional Ricci [20] [21] scalar) as the trace of $\mathbf{Ric}_M(p) = [(\mathbf{Ric}_{\varepsilon,M}(p))_\varepsilon]$, i.e. $\mathbf{R}_M(p) = \text{tr}[(\mathbf{Ric}_{\varepsilon,M}(p))_\varepsilon]$. Assume that $\mathbf{R}_{M'}(p) \in \mathbf{G}(M') \setminus C^\infty(M')$.

Then we say that: (i) gravitational field $(g_\varepsilon(p))_\varepsilon$ (or corresponding distributional spacetime) has a gravitational singularity on a smooth compact submanifold $M_c \subset M$ iff $\mathbf{R}_{M_c}(p) \in \mathbf{G}(M_c) \setminus C^\infty(M_c)$; (ii) gravitational field $(g_\varepsilon(p))_\varepsilon$ has a gravitational singularity with compact support iff $\mathbf{R}_{M_c}(p) \in D'(\mathbb{R}^3)$.

Remark 1.1.5. It turns out that the distributional Schwarzschild spacetime has a gravitational singularity with compact support at origin $\{r = 0\}$ [6]-[11] and at Schwarzschild horizon $\mathbf{S}^2 \times \{r = 2m\}$ [18] [19].

Definition 1.1.2. (i) Let $\mathbf{G}(M)$ be algebra of Colombeau generalized functions on M , and let $(g_\varepsilon(p))_\varepsilon$ be Colombeau generalized metric tensor on M such that $(g_\varepsilon(p))_\varepsilon$ is the Colombeau solution of the generalized Einstein field Equation (1.3.19). The generalized point value of $(g_\varepsilon(p))_\varepsilon$ at generalized point $((p_\varepsilon))_\varepsilon$ is $(g_\varepsilon(p_\varepsilon))_\varepsilon$. (ii) We define now the generalized point value of the distributional scalar curvature $\mathbf{R}_M(p)$ at generalized point $\mathbf{p} = [((p_\varepsilon))_\varepsilon]$ by formula $\mathbf{R}_M(\mathbf{p}) = [(\mathbf{R}_{\varepsilon,M}(p_\varepsilon))_\varepsilon]$.

1.2. Distributional Møller’s Geometry as Colombeau Extension of the Classical Møller’s Spacetime

As important example of Colombeau extension of the singular semi-Riemannian geometry mentioned above, we consider now Møller’s uniformly accelerated frame given by Møller’s line element [23]:

$$ds^2 = -(a + gx)^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1.2.1}$$

Of course Møller’s metric (1.2.1) degenerate at Møller horizon $x_{hor}^{Mol} = -(a/g)^{-1}$. Note that formally corresponding to the metric (1.2.1) classical Levi-Civita connection is [23]

$$\Gamma_{44}^1(x) = (a + gx), \Gamma_{14}^4(x) = \Gamma_{41}^4(x) = \frac{1}{a + gx} \tag{1.2.2}$$

and therefore classical Levi-Civita connection (1.2.2) of course is not available at Møller horizon $x_{hor}^{Mol} = -a \cdot g^{-1}$. Recall that fundamental tensor corresponding to

the metric (1.2.1) was obtained in Moller’s paper [23] as a vacuum solution of the classical Einstein’s field equations

$$G_i^k = R_i^k - \frac{1}{2} \delta_i^k R = 0, \tag{1.2.3}$$

where R_i^k is the contracted Riemann-Christoffel tensor formally calculated by canonical way by using classical Levi-Civita connection (1.2.2) and $R = R_i^i$. Using Dingle’s formula [23] in case of the metric (1.2.1) we get

$$G_2^2(x) = G_3^3(x) = -\frac{1}{2\Delta(x)} \left\{ \Delta''(x) - \frac{[\Delta'(x)]^2}{2\Delta(x)} \right\}, \tag{1.2.4}$$

$$\Delta(x) = (a + gx)^2,$$

where $\Delta'(x) = \partial\Delta(x)/\partial x$ and all other components of G_i^k vanishes identically. Note that

$$\Delta'(x) = 2g(a + gx), \Delta''(x) = 2g^2. \tag{1.2.5}$$

Thus for any $x \neq -a \cdot g^{-1}$ we get a classical result

$$G_2^2(x) = G_3^3(x) = -\frac{1}{2\Delta(x)} \left\{ 2g^2 - \frac{4g^2(a + gx)^2}{2\Delta(x)} \right\} \equiv 0. \tag{1.2.6}$$

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \rightarrow \infty} x_n = -a \cdot g^{-1}, x_n \neq -a \cdot g^{-1}, n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ we get

$$\mathfrak{I}(x_n) = G_2^2(x_n) = G_3^3(x_n) = -\frac{1}{2\Delta(x_n)} \left\{ 2g^2 - \frac{4g^2(a + gx_n)^2}{2\Delta(x_n)} \right\} \equiv 0, \tag{1.2.7}$$

and therefore $\lim_{n \rightarrow \infty} \mathfrak{I}(x_n) \equiv 0$. However

$$\lim_{n \rightarrow \infty} \Gamma_{14}^4(x_n) = \lim_{n \rightarrow \infty} \Gamma_{41}^4(x_n) = \lim_{n \rightarrow \infty} \frac{1}{a + gx_n} = \infty, \tag{1.2.8}$$

i.e. classical Levi-Civita connection given by (1.2.2) unavailable at Moller horizon.

Remark 1.2.1. In order to avoid difficultness mentioned above, we consider now the regularized Moller’s metric

$$d_\varepsilon s^2 = -\Delta_\varepsilon(x) dt^2 + dx^2 + dy^2 + dz^2, \tag{1.2.9}$$

$$\Delta_\varepsilon(x) = [(a + gx)^2 + \varepsilon^2], \varepsilon \in (0, 1].$$

Using now Dingle’s formula [23] for the case of (1.2.9) we get

$$\mathfrak{I}(x; \varepsilon) = G_2^2(x; \varepsilon) = G_3^3(x; \varepsilon) = -\frac{1}{2\Delta_\varepsilon(x)} \left\{ \Delta_\varepsilon''(x) - \frac{[\Delta_\varepsilon'(x)]^2}{2\Delta_\varepsilon(x)} \right\}, \tag{1.2.10}$$

$$\Delta_\varepsilon(x) = [(a + gx)^2 + \varepsilon^2].$$

Note that

$$\Delta_\varepsilon'(x) = 2g(a + gx), \Delta_\varepsilon''(x) = 2g^2 \tag{1.2.11}$$

and therefore

$$\begin{aligned} \mathfrak{I}(x; \varepsilon) &= -\frac{1}{2\Delta_\varepsilon(x)} \left\{ 2g^2 - \frac{2g^2(a+gx)^2}{\Delta_\varepsilon(x)} \right\} \\ &= -\frac{1}{2\Delta_\varepsilon(x)} \left\{ 2g^2 - \frac{2g^2[(a+gx)^2 + \varepsilon^2] - 2g^2\varepsilon^2}{\Delta_\varepsilon(x)} \right\} \\ &= -\frac{g^2\varepsilon^2}{\Delta_\varepsilon^2(x)}. \end{aligned} \tag{1.2.12}$$

Remark 1.2.2. (i) Note that $(\mathfrak{I}(x; \varepsilon))_\varepsilon, \varepsilon \in (0, 1]$ is Colombeau generalized function such that

$$\mathbf{cl}\left[(\mathfrak{I}(x; \varepsilon))_\varepsilon\right] \in \mathbf{G}(\mathbb{R}) \text{ and } \mathbf{cl}\left[(\mathfrak{I}(-(a/g)^{-1}; \varepsilon))_\varepsilon\right] = \mathbf{cl}\left[(\varepsilon^{-2})_\varepsilon\right] \in \tilde{\mathbb{R}}.$$

Remark 1.2.3. Note that: (i) at any point $\tilde{x} \in \tilde{\mathbb{R}}$ such that $x \in \mathbb{R}$ and $x \neq -a \cdot g^{-1}$ one obtains $(\mathfrak{I}(\tilde{x}; \varepsilon))_\varepsilon \approx_{\tilde{\mathbb{R}}} \tilde{0}$ (see Definition 1.5.0 (i)) and therefore the Ricci tensor as well as the Ricci scalar are infinite small beyond Moller horizon $x_{hor}^M = -a \cdot g^{-1}$. Thus at any point \tilde{x} such that $x \in \mathbb{R}$ and $x \neq -a \cdot g^{-1}$ we obtain the disered result in a good agriment with formall canonical calculation (see for example [24], subsect. 2.1.6), (ii) obviously at any finite point $x_{fin} \in \tilde{\mathbb{R}}$ (see Definition 1.5.0 (iii)) one obtains again $(\mathfrak{I}(\tilde{x}; \varepsilon))_\varepsilon \approx_{\tilde{\mathbb{R}}} \tilde{0}$.

Remark 1.2.4. (I) Thus Colombeau generalized fundamental tensor $(g_{ik}(\varepsilon))_\varepsilon$ corresponding to Colombeau metric

$$\begin{aligned} (d_\varepsilon s^2) &= -(\Delta_\varepsilon(x) dt^2)_\varepsilon + dx^2 + dy^2 + dz^2, \\ (\Delta_\varepsilon(x))_\varepsilon &= \left(\left[(a+gx)^2 + \varepsilon^2 \right] \right)_\varepsilon, \varepsilon \in (0, 1] \end{aligned} \tag{1.2.13}$$

that is non vacuum Colombeau solution (see [18] section 6 and [19] subsection 2.3 Distributional general relativity) of the Einstein’s field equations

$$(G_i^k(\varepsilon))_\varepsilon = (R_i^k(\varepsilon))_\varepsilon - \frac{1}{2} \delta_i^k (R(\varepsilon))_\varepsilon = -g^2 \left(\frac{\varepsilon^2}{\Delta_\varepsilon^2(x)} \right)_\varepsilon. \tag{1.2.14}$$

For Rindler metric $a=0, g=1$ and we get

$$(G_i^k(\varepsilon))_\varepsilon = (R_i^k(\varepsilon))_\varepsilon - \frac{1}{2} \delta_i^k (R(\varepsilon))_\varepsilon = -\left(\frac{\varepsilon^2}{(x^2 + \varepsilon^2)^2} \right)_\varepsilon \in \mathbf{G}(\mathbb{R}). \tag{1.2.15}$$

Definition 1.2.1. Distributional Moller’s geometry that is Colombeau extension of the classical Moller’s spacetime given by Colombeau generalized fundamental tensor (1.2.13).

1.3. Distributional Schwarzschild Geometry as Colombeau Extension of the Classical Singular Schwarzschild Spacetime

1.3.1. Colombeau Extension of the Classical Singular Schwarzschild Spacetime Furnished with a Degenerate and Singular Schwarzschild Metric

As another important example of Colombeau extension of the singular

semi-Riemannian geometry we consider now classical singular Schwarzschild spacetime furnished with a degenerate and singular Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.3.1}$$

Remark 1.3.1. Note that formally corresponding to the metric (1.3.1) classical Levi-Civita connection given by canonical Christoffel symbols are [24]:

$$\begin{aligned} \Gamma_{00}^1(r)|_{r=2m} &= \lim_{r \rightarrow 2m} \frac{m(r-2m)}{r^3} = 0, \Gamma_{11}^1(r)|_{r=2m} = \lim_{r \rightarrow 2m} \frac{-m}{r(r-2m)} = \infty, \\ \Gamma_{01}^0(r)|_{r=2m} &= \lim_{r \rightarrow 2m} \frac{m}{r(r-2m)} = \infty, \\ \Gamma_{12}^2(r)|_{r=2m} &= \lim_{r \rightarrow 2m} \frac{1}{r} = 2^{-1}m^{-1}, \Gamma_{22}^1|_{r=2m} = -\lim_{r \rightarrow 2m} (r-2m) = 0, \\ \Gamma_{13}^3|_{r=2m} &= \lim_{r \rightarrow 2m} \frac{1}{r} = 2^{-1}m^{-1}, \Gamma_{33}^1|_{r=2m} = -\lim_{r \rightarrow 2m} (r-2m)\sin^2 \theta = 0, \\ \Gamma_{00}^1(r)|_{r=0} &= \lim_{r \rightarrow 0} \frac{m(r-2m)}{r^3} = \infty, \Gamma_{11}^1(r)|_{r=0} = \lim_{r \rightarrow 0} \frac{-m}{r(r-2m)} = \infty, \\ &\vdots \\ \Gamma_{33}^2 &= -\sin \theta \cos \theta, \Gamma_{23}^3 = \frac{\cos \theta}{\sin \theta}. \end{aligned} \tag{1.3.2}$$

i.e. classical Levi-Civita connection given by Equation (1.3.2) unavailable at Schwarzschild horizon.

Remark 1.3.2. Nevertheless in classical handbooks [24]-[37] were mistakenly assumed that classical semi-Riemannian geometry holds on whole Schwarzschild manifold and therefore canonical formal calculation gives

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6}. \tag{1.3.3}$$

By Equation (1.3.2) it is mistakenly pointed out that the Schwarzschild metric has only a coordinate singularity at $r = 2m$ and there is no gravitational singularity at Schwarzschild horizon.

Remark 1.3.3. Note that canonical formal calculation gives

$$\begin{aligned} R_{abcd}(r)R^{abcd}(r) &= \frac{16m^2}{r^6} + 4 \left[-\frac{m}{r\left(1 - \frac{2m}{r}\right)} \left[-\frac{m}{r^5} \left(1 - \frac{2m}{r}\right) \right] \right. \\ &+ 4 \left[-\frac{m}{r\left(1 - \frac{2m}{r}\right)} \sin^2 \theta \left[-\frac{m}{r^5 \sin^2 \theta} \left(1 - \frac{2m}{r}\right) \right] \right. \\ &+ \frac{4m}{r} \left(1 - \frac{2m}{r}\right) \frac{m}{r^5 \left(1 - \frac{2m}{r}\right)} + 8mr \sin^2 \theta \frac{2m}{r^7 \sin^2 \theta} \\ &+ \left. \frac{4m}{r} \left(1 - \frac{2m}{r}\right) \sin^2 \theta \frac{m}{r^5 \sin^2 \theta \left(1 - \frac{2m}{r}\right)} \right] \end{aligned} \tag{1.3.4}$$

Assume that $r \neq 0$ and $r \neq 2m$, i.e. $1 - \frac{2m}{r} \neq 0, \sin^2 \theta \neq 0$, then from Equation (1.3.4) one obtains directly

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} + 4\left[-\frac{m}{r}\right]\left[-\frac{m}{r^5}\right] + 4\left[-\frac{m}{r}\right]\left[-\frac{m}{r^5}\right] + \frac{4m^2}{r^6} + \frac{16m^2}{r^6} + \frac{4m^2}{r^6} = \frac{48m^2}{r^6} = \frac{12r_s^2}{r^6}. \tag{1.3.5}$$

Remark 1.3.4. Notice that: if $r = 2m$ then RHS of the Equation (1.3.4) become uncertainty

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} + 4\left[-\frac{m}{r}\right]\left[-\frac{m}{r^5}\right] + 4\left[-\frac{m}{r}\sin^2 \theta\right]\left[-\frac{m}{r^5\sin^2 \theta}\right] + \frac{4m}{r}\frac{m}{r^5} + 8mr\sin^2 \theta\frac{2m}{r^7\sin^2 \theta} + \frac{4m}{r}\sin^2 \theta\frac{m}{r^5\sin^2 \theta} = \frac{16m^2}{r^6} + \frac{0}{0}. \tag{1.3.6}$$

A. Einstein emphasized that uncertainty of the form 0/0 mentioned above that is a fundamental mathematical problem, see [38], p. 74. However in order to avoid this difficulty mentioned above in physical literature [24]-[37] one mistakenly defines

$$\begin{aligned} & \lim_{r \rightarrow 2m} R_{abcd}(r)R^{abcd}(r) \\ &= \lim_{r \rightarrow 2m} \left\{ \frac{16m^2}{r^6} + 4\left[-\frac{m}{r\left(1-\frac{2m}{r}\right)}\right]\left[-\frac{m}{r^5\left(1-\frac{2m}{r}\right)}\right] \right. \\ & \quad + 4\left[-\frac{m}{r\left(1-\frac{2m}{r}\right)}\sin^2 \theta\right]\left[-\frac{m}{r^5\sin^2 \theta\left(1-\frac{2m}{r}\right)}\right] \\ & \quad + \frac{4m}{r}\left(1-\frac{2m}{r}\right)\frac{m}{r^5\left(1-\frac{2m}{r}\right)} + 8mr\sin^2 \theta\frac{2m}{r^7\sin^2 \theta} \\ & \quad \left. + \frac{4m}{r}\left(1-\frac{2m}{r}\right)\sin^2 \theta\frac{m}{r^5\sin^2 \theta\left(1-\frac{2m}{r}\right)} \right\} = \frac{48m^2}{r^6} \end{aligned} \tag{1.3.7}$$

However Equation (1.3.7) doesn't holds at $r = 2m$ because classical Levi-Civita connection (1.3.2) of course is not available at Schwarzschild horizon, see Remark 1.3.1.

Remark 1.3.5. Thus from Equation (1.3.4) for $r \neq 0$ and $r \neq 2m$ we get

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} \Leftrightarrow (r \neq 0) \wedge (r \neq 2m), \tag{1.3.8}$$

and we get nothing at Schwarzschild horizon. Therefore semi-Riemannian geometry break down at Schwarzschild horizon [18] [19].

Remark 1.3.6. Recall that canonical derivation of the canonical singular Schwarzschild metric in classical handbooks is always based on assumption that:

Assumption 1.3.1. Classical semi-Riemannian geometry holds on the whole semi-Riemannian manifold, see for example [26].

Let ds^2 be the metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega^2, \tag{1.3.9}$$

where $A, B \rightarrow 1$ as $r \rightarrow \infty$. Then under Assumption 1.3.1 one obtains [26]:

(i) all $\Gamma_{\mu\nu}^1$ are zero except

$$\Gamma_{00}^1 = A'/2B, \Gamma_{11}^1 = B'/2B, \Gamma_{22}^1 = -r/B, \Gamma_{33}^1 = -(r/B)\sin^2\theta, \tag{1.3.10}$$

The equations $R_{\mu\nu} = 0, \mu, \nu = 0, 1, 2, 3$ are

$$\begin{aligned} R_{00} &= \Gamma_{00,1}^1 - 2\Gamma_{00}^1\Gamma_{01}^0 + \Gamma_{00}^1(\log\sqrt{-g})_{,1} \\ &= \left(\frac{A'}{2B}\right)' - \frac{A'^2}{2AB} + \frac{A'}{2B}\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right) \\ &= \frac{1}{2B}\left(A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r}\right) = 0, \end{aligned} \tag{1.3.11}$$

and

$$\begin{aligned} R_{11} &= -(\log\sqrt{-g})_{,1,1} + \Gamma_{11,1}^1 - (\Gamma_{10}^0)^2 - (\Gamma_{11}^1)^2 \\ &\quad - (\Gamma_{21}^2)^2 - (\Gamma_{31}^3)^2 + \Gamma_{11}^1(\log\sqrt{-g})_{,1} \\ &= \frac{1}{2A}\left(-A'' + \frac{A'B'}{2B} + \frac{A'^2}{2A} + \frac{2AB'}{rB}\right) = 0, \end{aligned} \tag{1.3.12}$$

and

$$\begin{aligned} R_{22} &= -(\log\sqrt{-g})_{,2,2} + \Gamma_{22,1}^1 - 2\Gamma_{22}^1\Gamma_{21}^2 - (\Gamma_{23}^3)^2 + \Gamma_{22}^1(\log\sqrt{-g})_{,1} \\ &= -\frac{d\cot\theta}{d\theta} - \left(\frac{r}{B}\right)' + \frac{2}{B} - \cot^2\theta - \frac{r}{B}\left(\frac{2}{r} + \frac{(AB)'}{2AB}\right) = 0. \end{aligned} \tag{1.3.13}$$

From Equation (1.3.11)-Equation (1.3.12) one obtains

$$\frac{2(AB)'}{rB} = 0. \tag{1.3.14}$$

Therefore $AB = \text{constant}$. Since at $r \rightarrow \infty$ we have A and $B \rightarrow 1$ one obtains $B = A^{-1}$. From Equation (1.3.13)-Equation (1.3.14) one obtains

$$\left(\frac{r}{B}\right)' = 1, \tag{1.3.15}$$

and by integration Equation (1.3.15) one obtains $r/B = r - 2m$, where $2m$ is an integration constant. Finally one obtains well known classical result

$$A(r) = 1 - \frac{2m}{r}, B(r) = \left(1 - \frac{2m}{r}\right)^{-1}. \tag{1.3.16}$$

From Equation (1.3.16) and consideration above (see Remark 1.3.4) Assumption 1.3.1 wrong, otherwise one obtains the contradiction.

Remark 1.3.7. In order to avoid this difficulty:

(i) we have introduced instead a classical Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi GT_{\mu\nu}, \tag{1.3.17}$$

[where the sign of the energy-momentum tensor is defined by (ρ is the energy density)]

$$T_{44} = -T_{00} = T_0^0 = \rho, \tag{1.3.18}$$

appropriate Colombeau generalization of the Equation (1.3.17)-Equation (1.3.18) such that

$$\left(R_{\mu\nu}(\varepsilon)\right)_\varepsilon - \frac{1}{2}\left(R(\varepsilon)g_{\mu\nu}(\varepsilon)\right)_\varepsilon = -8\pi G\left(T_{\mu\nu}(\varepsilon)\right)_\varepsilon, \tag{1.3.19}$$

where the sign of the distributional energy-momentum tensor is defined by

$$T_{44}(\varepsilon) = -T_{00}(\varepsilon) = T_0^0(\varepsilon) = \rho(\varepsilon) \in \mathbf{G}(M), \tag{1.3.20}$$

see [18] [19].

(ii) we have introduced instead of Assumption 1.3.1 the following assumption.

Assumption 1.3.2. Distributional semi-Riemannian geometry holds on the whole distributional semi-Riemannian manifold.

Definition 1.3.1. Let $A_\varepsilon^\pm(r), \varepsilon \in [0,1]$ and $B_\varepsilon^\pm(r), \varepsilon \in [0,1]$ the regularization of the functions $A^\pm(r)$ and $B^\pm(r)$ [defined above by Equation (1.3.16)] such that the following conditions are satisfied:

(i) $\left(A_\varepsilon^\pm(r)\right)_\varepsilon \in \mathbf{G}(\mathbb{R}_+)$ and $\left(B_\varepsilon^\pm(r)\right)_\varepsilon \in \mathbf{G}(\mathbb{R}_+), \varepsilon \in (0,1]$ are Colombeau generalized functions;

$$(ii) \quad A_0^\pm(r) = 1 - \frac{2m}{r}, B_0^\pm(r) = \left(1 - \frac{2m}{r}\right)^{-1}; \tag{1.3.21}$$

$$(iii) \quad \left(A_\varepsilon^\pm(2m)\right)_\varepsilon = \mp(\varepsilon) \in \tilde{\mathbb{R}}, \left(B_\varepsilon^\pm(2m)\right)_\varepsilon = \pm(\varepsilon^{-1}) \in \tilde{\mathbb{R}};$$

$$(iv) \quad \left(A_\varepsilon^\pm(0)\right)_\varepsilon = 1 - \frac{2m}{(\varepsilon)_\varepsilon} \in \tilde{\mathbb{R}}, \left(B_\varepsilon^\pm(0)\right)_\varepsilon = \left(1 - \frac{2m}{(\varepsilon)_\varepsilon}\right)^{-1} \in \tilde{\mathbb{R}}.$$

Let ds_ε^2 be the Colombeau metric

$$\left(ds_\varepsilon^2\right)_\varepsilon = -\left(A_\varepsilon^\pm(r)dt^2\right)_\varepsilon + \left(B_\varepsilon^\pm(r)dr^2\right)_\varepsilon + r^2d\Omega^2, \tag{1.3.22}$$

and let $\left(\Gamma_{\mu\nu}^\eta(\varepsilon)\right)_\varepsilon$ be the distributional Levi-Civita connection [18] [19] corresponding to Colombeau metric (1.3.22). Then under Assumption 1.3.2 one obtains:

(i) all $\left(\Gamma_{\mu 3bd}^1(\varepsilon)\right)_\varepsilon$ are zero except

$$\begin{aligned} \left(\Gamma_{00}^1(\varepsilon)\right)_\varepsilon &= \left(A_\varepsilon^t\right)_\varepsilon / 2\left(B_\varepsilon\right)_\varepsilon, \\ \left(\Gamma_{11}^1(\varepsilon)\right)_\varepsilon &= \left(B_\varepsilon^t\right)_\varepsilon / 2\left(B_\varepsilon\right)_\varepsilon, \\ \left(\Gamma_{22}^1(\varepsilon)\right)_\varepsilon &= -r/\left(B_\varepsilon\right)_\varepsilon, \\ \left(\Gamma_{33}^1(\varepsilon)\right)_\varepsilon &= -\left(r/B_\varepsilon\right)_\varepsilon \sin^2 \theta, \end{aligned} \tag{1.3.23}$$

$$\begin{aligned}
 (R_{00}(\varepsilon))_\varepsilon &= (\Gamma_{00,1}^1(\varepsilon))_\varepsilon - 2(\Gamma_{00}^1(\varepsilon))_\varepsilon (\Gamma_{01}^0(\varepsilon))_\varepsilon \\
 &\quad + (\Gamma_{00}^1(\varepsilon))_\varepsilon \left(\log \sqrt{-(g^\pm(\varepsilon))_\varepsilon} \right)_{,1} \\
 &= \left(\frac{(A'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} \right)' - \frac{(A_\varepsilon'^2)_\varepsilon}{2(A_\varepsilon)_\varepsilon (B_\varepsilon)_\varepsilon} + \frac{(A'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} \left(\frac{(A'_\varepsilon)_\varepsilon}{2(A_\varepsilon)_\varepsilon} + \frac{(B'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} + \frac{2}{r} \right) \quad (1.3.24) \\
 &= \frac{1}{2(B_\varepsilon)_\varepsilon} \left((A''_\varepsilon) - \frac{(A'_\varepsilon)_\varepsilon (B'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} - \frac{(A_\varepsilon'^2)_\varepsilon}{2(A_\varepsilon)_\varepsilon} + \frac{2(A'_\varepsilon)_\varepsilon}{r} \right),
 \end{aligned}$$

and

$$\begin{aligned}
 (R_{11}(\varepsilon))_\varepsilon &= -\left(\log \sqrt{-(g^\pm(\varepsilon))_\varepsilon} \right)_{,1,1} + (\Gamma_{11,1}^1(\varepsilon))_\varepsilon - (\Gamma_{10}^0(\varepsilon))^2 - \left((\Gamma_{11}^1(\varepsilon))^2 \right)_\varepsilon \\
 &\quad - \left((\Gamma_{21}^2(\varepsilon))^2 \right)_\varepsilon - \left((\Gamma_{31}^3(\varepsilon))^2 \right)_\varepsilon + (\Gamma_{11}^1(\varepsilon))_\varepsilon \left(\log \sqrt{-(g^\pm(\varepsilon))_\varepsilon} \right)_{,1} \quad (1.3.25) \\
 &= \frac{1}{2(A_\varepsilon)_\varepsilon} \left(-(A''_\varepsilon) + \frac{(A'_\varepsilon)_\varepsilon (B'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} + \frac{(A_\varepsilon'^2)_\varepsilon}{2(A_\varepsilon)_\varepsilon} + \frac{2(A_\varepsilon)_\varepsilon (B'_\varepsilon)_\varepsilon}{r(B_\varepsilon)_\varepsilon} \right),
 \end{aligned}$$

and

$$\begin{aligned}
 (R_{22}(\varepsilon))_\varepsilon &= -\left(\log \sqrt{-(g^\pm(\varepsilon))_\varepsilon} \right)_{,2,2} + \left((\Gamma_{22,1}^1(\varepsilon)) \right)_\varepsilon - 2\left((\Gamma_{22}^1(\varepsilon)) \right)_\varepsilon \left((\Gamma_{21}^2(\varepsilon)) \right)_\varepsilon \\
 &\quad - \left((\Gamma_{23}^3(\varepsilon))^2 \right)_\varepsilon + \Gamma_{22}^1(\varepsilon) \left(\log \sqrt{-(g^\pm(\varepsilon))_\varepsilon} \right)_{,1} \\
 &= -\frac{d \cot \theta}{d\theta} - \left(\frac{r}{(B_\varepsilon)_\varepsilon} \right)' + \frac{2}{(B_\varepsilon)_\varepsilon} - \cot^2 \theta - \frac{r}{(B_\varepsilon)_\varepsilon} \left(\frac{2}{r} + \frac{\left((A_\varepsilon)_\varepsilon (B_\varepsilon)_\varepsilon \right)'}{2(A_\varepsilon)_\varepsilon (B_\varepsilon)_\varepsilon} \right). \quad (1.3.26)
 \end{aligned}$$

Weak distributional limit in $D'(\mathbb{R}^3)$ of the RHS of the Equation (1.3.18), i.e. $w\text{-}\lim_{\varepsilon \rightarrow 0} T_{\mu\nu}(\varepsilon)$ is calculated in our papers [18] [19], see also **Appendix B**.

Remark 1.3.8. It turns out that the distributional Schwarzschild metric (1.3.22) has a gravitational singularity with compact support at origin $\{r=0\}$ [6]-[11] and at Schwarzschild horizon $\mathbf{S}^2 \times \{r=2m\}$ [18] [19].

1.3.2. Colombeau Extension of the Schwarzschild Spacetime in Isotropic Coordinates

Let us consider now nonclassical spacetime furnished with a degenerate at horizon $\bar{\rho} = r_s/4$ but nonsingular (at horizon) metric and known in physical literature as Schwarzschild spacetime in isotropic coordinates [24]:

$$ds^2 = -\left(\frac{1-r_s/4\rho}{1+r_s/4\rho} \right)^2 c^2 dt^2 + \left(1 + \frac{r_s}{4\rho} \right)^4 \left[d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (1.3.27)$$

Nonsingular metric (1.3.27) is obtained by the coordinate transformation: $r = \rho(1+r_s/4\rho)^2$, between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ . Under formal calculation one obtains [24]:

$$\Gamma_{t\rho}^t = \frac{8r_s}{16\rho^2 - r_s}, \quad (1.3.28)$$

i.e. classical Levi-Civita connection given by (1.3.28) of course unavaluble at horizon $\bar{\rho} = r_s/4$. However in physical literature under ubnormal calculation it was mistakenly pointed out that the Ricci tensor and the Ricci scalar vanish identically and Kretschman scalar is

$$K(\rho) = R_{abcd}(\rho)R^{abcd}(\rho) = \frac{3 \times 4^{13} \rho^6 r_s^2}{(4\rho + r_s)^{12}}. \tag{1.3.29}$$

Remark 1.3.9. In order to avoid difficulty with the degeneracy of the metric (1.3.27) mentioned above, we consider now the corresponding distributional Colombeau metric which reads

$$(ds_\varepsilon^2)_\varepsilon = -\frac{((\rho - \bar{\rho})^2 + \varepsilon^2)_\varepsilon}{(\rho + \bar{\rho})^2} c^2 dt^2 + \left(1 + \frac{\bar{\rho}}{\rho}\right)^4 \left[d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\varphi) \right], \tag{1.3.30}$$

where $\varepsilon \in (0, 1]$.

Definition 1.3.2. Distributional Schwarzschild geometry in isotropic coordinates which is Colombeau extension of the classical spacetime (1.3.27), given by Colombeau generalized fundamental tensor (1.3.30).

Colombeau generalized metric (1.3.30) nondegenerate at horizon in Colombeau sence and distributional Levi-Civita connection now available on the whole distributional Schwarzschild spacetime in isotropic coordinates. Notice that generalized metric (1.3.30) has the form given by Equation (A.1) (see **Appendix A**) with

$$\begin{aligned} A_\varepsilon(\rho) &= ((\rho - \bar{\rho})^2 + \varepsilon^2)(\rho + \bar{\rho})^{-2}, \\ B_\varepsilon(\rho) &= \left(1 + \frac{\bar{\rho}}{\rho}\right)^4, D_\varepsilon(\rho) = 0, C_\varepsilon(\rho) = 0 \end{aligned} \tag{1.3.31}$$

$$\text{Thus } \Delta_\varepsilon(\rho) = A_\varepsilon(\rho)B_\varepsilon(\rho) = ((\rho - \bar{\rho})^2 + \varepsilon^2)(\rho + \bar{\rho})^2 \rho^{-1}.$$

From Equation (A.2) (see **Appendix A2**) and Equation (1.3.31) in the limit $(|\rho_\varepsilon - \bar{\rho}|)_\varepsilon \approx_{\mathbb{R}} 0$, we get

$$\begin{aligned} (\mathbf{R}(\rho_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} -\left(\frac{O(\bar{\rho})\varepsilon^2}{\left[(\rho_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^2} \right)_\varepsilon, \\ (\mathbf{R}^{\mu\nu}(\rho_\varepsilon, \varepsilon)\mathbf{R}_{\mu\nu}(\rho_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} \left(\frac{O(\bar{\rho})\varepsilon^4}{\left[(\rho_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right)_\varepsilon + O(\bar{\rho}), \\ (\mathbf{R}^{\rho\sigma\mu\nu}(\rho_\varepsilon, \varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(\rho_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} \left(\frac{O(\bar{\rho})\varepsilon^4}{\left[(\rho_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right)_\varepsilon + O(\bar{\rho}). \end{aligned} \tag{1.3.32}$$

Compare the equation $(\mathbf{R}(\rho, \varepsilon))_\varepsilon \asymp -O(\bar{\rho})\varepsilon^2 \left[(\rho - \bar{\rho})^2 + \varepsilon^2 \right]^{-2}$ with Equation (1.2.12).

Remark 1.3.10. Notice that in contrast with result of naive formal calculation mentioned above (see Equation (1.3.29)) we get:

- (i) $\left(\mathbf{R}^{\rho\sigma\mu\nu}(\rho, \varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(\rho, \varepsilon)\right)_\varepsilon \in \mathbf{G}(\mathbb{R}^3)$
- (ii) $\left(\mathbf{R}^{\mu\nu}(\rho_\varepsilon, \varepsilon)\mathbf{R}_{\rho\nu}(\rho, \varepsilon)\right)_\varepsilon \in \mathbf{G}(\mathbb{R}^3) \setminus C^\infty(\mathbb{R}^3),$
- (iii) $\left(\mathbf{R}^{\rho\sigma\mu\nu}(\rho_\varepsilon, \varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(\rho_\varepsilon, \varepsilon)\right)_\varepsilon \in \mathbf{G}(\mathbb{R}^3) \setminus C^\infty(\mathbb{R}^3).$

1.4. On the Near Horizon Colombeau Approximation for the Classical Singular Schwarzschild Black Hole Geometry

Let us perform the following coordinate transformation

$$\bar{t} = \frac{t}{4m}, \quad \bar{r}_\varepsilon = \sqrt{8m(r-2m) + \varepsilon^2}, \quad \varepsilon \in (0,1] \tag{1.4.1}$$

to the classical singular Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.4.2}$$

we get

$$ds_\varepsilon^2 = -\bar{r}_\varepsilon^2 \left(1 + \frac{\bar{r}_\varepsilon^2}{16m^2}\right)^{-1} d\bar{t}^2 + \left(1 + \frac{\bar{r}_\varepsilon^2}{16m^2}\right) d\bar{r}^2 + 4m^2 \left(1 + \frac{\bar{r}_\varepsilon^2}{16m^2}\right)^2 d\Omega^2. \tag{1.4.3}$$

In Equation (1.4.2), m is the central mass, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $G = c = 1$. Taking the limit $m \rightarrow \infty$, the spherical horizon becomes planar and Equation (1.4.3) leads to the Colombeau type metric

$$\left(ds_\varepsilon^2\right)_\varepsilon = -\left\{\left(\bar{r}_\varepsilon^2\right)_\varepsilon\right\} d\bar{t}^2 + \left(d\bar{r}_\varepsilon^2\right)_\varepsilon + 4m^2 d\Omega^2 \tag{1.4.4}$$

which is distributional Rindler’s spacetime if we neglect the angular contribution. The condition $m \rightarrow \infty$ is equivalent to the “near horizon approximation” for the exterior geometry of a black hole: for $r \approx 2m$ ($r > 2m$) the line element (1.4.2) appears, indeed, as

$$ds^2 = -\frac{r-2m}{2m} dt^2 + \frac{2m}{r-2m} dr^2 + 4m^2 d\Omega^2. \tag{1.4.5}$$

By using simple coordinate transformations it could be shown that (1.4.5) again becomes the distributional Rindler metric when we take $\theta, \phi = \text{const}$ or $\Delta\theta$ and $\Delta\phi$ are negligible. We stress that the condition $r \approx 2m$ only is not enough to obtain Rindler’s spacetime which has no spherical symmetry as Schwarzschild.

Remark 1.4.1. At this stage of consideration, it is already clear that near horizon Schwarzschild black hole geometry has a gravitational singularity at horizon. Notice that in classical literature (see, for example, [24]-[36]) near horizon Schwarzschild black hole geometry were mistakenly accepted as regular with the Ricci tensor and the Ricci scalar vanish identically.

1.5. Colombeau Distributional Semi-Riemannian Geometry. Preliminaries

1.5.1. The Ring of Colombeau Generalized Numbers $\tilde{\mathbb{R}}$

Designation 1.5.1. We denote by $\tilde{\mathbb{R}}$ the ring of real, Colombeau generalized

numbers. Recall that [2] [3] by definition

$$\begin{aligned} \tilde{\mathbb{R}} &= \mathbf{E}_{\mathbb{R}}(\mathbb{R}) / \mathbf{N}(\mathbb{R}) \text{ where} \\ \mathbf{E}_{\mathbb{R}}(\mathbb{R}) &= \left\{ (x_{\varepsilon})_{\varepsilon} \in \mathbb{R}^{(0,1)} \mid (\exists a \in \mathbb{R}_+) (\exists \varepsilon_0 \in (0,1)) (\forall \varepsilon \leq \varepsilon_0) [|x_{\varepsilon}| \leq \varepsilon^{-a}] \right\}, \\ \mathbf{N}(\mathbb{R}) &= \left\{ (x_{\varepsilon})_{\varepsilon} \in \mathbb{R}^{(0,1)} \mid (\forall a \in \mathbb{R}_+) (\exists \varepsilon_0 \in (0,1)) (\forall \varepsilon \leq \varepsilon_0) [|x_{\varepsilon}| \leq \varepsilon^a] \right\}. \end{aligned} \tag{1.5.0}$$

Designation 1.5.2. In the sequel we denote by:

$\mathbf{E}[\Omega]$, where Ω an open subset of \mathbb{R}^n , the algebra of all the sequences $(u_{\varepsilon}(x))_{\varepsilon \in (0,1]}$ (for short, $(u_{\varepsilon}(x))_{\varepsilon}$) of smooth functions $u_{\varepsilon}(x) \in C^{\infty}(\Omega)$.

$\mathbf{E}_M[\Omega]$ is the differential subalgebra of the elements $(u_{\varepsilon}(x))_{\varepsilon} \in \mathbf{E}[\Omega]$ such that for all $K \subseteq \Omega$, for all $\alpha \in \mathbb{N}^n$ there exists $N \in \mathbb{N}$ with the following property: $\sup_{x \in K} |\partial^{\alpha} u_{\varepsilon}(x)| = O(\varepsilon^{-N})$ as $\varepsilon \rightarrow 0$.

$\mathbf{N}(\Omega)$ the differential subalgebra of the elements $(u_{\varepsilon}(x))_{\varepsilon} \in \mathbf{E}[\Omega]$ such that for all $K \subseteq \Omega$, for all $\alpha \in \mathbb{N}^n$ and $q \in \mathbb{N}$ the following property holds: $\sup_{x \in K} |\partial^{\alpha} u_{\varepsilon}(x)| = O(\varepsilon^q)$ as $\varepsilon \rightarrow 0$.

Definition 1.5.1. The elements of $\mathbf{E}_M[\Omega]$ and $\mathbf{N}(\Omega)$ are called moderate and negligible, respectively. The factor algebra

$$\mathbf{G}(\Omega) = \mathbf{E}_M[\Omega] / \mathbf{N}(\Omega) \tag{1.5.1}$$

is the algebra of Colombeau generalized functions on Ω .

Remark 1.5.0. Note that: (i) there exists natural embedding $\tilde{r} : \mathbb{R} \rightarrow \tilde{\mathbb{R}}$ such that for all $r \in \mathbb{R}$, $\tilde{r} : r \rightarrow (r_{\varepsilon})_{\varepsilon}$, $r_{\varepsilon} = r$, for all $\varepsilon \in [1,0)$, (ii) the ring $\tilde{\mathbb{R}}$ can be endowed with the structure of a partially ordered ring: for $\mathbf{r}, \mathbf{s} \in \tilde{\mathbb{R}}$, $\mathbf{r} \leq_{\tilde{\mathbb{R}}} \mathbf{s}$ if and only if there are representatives $(r_{\varepsilon})_{\varepsilon}$ and $(s_{\varepsilon})_{\varepsilon}$ with $r_{\varepsilon} \leq s_{\varepsilon}$ for all $\varepsilon \in [1,0)$.

Definition 1.5.2. (i) Let $\delta \in \tilde{\mathbb{R}}$. We say that δ is infinite small Colombeau generalized number and abbreviate

$$\delta \approx_{\tilde{\mathbb{R}}} \tilde{0}$$

or $\delta \approx_{\tilde{\mathbb{R}}} 0$, if there exists representative $(\delta_{\varepsilon})_{\varepsilon}$, $\varepsilon \in (0,1]$ and some $q \in \mathbb{N}$ such that $|\delta_{\varepsilon}| = O(\varepsilon^q)$, as $\varepsilon \rightarrow 0$. We abbreviate $\alpha \approx_{\tilde{\mathbb{R}}} \beta$ iff $\alpha - \beta \approx_{\tilde{\mathbb{R}}} 0$.

(ii) We say that $\Delta \in \tilde{\mathbb{R}}$ is infinite large Colombeau generalized number and abbreviate

$$\Delta \approx_{\tilde{\mathbb{R}}} \infty$$

if $\Delta^{-1} \approx_{\tilde{\mathbb{R}}} 0$.

(iii) We say that $\mathbf{r} \in \tilde{\mathbb{R}}$ is finite Colombeau generalized number and abbreviate \mathbf{r}_{fin} if there are $r_1, r_2 \in \mathbb{R}$ such that $\tilde{r}_1 \leq \mathbf{r} \leq \tilde{r}_2$.

Definition 1.5.3. Let $\Omega_{1,M}$ and $\Omega_{2,M}$ be a set

$$\begin{aligned} \Omega_{1,M} &= \left\{ (x_{\varepsilon})_{\varepsilon} \in \Omega^{(0,1]} \mid \exists N \in \mathbb{N}, |x_{\varepsilon}| = O(\varepsilon^{-N}), \text{ as } \varepsilon \rightarrow 0 \right\}, \\ \Omega_{2,M} &= \left\{ (x_{\varepsilon})_{\varepsilon} \in \Omega^{(0,1]} \mid \exists N \in \mathbb{N}, |x_{\varepsilon}| = O(\varepsilon^N), \text{ as } \varepsilon \rightarrow 0 \right\}. \end{aligned}$$

correspondingly. We introduce equivalence relation given by

$$(x_{\varepsilon})_{\varepsilon} \sim (y_{\varepsilon})_{\varepsilon} \Leftrightarrow \forall q \in \mathbb{N}, |x_{\varepsilon} - y_{\varepsilon}| = O(\varepsilon^q), \text{ as } \varepsilon \rightarrow 0,$$

and denote by $\tilde{\Omega} = (\Omega_{1,M} \cup \Omega_{2,M}) / \sim$ the set of generalized points. Moreover, if $[(x_\varepsilon)_\varepsilon]$ is the class of $(x_\varepsilon)_\varepsilon$ in $\tilde{\Omega}$ then the set of compactly generalized points is

$$\tilde{\Omega}_c = \left\{ x = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega} \mid \exists K (K \Subset \Omega) \exists \eta (\eta > 0) \forall \varepsilon (\varepsilon \in (0, \eta)) [x_\varepsilon \in K] \right\}.$$

Note that if the $\tilde{\Omega}_c$ -property holds for one representative of $x \in \tilde{\Omega}_c$ then it holds for every representative. Also, for $\Omega = \mathbb{R}$ we have that the factor \mathbb{R}_M / \sim is the usual algebra of real generalized numbers.

Definition 1.5.4. We denote by $\tilde{\Omega}_{IL} \triangleq \Omega_{1,M} / \sim$ and $\tilde{\Omega}_{ISM} \triangleq \Omega_{2,M} / \sim$ the set of inite large and infite small generalized points correspondingly. It is clear that the generalized point value of $u(x)$ at $x \in \tilde{\Omega}_{ISM}$ is

$$u(x) = (u_\varepsilon(x_\varepsilon))_\varepsilon + \mathbf{N}(\Omega).$$

Definition 1.5.5. Let $\delta \in \tilde{\mathbb{R}}$ be infinite small Colombeau generalized number with representative $\delta' = (\delta_\varepsilon)_\varepsilon, \varepsilon \in (0, 1]$. We introduce a norm $\|\delta'\|$ of such representative by formula $\|\delta'\| = \sup_{\varepsilon \in (0, 1]} |\delta_\varepsilon|$.

1.5.2. A Real Colombeau Vector Bundle

Definition 1.5.6. A real vector bundle consists of:

- 1) Topological spaces X (base space) and E (total space)
- 2) A continuous surjection $\pi : E \rightarrow X$ (bundle projection)
- 3) For every x in X , the structure of a finite-dimensional vector space over Colombeau ring $\tilde{\mathbb{R}}$ on the fiber $\pi^{-1}(\{x\})$ where the following compatibility condition is satisfied: for every point in X , there is an open neighborhood U , a natural number k , and a homeomorphism

$\varphi : U \times \tilde{\mathbb{R}}^k \rightarrow \pi^{-1}(U)$ such that for all $x \in U$, $(\pi \circ \varphi)(x, v) = x$ for all vectors v in $\tilde{\mathbb{R}}^k$, and the map $v \mapsto \varphi(x, v)$ is a linear isomorphism between the vector spaces $\tilde{\mathbb{R}}^k$ and $\pi^{-1}(\{x\})$.

The open neighborhood U together with the homeomorphism φ is called a local trivialization of the Colombeau vector bundle. The local trivialization shows that locally the map π “looks like” the projection of $U \times \tilde{\mathbb{R}}^k$ on U .

The Cartesian product $X \times \tilde{\mathbb{R}}^k$, equipped with the projection $X \times \tilde{\mathbb{R}}^k \rightarrow X$, is called the trivial bundle of rank k over X .

1.5.3. The Algebra of Colombeau Generalized Functions

The basic idea of Colombeau’s theory of generalized functions is a regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter $\varepsilon \in (0, 1]$. Let $(u_\varepsilon)_{\varepsilon \in (0, 1]}$ with $u_\varepsilon \in C^\infty(M)$ for all $\varepsilon \in (0, 1]$ (M a separable, smooth orientable Hausdorff manifold of dimension n). The algebra of Colombeau generalized functions on M is defined as the quotient

$$\mathbf{G}(M) = \mathbf{E}_M(M) / \mathbf{N}(M) \tag{1.5.2}$$

of the space $\mathbf{E}_M(M)$ of sequences of moderate growth modulo the space $\mathbf{N}(M)$ of negligible sequences. More precisely the notions of moderateness

resp. negligibility are defined by the following asymptotic estimates ($\tilde{\mathbf{X}}_{\mathbb{R}}(M)$ or $\tilde{\mathbf{X}}(M)$) denoting the space of smooth vector fields on M .

$$\begin{aligned} \mathbf{E}_M(M) &= \left\{ (u_\varepsilon)_\varepsilon \in C^\infty(M)^{(0,1]} \mid (\forall K \subset\subset M)(\forall k \in \mathbb{N}_0)(\exists n \in \mathbb{N}) \right. \\ &\quad \left. (\forall \xi_1 \in \tilde{\mathbf{X}}(M), \dots, \forall \xi_k \in \tilde{\mathbf{X}}(M)) \left[\sup_{p \in K} |L_{\xi_1} \dots L_{\xi_n} u_\varepsilon(p)| \leq O(\varepsilon^{-n}) \right] \right\} \\ \mathbf{N}(M) &= \left\{ (u_\varepsilon)_\varepsilon \in M^{(0,1]} \mid (\forall K \subset\subset M)(\forall k, q, l \in \mathbb{N}_0) \right. \\ &\quad \left. (\forall \xi_1 \in \tilde{\mathbf{X}}(M), \dots, \forall \xi_k \in \tilde{\mathbf{X}}(M)) \left[\sup_{p \in K} |L_{\xi_1} \dots L_{\xi_n} u_\varepsilon(p)| \leq O(\varepsilon^q \delta^l) \right] \right\}. \end{aligned} \tag{1.5.3}$$

Elements of $G(M)$ are denoted by

$$\bar{u} = \mathbf{cl}[(u_\varepsilon)_\varepsilon] = \overline{(u_\varepsilon)_\varepsilon} = (u_\varepsilon)_\varepsilon + \mathbf{N}(M). \tag{1.5.4}$$

With componentwise operations $G(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by

$$L_\xi \bar{u} := \mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon] = \overline{(L_\xi u_\varepsilon)_\varepsilon}. \tag{1.5.5}$$

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $\bar{u} \in G(M)$ iff $\bar{u} \circ \psi_\alpha \in G(\psi_\alpha(V_\alpha))$ for all charts (V_α, ψ_α) , where on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives. Smooth functions are embedded into $G(M)$ simply by the “constant” embedding σ , i.e., $\sigma(f) := \mathbf{cl}[(f)_\varepsilon]$, hence $C^\infty(M)$ is a faithful subalgebra of $G(M)$. On open sets of \mathbb{R}^n compactly supported distributions are embedded into G via convolution with a mollifier $\rho \in S(\mathbb{R}^n)$ with unit integral satisfying $\int \rho(x) x^\alpha dx = 0$ for all $|\alpha| \geq 1$; more precisely setting $\rho_\varepsilon(x) = (1/\varepsilon^n) \rho(x/\varepsilon)$ we have $\iota(w) := \mathbf{cl}[(w * \rho_\varepsilon)_\varepsilon]$. In case $\text{supp}(w)$ is not compact one uses a sheaf-theoretical construction.

1.5.4. Colombeau Tangent Vector

Let $\bar{f} = \mathbf{cl}[(f_\varepsilon(\mathbf{x}))_\varepsilon] = \overline{(f_\varepsilon(\mathbf{x}))_\varepsilon} \in G(\mathbb{R}^n)$, where $f_\varepsilon(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}, \varepsilon \in (0,1)$ is a differentiable function and let \mathbf{v} be a vector in \mathbb{R}^n . We define the Colombeau directional derivative in the \mathbf{v} direction at a point $x \in \mathbb{R}^n$ by

$$\begin{aligned} D_{\mathbf{v}}^{Col}[\bar{f}] &= D_{\mathbf{v}} \left[\overline{(f_\varepsilon(\mathbf{x}))_\varepsilon} \right] = \overline{(D_{\mathbf{v}} f_\varepsilon(\mathbf{x}))_\varepsilon} \\ &= \overline{\left[\left(\frac{d}{dt} f_\varepsilon(\mathbf{x} + t\mathbf{v}) \Big|_{t=0} \right)_\varepsilon \right]} = \overline{\left[\left(\sum_{i=1}^n \mathbf{v}_i \frac{\partial f_\varepsilon(\mathbf{x})}{\partial x_i} \right)_\varepsilon \right]}. \end{aligned} \tag{1.5.6}$$

The Colombeau tangent vector at the point x may then be defined as

$$\mathbf{v}^{Col}(\bar{f}) = \mathbf{v} \left[\overline{(f_\varepsilon(\mathbf{x}))_\varepsilon} \right] = \overline{(D_{\mathbf{v}} f_\varepsilon(\mathbf{x}))_\varepsilon}. \tag{1.5.7}$$

Let $\bar{f} = \overline{(f_\varepsilon(\mathbf{x}))_\varepsilon} \in G(\mathbb{R}^n), \bar{g} = \overline{(g_\varepsilon(\mathbf{x}))_\varepsilon} \in G(\mathbb{R}^n)$, where $f_\varepsilon, g_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}, \varepsilon \in (0,1)$ be differentiable functions, let \mathbf{v}, \mathbf{w} be tangent vectors in \mathbb{R}^n at $x \in \mathbb{R}^n$ and let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
 (a \cdot \mathbf{v} + b \cdot \mathbf{w})^{Col}(\bar{f}) &= (a \cdot \mathbf{v} + b \cdot \mathbf{w})\left(\overline{[(f_\varepsilon)_\varepsilon]}\right) \\
 1) \quad &= a\mathbf{v}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right) + b\mathbf{w}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right) \\
 &= a\mathbf{v}^{Col}(\bar{f}) + b\mathbf{w}^{Col}(\bar{f}); \\
 \mathbf{v}^{Col}(a \cdot \bar{f} + b \cdot \bar{g}) &= \mathbf{v}\left(a \cdot \overline{[(f_\varepsilon)_\varepsilon]} + b \cdot \overline{[(g_\varepsilon)_\varepsilon]}\right) \\
 2) \quad &= a \cdot \mathbf{v}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right) + b \cdot \mathbf{v}\left(\overline{[(g_\varepsilon)_\varepsilon]}\right) \\
 &= a \cdot \mathbf{v}^{Col}(\bar{f}) + b \cdot \mathbf{v}^{Col}(\bar{g}); \\
 \mathbf{v}^{Col}(\bar{f} \cdot \bar{g}) &= \mathbf{v}\left(\overline{[(f_\varepsilon \cdot g_\varepsilon)_\varepsilon]}\right) \\
 3) \quad &= \overline{[(f_\varepsilon(\mathbf{x}))_\varepsilon]} \cdot \mathbf{v}\left(\overline{[(g_\varepsilon)_\varepsilon]}\right) + \overline{[(g_\varepsilon(\mathbf{x}))_\varepsilon]} \cdot \mathbf{v}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right) \\
 &= \bar{f} \cdot \mathbf{v}^{Col}(\bar{g}) + \bar{g} \cdot \mathbf{v}^{Col}(\bar{f}).
 \end{aligned}$$

1.5.5. Colombeau Tangent Vector to Differentiable Manifold M

Let M be a differentiable manifold and let $\mathbf{G}(M)$ be the algebra of real-valued Colombeau generalized functions on M . Then the tangent vector to M at a point x in the manifold is given by the derivation $D_v : \mathbf{G}(M) \rightarrow \tilde{\mathbb{R}}$ which shall be linear—i.e., for any $\bar{f} = \overline{[(f_\varepsilon)_\varepsilon]}, \bar{g} = \overline{[(g_\varepsilon)_\varepsilon]} \in \mathbf{G}(M)$ and $a, b \in \tilde{\mathbb{R}}$ we have

$$\begin{aligned}
 D_v^{Col}(a \cdot \bar{f} + b \cdot \bar{g}) &= D_v\left(a \cdot \overline{[(f_\varepsilon)_\varepsilon]} + b \cdot \overline{[(g_\varepsilon)_\varepsilon]}\right) \\
 1) \quad &= a \cdot D_v\left(\overline{[(f_\varepsilon)_\varepsilon]}\right) + b \cdot D_v\left(\overline{[(g_\varepsilon)_\varepsilon]}\right) \\
 &= a \cdot D_v^{Col}(\bar{f}) + b \cdot D_v^{Col}(\bar{g}).
 \end{aligned}$$

Note that the derivation will by definition have the Leibniz property

$$\begin{aligned}
 D_v^{Col}(\bar{f} \cdot \bar{g}) &= D_v\left(\overline{[(f_\varepsilon \cdot g_\varepsilon)_\varepsilon]}\right) \\
 2) \quad &= D_v\left(\overline{[(f_\varepsilon)_\varepsilon]} \cdot \overline{[(g_\varepsilon(x))_\varepsilon]} + \overline{[(f_\varepsilon(x))_\varepsilon]} \cdot \overline{[(g_\varepsilon)_\varepsilon]}\right) \\
 &= D_v^{Col}(\bar{f}) \cdot \bar{g} + \bar{f} \cdot D_v^{Col}(\bar{g}).
 \end{aligned}$$

1.5.6. Colombeau Vector Fields on Distributional Manifolds

Colombeau vector field $\tilde{\mathbf{X}}_{\tilde{\mathbb{R}}}$ (denoted often by $\tilde{\mathbf{X}}$) on a manifold M is a linear map $\tilde{\mathbf{X}}_{\tilde{\mathbb{R}}} :$

$\mathbf{G}(M) \rightarrow \mathbf{G}(M)$ such that for all $\bar{f}, \bar{g} \in \mathbf{G}(M) :$

$$\tilde{\mathbf{X}}_{\tilde{\mathbb{R}}}(\bar{f} \cdot \bar{g}) = \bar{f} \cdot \tilde{\mathbf{X}}_{\tilde{\mathbb{R}}}(\bar{g}) + \tilde{\mathbf{X}}_{\tilde{\mathbb{R}}}(\bar{f}) \cdot \bar{g}. \tag{1.5.8}$$

1.5.7. Colombeau Tangent Space

Suppose now that M is a C^∞ manifold. A real-valued Colombeau generalized function $(f_\varepsilon)_\varepsilon : M \rightarrow \tilde{\mathbb{R}}, \varepsilon \in (0, 1]$ is said to belong to $\mathbf{G}(M)$ if and only if for every coordinate chart $\varphi : U \rightarrow \mathbb{R}^n$, the map $f_\varepsilon \circ \varphi^{-1} : \varphi[U] \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is infinitely differentiable. Note that $\mathbf{G}(M)$ is a real associative algebra with respect to the pointwise product and sum of Colombeau generalized functions.

Pick a point $x \in M$. A derivation at x is defined as a linear map $D : \mathbf{G}(M) \rightarrow \tilde{\mathbb{R}}$ that satisfies the Leibniz identity:

$$\bar{f} = \overline{(f_\varepsilon)}, \bar{g} = \overline{(g_\varepsilon)} \in \mathbf{G}(M) : D(\bar{f} \cdot \bar{g}) = D(\bar{f}) \cdot \bar{g}(x) + \bar{f}(x) \cdot D(\bar{g}),$$

which is modeled on the product rule of calculus.

If we define addition and scalar multiplication on the set of derivations at x by

$$(D_1 + D_2)(\bar{f}) = \bar{f} \cdot D_1(\bar{f}) + D_2(\bar{f}) \quad \text{and}$$

$$(\lambda \cdot D)(\bar{f}) = \bar{f} \cdot \lambda \cdot D(\bar{f}),$$

where $\lambda \in \tilde{\mathbb{R}}$, then we obtain a real vector space over $\tilde{\mathbb{R}}$, which we define as the Colombeau tangent space $T_x^{Col} M$ of M at x .

1.5.8. Colombeau Dual Space

Given any vector space $V_{\tilde{\mathbb{R}}}$ over Colombeau algebra $\tilde{\mathbb{R}}$, the (algebraic) Colombeau dual space $V_{\tilde{\mathbb{R}}}^*$ (also denoted for a short by V^*) is defined as the set of all linear maps $\varphi : V_{\tilde{\mathbb{R}}} \rightarrow \tilde{\mathbb{R}}$. Since linear maps are vector space homomorphisms, the Colombeau dual space is also sometimes denoted by $Hom(V, \tilde{\mathbb{R}})$. The Colombeau dual space $V_{\tilde{\mathbb{R}}}^*$ itself becomes a vector space over $\tilde{\mathbb{R}}$ when equipped with an addition and scalar multiplication satisfying: (i) $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ and (ii) $\alpha\varphi(x) = \varphi(\alpha x)$, where $\varphi(x), \psi(x) \in V_{\tilde{\mathbb{R}}}^*, \alpha \in \tilde{\mathbb{R}}$.

1.5.9. Colombeau Cotangent Space

Let M be a smooth manifold and let x be a point in M . Let $T_x M$ be Colombeau tangent space at x . Then Colombeau cotangent space at x is defined as the Colombeau dual space of $T_x M : T_x^* M = (T_x M)_{\tilde{\mathbb{R}}}^*$.

Suppose now that M is a C^∞ manifold and let $f \in \mathbf{G}(M)$. The differential of f at a point x is the map: $df_x(X_x) = X_x(f)$ where X_x is a tangent vector at x , thought of as a derivation. In either case, df_x is a linear map on $T_x M$ and hence it is a tangent covector at x .

We can then define the differential map $d : \mathbf{G}(M) \rightarrow T_x^* M$ at a point x as the map which sends f to df_x . Properties of the differential map include:

$$(i) \quad d(af + bg) = adf + bdf \quad \text{for } a, b \in \tilde{\mathbb{R}}, \quad (ii) \quad d(fg) = f(x)dg_x + g(x)df_x.$$

Let $\varphi_\varepsilon : M \rightarrow N$ for any $\varepsilon \in (0, 1]$ be a smooth map of smooth manifolds. Given some $x \in M$, the Colombeau differential of $(\varphi_\varepsilon)_\varepsilon$ at x is a linear map $(d\varphi_{\varepsilon,x})_\varepsilon : T_x M \rightarrow T_{(\varphi_\varepsilon(x))_\varepsilon} N$ from Colombeau tangent space of M at x to Colombeau tangent space of N at $(\varphi_\varepsilon(x))_\varepsilon$. The application of $(d\varphi_{\varepsilon,x})_\varepsilon$ to a tangent vector X is called the pushforward of X by $(\varphi_\varepsilon)_\varepsilon$.

1.5.10. $\mathbf{G}(M)$ -Module of Generalized Sections $\mathbf{G}(M, E)$ of a Vector Bundle $E \rightarrow M$

The $\mathbf{G}(M)$ -module of generalized sections $\mathbf{G}(M, E)$ of a vector bundle $E \rightarrow M$ and in particular the space of generalized tensor fields $\mathbf{G}_s^r(M)$ is defined along the same lines using analogous asymptotic estimates with respect

to the norm induced by any Riemannian metric on the respective fibers. We denote generalized sections by $S = \mathbf{cl}[(s_\varepsilon)_\varepsilon] = (s_\varepsilon)_\varepsilon + N(M, E)$. Alternatively we may describe a section $S \in \mathbf{G}(M, E)$ by a family $(S_\alpha)_\alpha = \left((S_\alpha^i) \right)_{i=1}^N$, where S_α is called the local expression of S with its components $S_\alpha^i \triangleq \Psi_\alpha^i \circ S \circ \Psi_\alpha^{-1} \in \mathbf{G}(\Psi_\alpha(V_\alpha))$ ($(V_\alpha, \Psi_\alpha)_\alpha$ a vector bundle atlas and $i = 1, \dots, N$, with N denoting the dimension of the fibers) satisfying $S_\alpha^i(x) = (\Psi_{\alpha\beta})_j^i(\Psi_\beta \circ \Psi_\alpha^{-1}(x)) S_\beta^j(\Psi_\beta \circ \Psi_\alpha^{-1}(x))$ for all $x \in (V \cap V)$, where $\Psi_{\alpha\beta}$ denotes the transition functions of the bundle.

Remark 1.5.1. Smooth sections of $E \rightarrow M$ again may be embedded as constant nets, *i.e.*,

$$\Theta(s) : s \rightarrow \mathbf{cl}[(s)_\varepsilon].$$

Since $C^\infty(M)$ is a subring of $\mathbf{G}(M), \mathbf{G}(M, E)$ also may be viewed as $C^\infty(M)$ -module and the two respective module structures are compatible with respect to the embeddings.

Moreover we have the following algebraic characterization of the space of generalized sections

$$\mathbf{G}(M, E) = \mathbf{G}(M) \otimes \Gamma(M, E), \tag{1.5.9}$$

where $\Gamma(M, E)$ denotes the space of smooth sections and the tensor product is taken over the module $C^\infty(M)$. Generalized tensor fields may be viewed likewise as C^∞ -resp. \mathbf{G} -multilinear mappings, *i.e.*, as $C^\infty(M)$ -resp. $\mathbf{G}(M)$ -modules we have

$$\begin{aligned} \mathbf{G}_s^r(M) &\cong L_{C^\infty(M)}(\tilde{X}(M)^r, \tilde{X}^*(M)^s; \mathbf{G}(M)), \\ \mathbf{G}_s^r(M) &\cong L_{\mathbf{G}(M)}(\mathbf{G}_1^0(M)^r, \mathbf{G}_0^1(M)^s; \mathbf{G}(M)). \end{aligned} \tag{1.5.10}$$

Here $\tilde{X}(M)$ resp. $\tilde{X}^*(M)$ denotes the space of smooth vector resp. covector fields on M .

1.5.11. Generalized Pseudo-Riemannian Manifold

A generalized (0,2) tensor field $\bar{g} \in \mathbf{G}_2^0(M)$ is called a generalized Pseudo-Riemannian metric if it has a representative $(g_\varepsilon)_\varepsilon$ satisfying

- (i) g_ε is a smooth Pseudo-Riemannian metric for all $\varepsilon \in (0, 1]$, and
- (ii) $(\det g_\varepsilon(p))_\varepsilon$ is strictly nonzero on compact sets, *i.e.*,

$$\forall K (K \subset\subset X) \exists m (m \in \mathbb{N}) \left[\inf_{p \in K} |\det g_\varepsilon(p)| \geq \varepsilon^m \right].$$

We call a separable, smooth Hausdorff manifold M furnished with a generalized pseudo-Riemannian metric $\mathbf{cl}[(g_\varepsilon)_\varepsilon] \triangleq \bar{g}$ generalized pseudo-Riemannian manifold or generalized spacetime and denote it by (M, \bar{g}) . The action of the metric on a pair $\{H_1, H_2\}$ of generalized vector fields will be denoted by $\bar{g}(H_1, H_2)$ or $\langle H_1, H_2 \rangle$ [20] [21].

A generalized metric \bar{g} is non-degenerate in the following sense:

$$[H_1 \in \mathbf{G}_0^1(M)] \wedge \forall H_2 (H_2 \in \mathbf{G}_0^1(M)) [\bar{g}(H_1, H_2) = 0] H_1 = 0. \tag{1.5.11}$$

Note that condition (ii) above is precisely equivalent to invertibility of

$\det(\bar{g})$ in the generalized sense.

The inverse metric $\bar{g}^{-1} \triangleq \mathbf{cl}\left[\left(g_\varepsilon^{-1}\right)_\varepsilon\right]$ is a well defined element of $\mathbf{G}_0^2(M)$, depending exclusively on \bar{g} (i.e., independent of the particular representative $(g_\varepsilon)_\varepsilon$).

Moreover if $\bar{g} \approx_k g$, where g is a classical C^k -pseudo-Riemannian metric then $\bar{g}^{-1} \approx_k g^{-1}$.

From now on we denote the inverse metric by \bar{g}^{ab} , its components by \bar{g}^{ij} and the components of a representative by $\left(g_\varepsilon^{ij}\right)_\varepsilon$. Also we shall denote the generalized metric \bar{g}_{ab} by $\bar{ds}^2 = \mathbf{cl}\left[\left(ds_\varepsilon^2\right)_\varepsilon\right]$ and its representative by $\left(ds_\varepsilon^2\right)_\varepsilon = \left(g_{ij}(\varepsilon)dx^i dx^j\right)_\varepsilon$ and use summation convention.

Notice that \bar{g} induces a $\mathbf{G}(M)$ -linear isomorphism $\mathbf{G}_0^1(M) \rightarrow \mathbf{G}_1^0(M)$ by $\Phi \mapsto \bar{g}(\Phi, \cdot)$, which as in the classical context extends naturally to generalized tensor fields of all types.

1.5.12. Colombeau Isometric Embedding

Let (M, \bar{g}) and (N, \bar{h}) be generalized pseudo-Riemannian manifolds. An isometric Colombeau embedding is a Colombeau generalized function $(f_\varepsilon)_\varepsilon : M \rightarrow N$ which preserves the metric in the sense that $(g_\varepsilon)_\varepsilon$ is equal to the pullback of $(h_\varepsilon)_\varepsilon$ by $(f_\varepsilon)_\varepsilon$, i.e. $(g_\varepsilon)_\varepsilon = (f_\varepsilon^* h_\varepsilon)_\varepsilon$. Explicitly, for any two tangent vectors $\mathbf{v}, \mathbf{w} \in T_x(M)$ we have $(g_\varepsilon(\mathbf{v}, \mathbf{w}))_\varepsilon = (h_\varepsilon(df_\varepsilon(\mathbf{v}), df_\varepsilon(\mathbf{w})))_\varepsilon$.

1.5.13. Generalized Connection on a Generalized Pseudo-Riemannian Manifold

Generalized connection $\bar{D}_{H_1} H_2$ on a manifold M is a map

$\mathbf{G}_0^1(M) \times \mathbf{G}_0^1(M) \rightarrow \mathbf{G}_0^1(M)$ satisfying:

- (D1) $\bar{D}_{H_1}(H_2)$ is \mathbb{R} -linear in H_2 .
- (D2) $\bar{D}_{H_1}(H_2)$ is $\mathbf{G}(M)$ -linear in H_1 .
- (D3) $\bar{D}_{H_1}(\bar{\mathbf{f}} \cdot H_2) = \bar{\mathbf{f}} \cdot \bar{D}_{H_1}(H_2) + H_1(\bar{\mathbf{f}})H_2$ for all $\bar{\mathbf{f}} \in \mathbf{G}(M)$.

Let (V_α, ψ_α) be a chart on M with coordinates x^i . The generalized Christoffel symbols for this chart are given by the $[\dim(M)]^3$ generalized functions $\bar{\Gamma}_{ij}^k \in \mathbf{G}(V_\alpha)$ defined by

$$\bar{D}_{\partial_i}(\partial_j) = \sum_k \bar{\Gamma}_{ij}^k \partial_k. \tag{1.5.12}$$

Theorem. [21]. (I) Let (M, \bar{g}) be a generalized pseudo-Riemannian manifold. Then there exists a unique generalized connection $\bar{D}_{H_1}(H_2)$ such that

- (D4) $[H_1, H_2] = \bar{D}_{H_1}(H_2) - \bar{D}_{H_2}(H_1)$ and
- (D5) $H_1 \langle H_2, H_3 \rangle = \langle \bar{D}_{H_1}(H_2), H_3 \rangle + \langle H_2, \bar{D}_{H_1}(H_3) \rangle$

hold for all H_1, H_2, H_3 in $\mathbf{G}_0^1(M)$.

$\bar{D}_{H_1}(H_2)$ is called generalized Levi-Civita connection of M and characterized by the so-called Koszul formula

$$2 \langle \bar{D}_{H_1}(H_2), H_3 \rangle = H_1 \langle H_2, H_3 \rangle + H_2 \langle H_3, H_1 \rangle - H_3 \langle H_1, H_2 \rangle - \langle H_1, [H_2, H_3] \rangle + \langle H_2, [H_3, H_1] \rangle + \langle H_3, [H_1, H_2] \rangle. \tag{1.5.13}$$

(II) On every chart (V_α, ψ_α) we have for the generalized Levi-Civita connection $\bar{D}_{H_1}(H_2)$ of (M, \bar{g}) and any vector field $H \in \mathbf{G}_0^1(M)$

$$\bar{D}_{\partial_i}(H^j \partial_j) = \left(\frac{\partial H^k}{\partial x^i} + \bar{\Gamma}_{ij}^k H^j \right) \partial_k. \tag{1.5.14}$$

The generalized Christoffel symbols are given by

$$\bar{\Gamma}_{ij}^k = \frac{1}{2} \bar{g}^{km} \left(\frac{\partial \bar{g}_{jm}}{\partial x^i} + \frac{\partial \bar{g}_{im}}{\partial x^j} + \frac{\partial \bar{g}_{ij}}{\partial x^m} \right), \tag{1.5.15}$$

or by using representative

$$\left(\bar{\Gamma}_{ij}^k(\varepsilon) \right)_\varepsilon = \frac{1}{2} \left[\left(\bar{g}^{km}(\varepsilon) \right)_\varepsilon \right] \left[\left(\frac{\partial \bar{g}_{jm}(\varepsilon)}{\partial x^i} \right)_\varepsilon + \left(\frac{\partial \bar{g}_{im}(\varepsilon)}{\partial x^j} \right)_\varepsilon + \left(\frac{\partial \bar{g}_{ij}(\varepsilon)}{\partial x^m} \right)_\varepsilon \right]. \tag{1.5.16}$$

We define now the action of a classical (smooth) connection D on generalized vector fields $H_1 = \mathbf{cl}[(\xi_\varepsilon)_\varepsilon]$ and $H_2 = \mathbf{cl}[(\eta_\varepsilon)_\varepsilon]$ by

$$D_{H_1}(H_2) = \mathbf{cl} \left[\left(D_{\xi_\varepsilon}(\eta_\varepsilon) \right)_\varepsilon \right]$$

(III) Let (M, \bar{g}) be a generalized pseudo-Riemannian manifold.

(i) If $\bar{g}_{ab} = \Theta(g_{ab})$ where g_{ab} is a classical smooth metric then we have, in any chart, $\bar{\Gamma}_{jk}^i = \Theta(\Gamma_{jk}^i)$ (with Γ_{jk}^i denoting the Christoffel Symbols of g_{ab}). Hence for all $H \in \mathbf{G}_0^1(M)$:

$$\bar{D}_{H_1}(H_2) = D_{H_1}(H_2).$$

(ii) If $\bar{g}_{ab} \approx_\infty g_{ab}$, where g_{ab} a classical smooth metric, $H_1, H_2 \in \mathbf{G}_0^1(M)$ and $H_1 \approx_\infty \xi \in \tilde{X}(M)$, $H_2 \approx \eta \in D_0^1(M)$ or $H_1 \approx \xi \in D_0^1(M)$, $H_2 \approx_\infty \eta \in \tilde{X}(M)$, then $\bar{D}_{H_1}(H_2) \approx D_\xi(\eta)$.

(iii) Let $g_{ab} \approx_k g_{ab}$, where g_{ab} a classical C^k -metric, then, in any chart, $\bar{\Gamma}_{jk}^i \approx_{k-1} \Gamma_{jk}^i$. If in addition, $H_1, H_2 \in \mathbf{G}_0^1(M)$, $H_1 \approx \xi_{k-1} \in \Gamma^{k-1}(M, TM)$ and $H_2 \approx_k \eta \in \Gamma^k(M, TM)$ then

$$\bar{D}_{H_1}(H_2) \approx_k D_\xi(\eta).$$

1.5.14. The Generalized Riemannian Curvature Tensor

Let (M, \bar{g}) be a generalized pseudo-Riemannian manifold with a generalized Levi-Civita connection \bar{D} .

(i) The generalized Riemannian curvature tensor $\bar{R}_{abc}{}^d \in \mathbf{G}_3^1(M)$ is defined by

$$\bar{R}_{H_1, H_2} H_3 \triangleq \bar{D}_{[H_1, H_2]} H_3 - [\bar{D}_{H_1}, \bar{D}_{H_2}] H_3. \tag{1.5.17}$$

(ii) The generalized Ricci curvature tensor is defined by

$$\bar{R}_{ab} \triangleq \bar{R}_{cab}{}^c. \tag{1.5.18}$$

(iii) The generalized Ricci scalar is defined by

$$R \triangleq R_a{}^a. \tag{1.5.19}$$

(iv) Finally we define the generalized Einstein tensor by

$$\bar{G}_{ab} \triangleq \bar{R}_{ab} - \frac{1}{2} \bar{R} \bar{g}_{ab}. \tag{1.5.20}$$

1.6. Super Generalized Functions

1.6.1. The Nonsmooth Regularization via Horizon

Examining now the Schwarzschild metric (1.3.1) (note that the origin is now excluded from our considerations, the space we are working on is $\mathbb{R}_+^3 \triangleq \mathbb{R}^3 \setminus \{0\}$) in a neighborhood of the horizon, we see that, whereas $h(r)$ is smooth, $h^{-1}(r)$ is not even L_{loc}^1 . Thus, regularizing the Schwarzschild metric amounts to embedding h^{-1} into $\mathbf{G}(\mathbb{R}_+^3)$, for example as that given in paper [17], one obtains Colombeau generalized metric

$$\begin{aligned} (ds_\varepsilon^2)_\varepsilon &= h(r)dt^2 - (h^{-1}(r, \varepsilon)dr^2)_\varepsilon + r^2d\Omega^2, \\ h^{-1}(r, \varepsilon) &= h_\varepsilon^{-1} = -1 - 2m \left(\mathbf{vp} \left(\frac{1}{r-2m} \right) \right) * \rho_\varepsilon(r) \\ &= -1 - 2m \cdot \iota_\varepsilon \left(\mathbf{vp} \left(\frac{1}{r-2m} \right) \right). \end{aligned} \tag{1.6.1}$$

Here $\rho_\varepsilon(r) = \varepsilon^{-3} \rho(r\varepsilon^{-3})$ and $\rho(r)$ is a mollifier.

Obviously, (1.6.1) is degenerate at $r = 2m$, because $h(r)$ is zero at the horizon. Due to the degeneracy of (33), the Levi-Civita connection is not available. In order avoid this difficultness in literature the following generalized pseudo-connection $\bar{\Gamma}_{kj}^i$ was considered $\bar{\Gamma}_{kj}^i = \mathbf{cl} \left[\left(\Gamma_{kj}^i(\varepsilon) \right)_\varepsilon \right] \in \mathbf{G}(\mathbb{R}_+^3)$ [17]:

$$\left(\Gamma_{kj}^i(\varepsilon) \right)_\varepsilon = \frac{1}{2} \left(\iota_\varepsilon(g^{im}) \left[\iota_\varepsilon(g)_{mk,j} + \iota_\varepsilon(g)_{mj,k} - \iota_\varepsilon(g)_{kj,m} \right] \right)_\varepsilon. \tag{1.6.2}$$

Obviously the generalized pseudo-connection $\bar{\Gamma}_{kj}^i$ coincides with the classical Levi-Civita connection Γ_{kj}^i on $\mathbb{R}_+^3 \setminus \{r = 2m\}$ since $\left(\iota_\varepsilon(g^{im}) \right)_\varepsilon = g^{im}$, $\left(\iota_\varepsilon(g_{im}) \right)_\varepsilon = g_{im}$ there. However the generalized pseudo-connection $\bar{\Gamma}_{kj}^i$ is not a true generalized Levi-Civita connection on \mathbb{R}_+^3 since $\bar{\Gamma}_{kj}^i$ does not respect the Colombeau generalized metric (1.6.1), i.e., $\left(\iota_\varepsilon(g)_{ij,k} \right)_\varepsilon \neq 0$, e.g., $\left(\iota_\varepsilon(g)_{00,1} \right)_\varepsilon = \left(1 - h(h_\varepsilon^{-1}) \right)_\varepsilon h'$. Compatibility with the metric $\left(\iota_\varepsilon(g) \right)_\varepsilon$ is a priori ruled out by the following statement: there exists no connection whatsoever under which $\left(\iota_\varepsilon(g) \right)_\varepsilon$ would be a parallel tensor. However in a weak sense, the connection (1.6.2) is metric compatible: $\left(\iota_\varepsilon(g)_{ij,k} \right)_\varepsilon \approx 0$. In additional [17]:

$$\left(R_{ij}(\varepsilon) \right)_\varepsilon \approx 0, \tag{1.6.3}$$

where $\left(R_{ij}(\varepsilon) \right)_\varepsilon$ is a Ricci tensor corresponding to generalized pseudo-connection (1.6.2), and therefore Colombeau object $\left(R_{ij}(\varepsilon) \right)_\varepsilon$ viewed as a classical distribution on $\mathbb{R}^3 \setminus \{0\}$ gives

$$R_{ij} = 0 \text{ on } \mathbb{R}^3 \setminus \{0\}. \tag{1.6.4}$$

Remark 1.6.1. In paper [17] the equality (1.6.4) mistakenly considered as a proof that the metric singularity at the Schwarzschild horizon is only a coordinate singularity.

Remark 1.6.2. Due to the degeneracy of any smooth regularization of the metric (1.3.1) no canonical Levi-Civita connection could be defined. In order to avoid such difficultnes in our papers [18] [19] the nonsmooth regularization via horizon is considered, see Section 2 below. However such regularization

demands appropriate extension of the Colombeau algebra $\mathbf{G}(M)$.

1.6.2. The Super Generalized Functions

The basic idea of the theory of super generalized functions is regularization by sequences (nets) of appropriate classes of non smooth and discontinuous functions or classical distributions and the use of asymptotic estimates in terms of a regularization parameter $\varepsilon \in (0, 1]$. Let $(u_\varepsilon)_{\varepsilon \in (0, 1]}$ with $u_\varepsilon \in D'(M)$ for all $\varepsilon \in (0, 1]$ (M a separable, smooth orientable Hausdorff manifold of dimension n). Such sequences are called super generalized functions. The algebra of super generalized functions on M is defined as the quotient

$$\mathbf{SG}(M) = \mathbf{SE}'_M(M) / \mathbf{SN}'(M) \tag{1.6.5}$$

of the space $\mathbf{SE}'_M(M)$ of sequences of moderate growth modulo the space $\mathbf{N}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates ($\tilde{\mathbf{X}}_{\mathbb{R}}(M)$ or $\tilde{\mathbf{X}}(M)$ denoting the space of smooth vector fields on M) [18] [19]:

$$\begin{aligned} \mathbf{SE}'_M(M) &= \left\{ (u_\varepsilon)_\varepsilon \in (D'(M))^{(0,1]} \mid (\forall K \subset\subset M) (\forall k \in \mathbb{N}_0) (\exists n \in \mathbb{N}) \right. \\ &\quad \left. (\forall \xi_1 \in \tilde{\mathbf{X}}(M), \dots, \forall \xi_k \in \tilde{\mathbf{X}}(M)) (\forall f \in C_0^\infty(K)) \left[\left| L_{\xi_1} \dots L_{\xi_n} u_\varepsilon(f) \right| \leq O_f(\varepsilon^{-n}) \right] \right\}, \\ \mathbf{SN}'(M) &= \left\{ (u_\varepsilon)_\varepsilon \in (D'(M))^{(0,1]} \mid (\forall K \subset\subset M) (\forall k, q \in \mathbb{N}_0) (\exists n \in \mathbb{N}) \right. \\ &\quad \left. (\forall \xi_1 \in \tilde{\mathbf{X}}(M), \dots, \forall \xi_k \in \tilde{\mathbf{X}}(M)) (\forall f \in C_0^\infty(K)) \left[\left| L_{\xi_1} \dots L_{\xi_n} u_\varepsilon(f) \right| \leq O_f(\varepsilon^q) \right] \right\}. \end{aligned} \tag{1.6.6}$$

Here $L_{\xi_k}^w, k = 1, 2, \dots, n$ denoting the weak Lie derivative in L.Schwartz sense and where Landau symbol $a_\varepsilon = O_f(\varphi(\varepsilon))$ appears, having the following meaning:

$$\forall f \exists C_f (C_f > 0) \exists \varepsilon_0 (\varepsilon_0 \in (0, 1]) \forall \varepsilon (\varepsilon < \varepsilon_0) [a_\varepsilon \leq C_f \varphi(\varepsilon)].$$

We denote by $\mathbf{S}\tilde{\mathbb{R}}$ the ring of real, Colombeau super generalized numbers. $\mathbf{S}\tilde{\mathbb{R}} = \mathbf{SE}_{\mathbb{R}}(\mathbb{R}) / \mathbf{SN}(\mathbb{R})$, where

$$\begin{aligned} \mathbf{SE}_{\mathbb{R}}(\mathbb{R}) &= \left\{ (x_{\varepsilon, \delta})_{\varepsilon, \delta} \in \mathbb{R}^{(0,1] \times (0,1]} \mid (\exists a, b \in \mathbb{R}_+) (\exists \varepsilon_0, \delta_0 \in (0, 1]) \right. \\ &\quad \left. (\forall \varepsilon \leq \varepsilon_0) (\forall \delta \leq \delta_0) \left[|x_{\varepsilon, \delta}| \leq \varepsilon^{-a} \delta^{-b} \right] \right\}, \\ \mathbf{SN}(\mathbb{R}) &= \left\{ (x_{\varepsilon, \delta})_{\varepsilon, \delta} \in \mathbb{R}^{(0,1] \times (0,1]} \mid (\forall a, b \in \mathbb{R}_+) (\exists \varepsilon_0, \delta_0 \in (0, 1]) \right. \\ &\quad \left. (\forall \varepsilon \leq \varepsilon_0) (\forall \delta \leq \delta_0) \left[|x_{\varepsilon, \delta}| \leq \varepsilon^a \delta^b \right] \right\}. \end{aligned} \tag{1.6.7}$$

Let $(u_{\varepsilon, \delta})_{\varepsilon, \delta \in (0, 1]}$ with $u_{\varepsilon, \delta} \in C^\infty(M)$ for all $\varepsilon, \delta \in (0, 1]$ (M a separable, smooth orientable Hausdorff manifold of dimension n). By using canonical imbedding $D'(M) \rightarrow \mathbf{G}(M)$ the algebra $\mathbf{SG}(M)$ of Colombeau super generalized functions on M is defined also in the equivalent form as the quotient

$$\mathbf{SG}(M) = \mathbf{SE}_M(M) / \mathbf{SN}(M) \tag{1.6.8}$$

of the space $\mathbf{SE}_M(M)$ of sequences of moderate growth modulo the space

$\text{SN}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates ($\tilde{\mathbf{X}}_{\mathbb{R}}(M)$) or $\tilde{\mathbf{X}}(M)$ denoting the space of smooth vector fields on M).

$$\begin{aligned} \mathbf{SE}_M(M) &= \left\{ (u_{\varepsilon,\delta})_{\varepsilon,\delta} \in C^\infty(M)^{(0,1] \times (0,1]} \mid (\forall K \subset\subset M)(\forall k \in \mathbb{N}_0)(\exists n, m \in \mathbb{N}) \right. \\ &\quad \left. (\forall \xi_1 \in \tilde{\mathbf{X}}(M), \dots, \forall \xi_k \in \tilde{\mathbf{X}}(M)) \left[\sup_{p \in K} |L_{\xi_1} \cdots L_{\xi_n} u_{\varepsilon,\delta}(p)| \leq O(\varepsilon^{-n} \delta^{-m}) \right] \right\}, \\ \mathbf{SN}(M) &= \left\{ (u_{\varepsilon,\delta})_{\varepsilon,\delta} \in M^{(0,1] \times (0,1]} \mid (\forall K \subset\subset M)(\forall k, q, l \in \mathbb{N}_0) \right. \\ &\quad \left. (\forall \xi_1 \in \tilde{\mathbf{X}}(M), \dots, \forall \xi_k \in \tilde{\mathbf{X}}(M)) \left[\sup_{p \in K} |L_{\xi_1} \cdots L_{\xi_n} u_{\varepsilon,\delta}(p)| \leq O(\varepsilon^q \delta^l) \right] \right\}. \end{aligned} \tag{1.6.9}$$

The $\mathbf{SG}(M)$ -module of super generalized sections $\mathbf{SG}(M, E)$ of a vector bundle $E \rightarrow M$ and in particular the space of super generalized tensor fields $\mathbf{SG}_s^r(M)$ is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. We denote super generalized sections by

$$S = \mathbf{cl} \left[(s_{\varepsilon,\delta})_{\varepsilon,\delta} \right] = (s_{\varepsilon,\delta})_{\varepsilon,\delta} + \mathbf{SN}(M, E). \text{ Alternatively we may describe a section}$$

$S \in \mathbf{SG}(M, E)$ by a family $(S_\alpha)_\alpha = \left((S_\alpha^i)_{i=1}^N \right)$, where S_α is called the local

expression of S with its components $S_\alpha^i \triangleq \Psi_\alpha^i \circ S \circ \psi_\alpha^{-1} \in \mathbf{SG}(\psi_\alpha(V_\alpha))$

$((V_\alpha, \Psi_\alpha)_\alpha$ a vector bundle atlas and $i = 1, \dots, N$, with N denoting the dimension of the fibers) satisfying $S_\alpha^i(x) = (\psi_{\alpha\beta})_j^i (\psi_\beta \circ \psi_\alpha^{-1}(x)) S_\beta^j(\psi_\beta \circ \psi_\alpha^{-1}(x))$ for all $x \in (V \cap V)$ where $\psi_{\alpha\beta}$ denotes the transition functions of the bundle.

Remark 1.6.3. Smooth sections of $E \rightarrow M$ again may be embedded as constant nets, *i.e.*,

$$\Theta(s) : s \rightarrow \mathbf{cl} \left[(s)_{\varepsilon,\delta} \right].$$

Since $C^\infty(M)$ is a subring of $\mathbf{SG}(M), \mathbf{SG}(M, E)$ also may be viewed as $C^\infty(M)$ -module and the two respective module structures are compatible with respect to the embeddings.

Moreover we have the following algebraic characterization of the space of super generalized sections

$$\mathbf{SG}(M, E) = \mathbf{SG}(M) \otimes \Gamma(M, E), \tag{1.6.10}$$

where $\Gamma(M, E)$ denotes the space of smooth sections and the tensor product is taken over the module $C^\infty(M)$. Generalized tensor fields may be viewed likewise as C^∞ -resp. \mathbf{SG} -multilinear mappings, *i.e.*, as $C^\infty(M)$ -resp. $\mathbf{SG}(M)$ -modules we have

$$\begin{aligned} \mathbf{SG}_s^r(M) &\cong L_{C^\infty(M)} \left(\tilde{\mathbf{X}}(M)^r, \tilde{\mathbf{X}}^*(M)^s; \mathbf{SG}(M) \right), \\ \mathbf{SG}_s^r(M) &\cong L_{\mathbf{SG}(M)} \left(\mathbf{SG}_1^0(M)^r, \mathbf{SG}_0^1(M)^s; \mathbf{SG}(M) \right). \end{aligned} \tag{1.6.11}$$

Here $\tilde{\mathbf{X}}(M)$ resp. $\tilde{\mathbf{X}}^*(M)$ denotes the space of smooth vector resp.

covector fields on M .

1.6.3. Super Generalized Pseudo-Riemannian Manifold

A super generalized $(0,2)$ tensor field $\bar{g} \in \mathbf{SG}_2^0(M)$ is called a super generalized Pseudo-Riemannian metric if it has a representative $(g_{\varepsilon,\delta})_{\varepsilon,\delta}$ satisfying:

- (i) $g_{\varepsilon,\delta}$ is a smooth Pseudo-Riemannian metric for all $\varepsilon, \delta \in (0,1]$, and
- (ii) $(\det g_{\varepsilon,\delta}(p))_{\varepsilon,\delta}$ is strictly nonzero on compact sets, i.e.,

$$\forall K (K \subset\subset X) \exists m (m \in N) \exists l (l \in N) \left[\inf_{p \in K} |\det g_{\varepsilon,\delta}(p)| \geq \varepsilon^m \delta^l \right].$$

We call a separable, smooth Hausdorff manifold M furnished with a super generalized pseudo-Riemannian metric $\mathbf{cl}[(g_{\varepsilon,\delta})_{\varepsilon,\delta}] \triangleq \bar{g}$ super generalized pseudo-Riemannian manifold or super generalized spacetime and denote it by (M, \bar{g}) . The action of the metric on a pair $\{H_1, H_2\}$ of super generalized vector fields will be denoted by $\bar{g}(H_1, H_2)$ or $\langle H_1, H_2 \rangle$.

A super generalized metric \bar{g} is non-degenerate in the following sense:

$$[H_1 \in \mathbf{SG}_0^1(M)] \wedge \forall H_2 (H_2 \in \mathbf{SG}_0^1(M)) [\bar{g}(H_1, H_2) = 0] \Rightarrow H_1 = 0. \quad (1.6.12)$$

Note that condition (ii) above is precisely equivalent to invertibility of $\det(\bar{g})$ in the super generalized sense. The inverse metric $\bar{g}^{-1} \triangleq \mathbf{cl}[(g_{\varepsilon,\delta}^{-1})_{\varepsilon,\delta}]$ is a well defined element of $\mathbf{SG}_0^2(M)$, depending exclusively on \bar{g} (i.e., independent of the particular representative $(g_{\varepsilon,\delta})_{\varepsilon,\delta}$). Moreover if $\bar{g} \approx_k g$, where g is a classical C^k -pseudo-Riemannian metric then $\bar{g}^{-1} \approx_k g^{-1}$. From now on we denote the inverse metric by \bar{g}^{ab} , its components by \bar{g}^{ij} and the components of a representative by $(g_{\varepsilon,\delta}^{ij})_{\varepsilon,\delta}$. Also we shall denote the super generalized metric \bar{g}_{ab} by $\bar{ds}^2 = \mathbf{cl}[(ds_{\varepsilon,\delta}^2)_{\varepsilon,\delta}]$ and its representative by $(ds_{\varepsilon,\delta}^2)_{\varepsilon,\delta} = (g_{ij}(\varepsilon, \delta) dx^i dx^j)_{\varepsilon,\delta}$ and use summation convention. Notice that \bar{g} induces a $\mathbf{SG}(M)$ -linear isomorphism $\mathbf{SG}_0^1(M) \rightarrow \mathbf{SG}_1^0(M)$ by $\Phi \mapsto \bar{g}(\Phi, \cdot)$, which as in the classical context extends naturally to generalized tensor fields of all types.

Let (M, \bar{g}) and (N, \bar{h}) be super generalized pseudo-Riemannian manifolds. An isometric Colombeau embedding is a Colombeau super generalized function $(f_{\varepsilon,\delta})_{\varepsilon,\delta} : M \rightarrow N$ which preserves the metric in the sense that $(g_{\varepsilon,\delta})_{\varepsilon,\delta}$ is equal to the pullback of $(h_{\varepsilon,\delta})_{\varepsilon,\delta}$ by $(f_{\varepsilon,\delta})_{\varepsilon,\delta}$, i.e. $(g_{\varepsilon,\delta})_{\varepsilon,\delta} = (f_{\varepsilon,\delta}^* h_{\varepsilon,\delta})_{\varepsilon,\delta}$. Explicitly, for any two tangent vectors $\mathbf{v}, \mathbf{w} \in T_x(M)$ we have

$$(g_{\varepsilon,\delta}(\mathbf{v}, \mathbf{w}))_{\varepsilon,\delta} = (h_{\varepsilon,\delta}(df_{\varepsilon,\delta}(\mathbf{v}), df_{\varepsilon,\delta}(\mathbf{w})))_{\varepsilon,\delta}.$$

1.6.4. Super Generalized Connection on a Super Generalized Pseudo-Riemannian Manifold

Super generalized connection $\bar{D}_{H_1} H_2$ on a manifold M is a map $\mathbf{SG}_0^1(M) \times \mathbf{SG}_0^1(M) \rightarrow \mathbf{SG}_0^1(M)$ satisfying:

- (D1) $\bar{D}_{H_1}(H_2)$ is $\mathbf{S}\tilde{\mathbb{R}}$ -linear in H_2 .
- (D2) $\bar{D}_{H_1}(H_2)$ is $\mathbf{SG}(M)$ -linear in H_1 .
- (D3) $\bar{D}_{H_1}(\bar{\mathbf{f}} \cdot H_2) = \bar{\mathbf{f}} \cdot \bar{D}_{H_1}(H_2) + H_1(\bar{\mathbf{f}})H_2$ for all $\bar{\mathbf{f}} \in \mathbf{SG}(M)$.

Let (V_α, ψ_α) be a chart on M with coordinates x^i . The super generalized

Christoffel symbols for this chart are given by the $[\dim(M)]^3$ super generalized functions $\bar{\Gamma}_{ij}^k \in \mathbf{SG}(V_\alpha)$ defined by

$$\bar{D}_{\partial_i}(\partial_j) = \sum_k \bar{\Gamma}_{ij}^k \partial_k. \tag{1.6.13}$$

Theorem. (I) Let (M, \bar{g}) be a super generalized pseudo-Riemannian manifold. Then there exists a unique super generalized connection $\bar{D}_{H_1}(H_2)$ such that

$$(D4) \quad [H_1, H_2] = \bar{D}_{H_1}(H_2) - \bar{D}_{H_2}(H_1) \quad \text{and}$$

$$(D5) \quad H_1 \langle H_2, H_3 \rangle = \langle \bar{D}_{H_1}(H_2), H_3 \rangle + \langle H_2, \bar{D}_{H_1}(H_3) \rangle$$

hold for all H_1, H_2, H_3 in $\mathbf{SG}_0^1(M)$.

$\bar{D}_{H_1}(H_2)$ is called super generalized Levi-Civita connection of M and characterized by the so-called Koszul formula

$$2 \langle \bar{D}_{H_1}(H_2), H_3 \rangle = H_1 \langle H_2, H_3 \rangle + H_2 \langle H_3, H_1 \rangle - H_3 \langle H_1, H_2 \rangle - \langle H_1, [H_2, H_3] \rangle + \langle H_2, [H_3, H_1] \rangle + \langle H_3, [H_1, H_2] \rangle. \tag{1.6.14}$$

(II) On every chart (V_α, ψ_α) we have for the super generalized Levi-Civita connection $\bar{D}_{H_1}(H_2)$ of (M, \bar{g}) and any vector field $H \in \mathbf{SG}_0^1(M)$

$$\bar{D}_{\partial_i}(H^j \partial_j) = \left(\frac{\partial H^k}{\partial x^i} + \bar{\Gamma}_{ij}^k H^j \right) \partial_k. \tag{1.6.15}$$

The super generalized Christoffel symbols are given by

$$\bar{\Gamma}_{ij}^k = \frac{1}{2} \bar{g}^{km} \left(\frac{\partial \bar{g}_{jm}}{\partial x^i} + \frac{\partial \bar{g}_{im}}{\partial x^j} + \frac{\partial \bar{g}_{ij}}{\partial x^m} \right), \tag{1.6.16}$$

or by using representative

$$\begin{aligned} (\bar{\Gamma}_{ij}^k(\varepsilon, \delta))_{\varepsilon, \delta} &= \frac{1}{2} \left[(\bar{g}^{km}(\varepsilon, \delta))_{\varepsilon, \delta} \right] \left[\left(\frac{\partial \bar{g}_{jm}(\varepsilon, \delta)}{\partial x^i} \right)_{\varepsilon, \delta} \right. \\ &\quad \left. + \left(\frac{\partial \bar{g}_{im}(\varepsilon, \delta)}{\partial x^j} \right)_{\varepsilon, \delta} + \left(\frac{\partial \bar{g}_{ij}(\varepsilon, \delta)}{\partial x^m} \right)_{\varepsilon, \delta} \right]. \end{aligned} \tag{1.6.17}$$

We define now the action of a classical (smooth) connection D on super generalized vector fields $H_1 = \mathbf{cl} \left[(\xi_{\varepsilon, \delta})_{\varepsilon, \delta} \right]$ and $H_2 = \mathbf{cl} \left[(\eta_{\varepsilon, \delta})_{\varepsilon, \delta} \right]$ by

$$D_{H_1}(H_2) = \mathbf{cl} \left[\left(D_{\xi_{\varepsilon, \delta}}(\eta_{\varepsilon, \delta}) \right)_{\varepsilon, \delta} \right]$$

(III) Let (M, \bar{g}) be a super generalized pseudo-Riemannian manifold.

(i) If $\bar{g}_{ab} = \Theta(g_{ab})$ where g_{ab} is a classical smooth metric then we have, in any chart, $\bar{\Gamma}_{jk}^i = \Theta(\Gamma_{jk}^i)$ (with Γ_{jk}^i denoting the Christoffel Symbols of g_{ab}). Hence for all $H \in \mathbf{SG}_0^1(M)$: $\bar{D}_{H_1}(H_2) = D_{H_1}(H_2)$.

(ii) If $\bar{g}_{ab} \approx_\infty g_{ab}$, where g_{ab} a classical smooth metric, $H_1, H_2 \in \mathbf{SG}_0^1(M)$ and $H_1 \approx_\infty \xi \in \tilde{X}(M)$, $H_2 \approx \eta \in D_0^1(M)$ or $H_1 \approx \xi \in D_0^1(M)$, $H_2 \approx_\infty \eta \in \tilde{X}(M)$, then $\bar{D}_{H_1}(H_2) \approx D_\xi(\eta)$.

(iii) Let $g_{ab} \approx_k g_{ab}$, where g_{ab} a classical C^k -metric, then, in any chart, $\bar{\Gamma}_{jk}^i \approx_{k-1} \Gamma_{jk}^i$. If in addition, $H_1, H_2 \in \mathbf{SG}_0^1(M)$, $H_1 \approx \xi_{k-1} \in \Gamma^{k-1}(M, TM)$ and $H_2 \approx_k \eta \in \Gamma^k(M, TM)$ then $\bar{D}_{H_1}(H_2) \approx_k D_\xi(\eta)$.

1.6.5. The Super Generalized Riemannian Curvature Tensor

Let (M, \bar{g}) be a super generalized pseudo-Riemannian manifold with a super generalized Levi-Civita connection \bar{D} .

(i) The super generalized Riemannian curvature tensor $\bar{R}_{abc}{}^d \in \mathbf{SG}_3^1(M)$ is defined by

$$\bar{R}_{H_1, H_2} H_3 \triangleq \bar{D}_{[H_1, H_2]} H_3 - [\bar{D}_{H_1}, \bar{D}_{H_2}] H_3. \tag{1.6.18}$$

(ii) The super generalized Ricci curvature tensor is defined by

$$\bar{R}_{ab} \triangleq \bar{R}_{cab}{}^c. \tag{1.6.19}$$

(iii) The super generalized Ricci scalar is defined by

$$R \triangleq R_a{}^a. \tag{1.6.20}$$

4) Finally we define the super generalized Einstein tensor by

$$\bar{G}_{ab} \triangleq \bar{R}_{ab} - \frac{1}{2} \bar{R} \bar{g}_{ab}. \tag{1.6.21}$$

2. Distributional Schwarzschild Geometry by Using Nonsmooth Regularization via Horizon

2.1. Distributional Schwarzschild Spacetime as Colombeau Extension of the Lorentzian Manifold with Nonregularity Conditions on Schwarzschild Horizon

Singular space-times present one of the major challenges in general relativity. Originally it was believed that their singular nature is due to the high degree of symmetry of the well-known examples ranging from the Schwarzschild geometry to the Friedmann-Robertson-Walker cosmological models. However, Penrose and Hawking [36] have shown in their classical singularity theorems that singularities are a phenomenon which is inherent to general relativity. Since the standard approach allows only smooth space-time metrics, one has to exclude the so called singular regions from the space-time manifold. In a recent work many authors advocated the use Colombeau distributional techniques [5]-[22] to calculate the energy-momentum tensor of the Schwarzschild geometry. It turns out that it is possible to include the singular region (*i.e.* the space-like line $r = 0$ with respect to Schwarzschild coordinates) in the space-time which now no longer is a vacuum geometry, and to identify it with the support of the energy-momentum tensor [5] [9] [11] [12] [13]. The same “physically expected” result for the distributional energy momentum tensor of the Schwarzschild geometry was obtained in papers [12]-[22], *i.e.*,

$$T_0^0 = 8\pi m \delta(\bar{x}), \tag{2.1.1}$$

in a conceptually satisfactory way.

Remark 2.1.1. The result (2.1.1) can be easily obtained by using appropriate nonsmooth regularization of the Schwarzschild singularity at the origin $r = 0$.

The nonsmooth regularization of the Schwarzschild singularity at the origin $r = 0$ was originally considered by N. R. Pantoja and H. Rago in paper [12].

Such non smooth regularization of the Schwarzschild singularity is

$$(h_\varepsilon(r))_\varepsilon = -1 + \left(\frac{r_s}{r} \Theta_\varepsilon(r - \varepsilon) \right), \quad \varepsilon \in (0, 1], r < r_s. \tag{2.1.2}$$

Here $(\Theta_\varepsilon(u))_\varepsilon$ is the generalized Heaviside function, where

$$\Theta_\varepsilon(u) = \begin{cases} (\varepsilon)_\varepsilon, & u < 0 \\ 1, & u \geq 0 \end{cases} \tag{2.1.3}$$

and the limit $\varepsilon \rightarrow 0$ is understood in a weak distributional sense. The equation

$$\begin{aligned} (ds_\varepsilon^2)_\varepsilon &= (h_\varepsilon(r)(dt)_\varepsilon^2 - (h_\varepsilon^{-1}(r)(dr)_\varepsilon^2) + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \\ h_0(r) &= -1 + \frac{r_s}{r}, \end{aligned} \tag{2.1.4}$$

with $h_\varepsilon, \varepsilon \in (0, 1]$, as given in (2.1.4) can be considered as Colombeau version of the Schwarzschild line element in curvature coordinates. From Equation (2.1.2), the calculation of the distributional Einstein tensor $(G_t^t(r, \varepsilon))_\varepsilon, (G_r^r(r, \varepsilon))_\varepsilon, (G_\theta^\theta(r, \varepsilon))_\varepsilon, (G_\phi^\phi(r, \varepsilon))_\varepsilon$ proceeds in a straightforward manner. By simple calculation one obtains [12]:

$$\begin{aligned} (G_t^t(r, \varepsilon))_\varepsilon &= (G_r^r(r, \varepsilon))_\varepsilon = -\left(\frac{h'_\varepsilon(r)}{r} \right)_\varepsilon - \left(\frac{1 + h_\varepsilon(r)}{r^2} \right)_\varepsilon \\ &= -r_s \left(\frac{\delta(r - \varepsilon)}{r^2} \right)_\varepsilon = -r_s \frac{\delta(r)}{r^2} \end{aligned} \tag{2.1.5}$$

and

$$\begin{aligned} (G_\theta^\theta(r, \varepsilon))_\varepsilon &= (G_\phi^\phi(r, \varepsilon))_\varepsilon = -\left(\frac{h''_\varepsilon(r)}{2} \right)_\varepsilon - \left(\frac{h'_\varepsilon(r)}{r^2} \right)_\varepsilon \\ &= r_s \left(\frac{\delta(r - \varepsilon)}{r^2} \right)_\varepsilon - r_s \left(\frac{\varepsilon}{r^2} \frac{d}{dr} \delta(r - \varepsilon) \right)_\varepsilon \approx -r_s \frac{\delta(r)}{r^2}. \end{aligned} \tag{2.1.6}$$

In papers [10] [27] Colombeau distributional techniques were extended to the general axisymmetric, stationary Kerr and Newman space-time family. This family also contains the Schwarzschild geometry and its charged extension the Reissner-Nordström solution as special cases of spherical symmetry. In the paper [22] it was shown that the solutions will satisfy the Einstein equations everywhere if the energy-momentum tensor has an appropriate singular addition of nonelectromagnetic origin. When this addition term is included, the total energy turns out to be finite and equal to mc^2 , while the angular momentum for the Kerr and Kerr-Newman solutions is mca .

Remark 2.1.2. The nonsmooth regularization of the Schwarzschild singularity above the horizon $r = r_s$ is

$$(h_\varepsilon^+(r))_\varepsilon = -1 + \left(\frac{r_s}{r} \Theta_\varepsilon((r - r_s) - \varepsilon) \right), \quad \varepsilon \in (0, 1], r \geq r_s. \tag{2.1.7}$$

Here $(\Theta_\varepsilon(u))_\varepsilon$ is the generalized Heaviside function and the limit $\varepsilon \rightarrow 0$ is understood in a weak distributional sense. The equation

$$\begin{aligned}
 (ds_{\epsilon}^{+2})_{\epsilon} &= (h_{\epsilon}^{+}(r)(dt)_{\epsilon}^2) - ([h_{\epsilon}^{+}(r)]^{-1}(dr)_{\epsilon}^2) + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \\
 h_0(r) &= -1 + \frac{r_s}{r},
 \end{aligned}
 \tag{2.1.8}$$

$h_{\epsilon}, \epsilon \in (0, 1]$, as given in (2.1.8) can be considered as Colombeau version of the Schwarzschild line element in curvature coordinates above horizon. From Equation (2.1.7), the calculation of the distributional Einstein tensor above horizon $(G_t^{+t}(r, \epsilon))_{\epsilon}, (G_r^{+r}(r, \epsilon))_{\epsilon}, (G_{\theta}^{+\theta}(r, \epsilon))_{\epsilon}, (G_{\phi}^{+\phi}(r, \epsilon))_{\epsilon}$ proceeds in a straightforward manner. By simple calculation one obtains

$$\begin{aligned}
 (G_t^{+t}(r, \epsilon))_{\epsilon} &= (G_r^{+r}(r, \epsilon))_{\epsilon} \\
 &= -\left(\frac{h'_{\epsilon}((r-r_s)-\epsilon)}{r}\right)_{\epsilon} - \left(\frac{1+h_{\epsilon}((r-r_s)-\epsilon)}{r^2}\right)_{\epsilon} \\
 &= -r_s \left(\frac{\delta((r-r_s)-\epsilon)}{r^2}\right)_{\epsilon} \approx -r_s \frac{\delta(r-r_s)}{r^2}.
 \end{aligned}
 \tag{2.1.9}$$

The truncated distributional Schwarzschild geometry.

There exist two different types of distributional Schwarzschild blackhole geometry corresponding to classical Schwarzschild solution. That is: (i) full distributional Schwarzschild blackhole geometry, given by Colombeau generalized object, for example by Equation (1.3.30), see **Figure 1(a)** and (ii) the truncated distributional Schwarzschild space-time given by Colombeau generalized object (2.1.7)-(2.1.8), *i.e.* in this case distributional spacetime ends just on the Schwarzschild horizon, see **Figure 1(b)**.

Remark 2.1.3. In a nutshell, there is a widespread but mistaken belief that there exist true gravitational singularities, for example at origin $r = 0$ of the

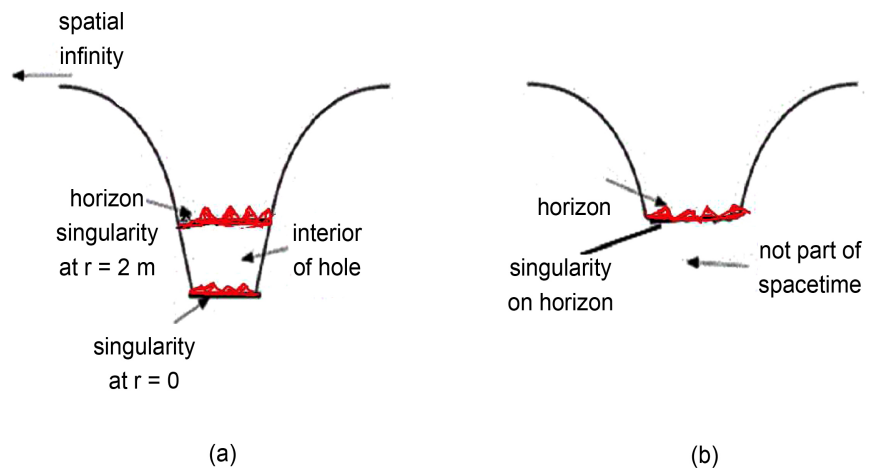


Figure 1. (a) The picture of a distributional Schwarzschild blackhole, given by Colombeau generalized object (1.3.30). Distributional spacetime ends just on the Schwarzschild singularity. (b) The truncated Schwarzschild distributional geometry, given by Colombeau generalized object (2.1.7)-(2.1.8). Distributional spacetime ends just on the Schwarzschild horizon.

Schwarzschild spacetime, and non principal and non gravitational, *i.e.* purely coordinate singularities, for example at horizon $r = r_g$ of the Schwarzschild spacetime. A coordinate singularity or coordinate degeneracy occurs when an apparent singularity or degeneracy occurs in one coordinate frame, which can be removed by choosing a different frame. Classical example of such mistake is ubnormal deletion of the gravitational singularity, for example from Schwarzschild spacetime

$$\text{Sch} = (S^2 \times \{r \geq 2m\}) \times \mathbb{R}, g_{ij}(r, \theta, \phi), \tag{2.1.10}$$

originally defined by singular and degenerate Schwarzschild metric [30],

$$ds^2 = -h(r)(dx^0)^2 + h^{-1}(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2 \theta(d\phi)^2], h(r) = 1 - \frac{r_g}{r}. \tag{2.1.11}$$

by using appropriate singular coordinate change [27]-[35].

Remark 2.1.4. Note that: (i) metric (2.1.11) is singular and degenerate at Schwarzschild horizon $r = r_g$, and thus metric (2.1.11) beyond canonical rigorous semi-Riemannian geometry.

(ii) however in physical literature (see for example [28] [29] [30]) singularity and degeneracy at Schwarzschild horizon $r = r_g$ are accepted as coordinate singularity and coordinate degeneracy.

Remark 2.1.5. (see [30] section 100, p. 296). “In the Schwarzschild metric (97.14), g_{00} goes to zero and g_{11} to infinity at $r = r_g$ (on the ‘Schwarzschild sphere’). This could give the basis for concluding that there must be a singularity of the space-time metric and that it is therefore impossible for bodies to exist that have a ‘radius’ (for a given mass) that is less than the gravitational radius. Actually, however, this conclusion would be wrong. This is already evident from the fact that the determinant $g(r) = -r^4 \sin^2 \theta$ has no singularity at $r = r_g$, so that the condition $g < 0$ (82.3) is not violated. We shall see that in fact we are dealing simply with the impossibility of establishing a suitable reference system for $r < r_g$.”

Remark 2.1.6. Notice that consideration above meant the following definition of the gravitational singularity.

Definition 2.1.1. There is no gravitational singularity at $r = \bar{r}$ iff the determinant $g(r, \theta) = \det(g_{ij}(r, \theta))$ has no singularity at $r = \bar{r}$.

Remark 2.1.7. Notice that at singular point $r = r_g$ the determinant $g(r_g)$ is well defined only by the limit

$$g(r_g) = \lim_{r \rightarrow r_g} \det(g_{ij}(r, \theta)) = -r_g^4 \sin^2 \theta. \tag{2.1.12}$$

however in the limit $r \rightarrow r_g = 2m$ the classical Levi-Civita connection Γ_{kj}^l becomes infinite

$$\Gamma_{11}^1(r) \Big|_{r=2m} = \lim_{r \rightarrow 2m} \frac{-m}{r(r-2m)} = \infty, \quad \Gamma_{01}^0(r) \Big|_{r=2m} = \lim_{r \rightarrow 2m} \frac{m}{r(r-2m)} = \infty, \tag{2.1.13}$$

and therefore the Definition 2.1.1 is not sound and even does not any sense

under canonical semi-Riemannian geometry.

Remark 2.1.8. Notice that:

(i) in order to fix the problem with singularity and degeneracy of the Schwarzschild metric (2.1.11) at Schwarzschild horizon $r = r_g$, in physical literature [27]-[35], many years one considers the abnormal formal change of coordinates obtained by replacing the canonical Schwarzschild time by “retarded time” $v(t, r)$, i.e., Eddington-Finkelstein coordinates, given by

$$\begin{aligned} dv(t, r) &= dt + [h(r)]^{-1} dr, \\ h(r) &= 1 - \frac{r_g}{r}; \end{aligned} \tag{2.1.14}$$

(ii) the change (2.1.14) of Schwarzschild coordinates is singular at Schwarzschild horizon $r = r_g$, as at Schwarzschild horizon $h(r_g) = \infty$ and therefore the change (2.1.14) does not hold on Schwarzschild horizon [36];

(iii) under the singular change (2.1.14) Schwarzschild metric (2.1.11) becomes to well known regular and nondegenerate Eddington-Finkelstein metric [27]-[35]:

$$ds_{\text{EF}}^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2drdv + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]; \tag{2.1.15}$$

(iv) in physical literature many years exist abnormal belief that by formal singular change (2.1.15) the singular and degenerate Schwarzschild spacetime $(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}$ was immersed in a larger Eddington-Finkelstein spacetime

$$\text{EF}_{\geq} = (\mathbf{S}^2 \times \{r \geq 2m\} \cup \{0 < r \leq 2m\}) \times \mathbb{R}, g_{\text{EF}_{\geq}}(r, \theta), \tag{2.1.16}$$

with regular and non degenerate metric tensor $g_{\text{EF}_{\geq}}(r, \theta)$, and whose manifold is not covered by the canonical Schwarzschild coordinate with $r \leq 2m$, and therefore singularity and degeneracy on Schwarzschild horizon $r = r_g$ are only coordinate singularity and coordinate degeneracy;

(v) from statement (iii) it was mistakenly assumed that there is no gravitational singularity at BH horizon.

We remind now canonical definitions.

Definition 2.1.2. Let (M, g) and (N, h) be semi-Riemannian manifolds. An isometric embedding is a smooth embedding $f : M \rightarrow N$ which preserves the metric in the sense that g is equal to the pullback of h by f , i.e. $g = f^*h$. Explicitly, for any two tangent vectors $\mathbf{v}, \mathbf{w} \in T_x(M)$ we have

$$g(\mathbf{v}, \mathbf{w}) = h(df(\mathbf{v}), df(\mathbf{w})). \tag{2.1.17}$$

Remark 2.1.9. Notice that such isometric embedding is a mathematical definition only and does not mean the equivalence $(M, g) \equiv (f(M), h)$ in absolute sense. Thus, it is not always appropriate as equivalence of the Lorentzian manifolds (M, g) and (N, h) corresponding to the physical frames $(M_{\text{ph}}, g_{\text{ph}})$ and $(N_{\text{ph}}, h_{\text{ph}})$.

Definition 2.1.3. [31]. In general, a Lorentzian manifold (M', h) is said to be an extension of a Lorentzian manifold (M, g) if there exists an isometric embedding $i : M \rightarrow M'$.

Remark 2.1.10. Notice that such extension is a mathematical definition only and therefore it is not always appropriate as extension of the Lorentzian manifolds (M, g) and (M', h) corresponding to the physical frames (M_{ph}, g_{ph}) and (M'_{ph}, h_{ph}) .

Remark 2.1.11. In order to obtain example for the statement mentioned and Remark 2.1.8 and Remark 2.1.9 we are going to prove below that the geometry of Schwarzschild spacetime $Sch_{>} \triangleq \{(S^2 \times \{r > 2m\}) \times \mathbb{R}, g_{Sch_{>}}\}$ above Schwarzschild horizon is essentially cardinally different in comparison with the geometry of Eddington-Finkelstein spacetime $EF_{>} \triangleq \{(S^2 \times \{r > 2m\}) \times \mathbb{R}, g_{EF_{>}}\}$ above Eddington-Finkelstein horizon.

We remind now canonical definitions.

Definition 2.1.4. Let $\Delta_{\Gamma} A_k$ be the change in a vector $A_i(\hat{x})$ after parallel displacement (as plotted in **Figure 2**) around closed contour Γ located in BH spacetime as plotted in **Figure 3**. This change $\Delta_{\Gamma} A_k$ can clearly be written in the form $\oint \delta A_k$. Substituting in place of δA_k the canonical expression $\delta A_k = \Gamma^i_{kl}(\hat{x}) A_k dx^l$ (see [31], Equation (85.5)) one obtains

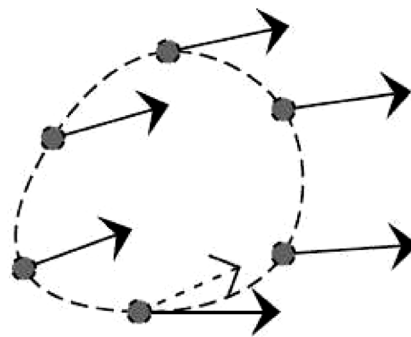


Figure 2. Paralel displacement along a closed contour Γ in a curved space.

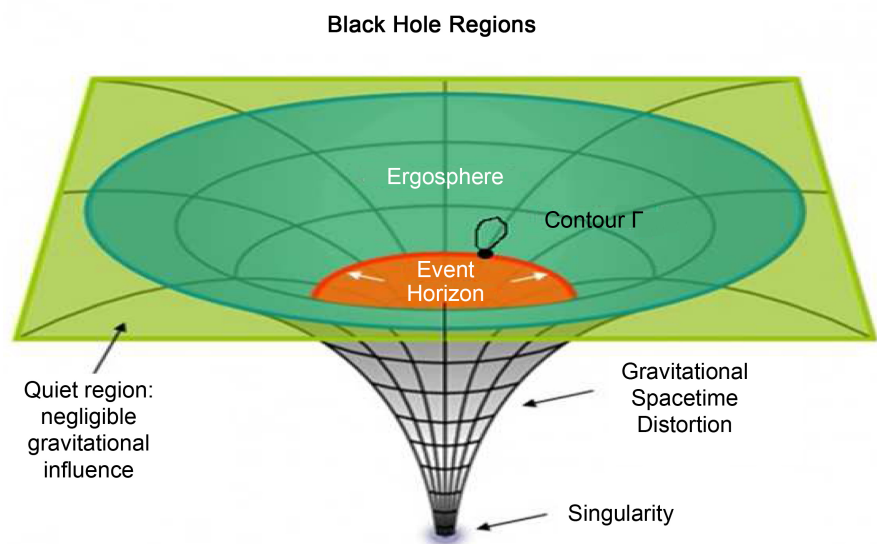


Figure 3. Paralel displacement along a closed contour Γ in BH spacetime.

$$\Delta_{\Gamma} A_k = \oint_{\Gamma} \delta A_k = \oint_{\Gamma} \Gamma_{kl}^i(\hat{x}) A_k dx^l. \tag{2.1.18}$$

Definition 2.1.5. (I) Let Σ_{Sch}^g be Schwarzschild horizon, let $\Gamma_{\hat{x}}$ be a contour located in Schwarzschild spacetime as plotted in **Figure 4** and such that (i) $\hat{x} \in \Gamma_{\hat{x}}$, (ii) $\Sigma_{Sch}^g \cap \Gamma_{\hat{x}} = \hat{x}$, and let $\Gamma^{\hat{x}}$ be a curve $\Gamma^{\hat{x}} = \Gamma_{\hat{x}} \setminus \hat{x}$. Let $\Delta_{\Gamma^{\hat{x}}} A_k$ be the integral

$$\Delta_{\Gamma^{\hat{x}}} A_k = \oint_{\Gamma_{\hat{x}} \setminus \hat{x}} \delta A_k = \oint_{\Gamma_{\hat{x}} \setminus \hat{x}} \Gamma_{kl}^i(\hat{x}) A_k dx^l. \tag{2.1.19}$$

(II) Let Σ_{EF}^g be Eddington-Finkelstein horizon, let $\Gamma_{\hat{x}}$ be a contour located in Eddington-Finkelstein spacetime as plotted in **Figure 5** and such that (i) $\hat{x} \in \Gamma_{\hat{x}}$, (ii) $\Sigma_{EF}^g \cap \Gamma_{\hat{x}} = \hat{x}$, and let $\Gamma^{\hat{x}}$ be a curve $\Gamma^{\hat{x}} = \Gamma_{\hat{x}} \setminus \hat{x}$. Let $\Delta_{\Gamma^{\hat{x}}} A_k$ be the integral

$$\Delta_{\Gamma^{\hat{x}}} A_k = \oint_{\Gamma_{\hat{x}} \setminus \hat{x}} \delta A_k = \oint_{\Gamma_{\hat{x}} \setminus \hat{x}} \Gamma_{kl}^i(\hat{x}) A_k dx^l. \tag{2.1.20}$$

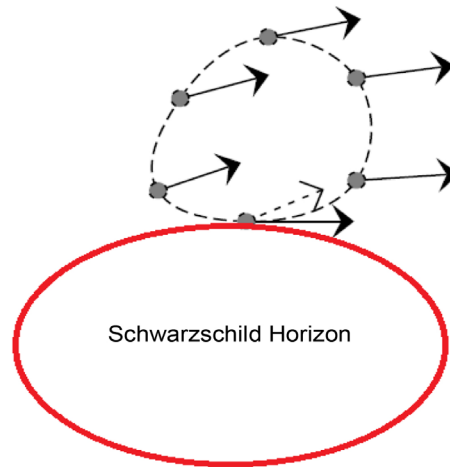


Figure 4. Parallel displacement $\Delta_{\Gamma^{\hat{x}}} A_k$ along a curve $\Gamma^{\hat{x}}$ in Schwarzschild spacetime such that $\Sigma_{Sch}^g \cap \Gamma_{\hat{x}} = \hat{x}$, then always $\Delta_{\Gamma^{\hat{x}}} A_k = \infty$.

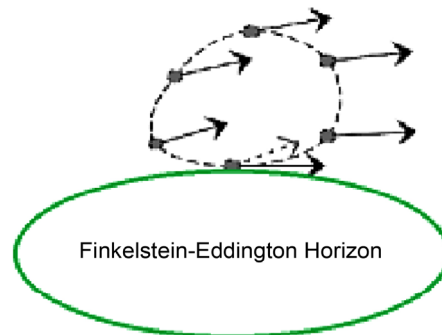


Figure 5. Parallel displacement along a curve $\Gamma^{\hat{x}}$ in Eddington-Finkelstein spacetime $\Sigma_{EF}^g \cap \Gamma_{\hat{x}} = \hat{x}$, then always $\Delta_{\Gamma^{\hat{x}}} A_k < \infty$.

Remark 2.1.12. (I) Note that the geometry of Schwarzschild spacetime $\mathbf{Sch}_>$

$$\mathbf{Sch}_> \triangleq \left\{ (\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{Sch}_>} \right\} \tag{2.1.21}$$

above Schwarzschild horizon $\Sigma_{\mathbf{Sch}}^g$, essentially cardinally different in comparizon with the geometry of Eddington-Finkelstein spacetime $\mathbf{EF}_>$

$$\mathbf{EF}_> \triangleq \left\{ (\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{EF}_>} \right\} \tag{2.1.22}$$

above Eddington-Finkelstein horizon $\Sigma_{\mathbf{EF}}^g$.

(II) Note that Schwarzschild spacetime $\mathbf{Sch}_>$ obviously satisfies a very strong nonregularity condition

$$\text{if } \Sigma_{\mathbf{Sch}}^g \cap \Gamma_{\hat{x}} = \hat{x}, \text{ then } \Delta_{\Gamma_{\hat{x}}} A_k = \infty. \tag{2.1.23}$$

Thus the geometry of spacetime $\mathbf{Sch}_>$ that is nonclassical geometry beyond apparatus of the classical semi-Riemannian geometry. Of course, the geometry any part of spacetime $\mathbf{Sch}_>$ located above some neighborhood of Schwarzschild horizon as plotted in **Figure 6** that is a classical semi-Riemannian geometry.

Remark 2.1.13. Note that from Remark 2.1.11 it follows that Eddington-Finkelstein spacetime does not hold in rigorous mathematical sense as extension of the Schwarzschild spacetime $\mathbf{Sch}_> \triangleq \left\{ (\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{Sch}_>} \right\}$ above Schwarzschild horizon.

Remark 2.1.14. It is clear that nonregularity condition (2.1.23) arises not only from singularity of the function $h^{-1}(r)$ at point $r = r_g$ but from degeneracy of the function $h(r)$ at point $r = r_g$.

Remark 2.1.15. We remind now that the relations (see [30] p. 234, Equation (84.7))

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}} \tag{2.1.24}$$

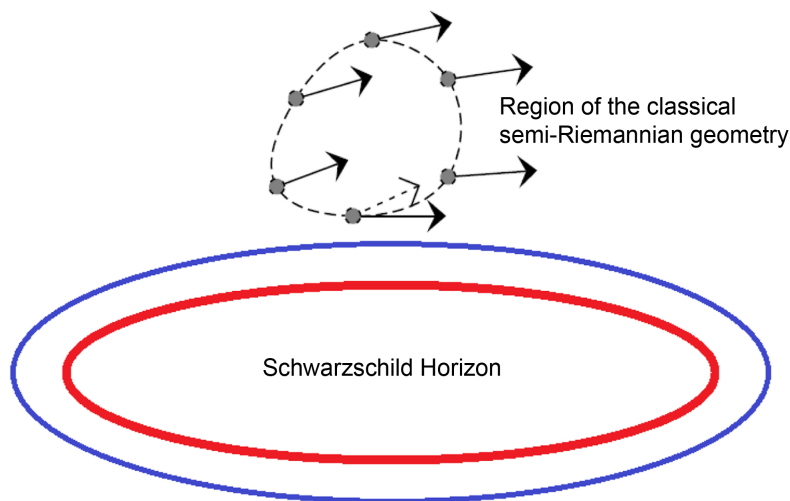


Figure 6. Parallel displacement along a closed contour Γ located in region of the classical semi-Riemannian geometry of the Schwarzschild spacetime such that $\Sigma_{\mathbf{sh}}^g \cap \Gamma = \emptyset$, then always $\Delta_{\Gamma} A_k < \infty$.

give the connection between the metric of real space

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta \tag{2.1.25}$$

and the metric of the four-dimensional space-time

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + 2g_{0\alpha} dx^0 dx^\alpha + g_{00} (dx^0)^2. \tag{2.1.26}$$

For Eddington-Finkelstein metric (2.1.15) metric of the corresponding real space is

$$dl_{EF}^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 \left[(d\theta)^2 + \sin^2 \theta (d\phi)^2 \right]. \tag{2.1.27}$$

Remark 2.1.16. Notice that the Eddington-Finkelstein metric (2.1.15) is regular at the horizon and therefore the infalling observer encounters nothing unusual at the horizon. However from Equation (2.1.17) it follows that the infalling observer encounters singularity on horizon. But this is a contradiction.

Remark 2.1.17. Note that in order to deal with singular Schwarzschild metric (2.1.11) using mathematically and logically soundness approach, one applies contemporary distributional geometry based on Colombeau generalized functions [2] [3] [4]. Distributional Schwarzschild geometry and distributional BHs geometry by using Colombeau generalized functions [2] [3] [4] was developed by many papers [4]-[22]. By appropriate regularization $g_{ij,\varepsilon}(r, \theta, \phi), \varepsilon \in (0, 1]$ of the singular Schwarzschild metric $g_{ij}(r, \theta, \phi)$ such that:

- (i) $g_{ij,0}(r, \theta, \phi) = g_{ij}(r, \theta, \phi)$ and
- (ii) for any $\varepsilon \in (0, 1]$ metric tensor $g_{ij,\varepsilon}(r, \theta, \phi)$ is regular and nondegenerate, one obtains Colombeau generalized object $\left[\left(g_{ij,\varepsilon}(r, \theta, \phi) \right)_\varepsilon \right] \in \mathbf{G}(\mathbb{R}^3)$ with an representative $(g_{ij,\varepsilon}(r, \theta, \phi))_\varepsilon$, for a more detailed explanation see [11] [18] [19]. Using rigorous Colombeau approach one obtains mathematically and logically soundness notion of singularity in

Distributional Schwarzschild spacetime.

Remark 2.1.18. Note that in the case of Schwarzschild spacetime the conditions (i) and (ii) mentioned above (see Remark 2.1.13) are satisfied only by using non smooth regularization of the singular and degenerate Schwarzschild metric $g_{ij}(r, \theta, \phi)$ via Schwarzschild horizon [18] [19].

By appropriate nonsmooth regularization one obtain Colombeau generalized object modeling the singular Schwarzschild metric above and below horizon [18] [19]:

$$\begin{aligned} (ds_\varepsilon^{+2})_\varepsilon &= -(h_\varepsilon^+(r) dt^2)_\varepsilon + \left([h_\varepsilon^+(r)]^{-1} dr^2 \right)_\varepsilon + r^2 d\Omega^2, \\ (ds_\varepsilon^{-2})_\varepsilon &= (h_\varepsilon^-(r) dt^2)_\varepsilon - \left([h_\varepsilon^-(r)]^{-1} dr^2 \right)_\varepsilon + r^2 d\Omega^2, \\ h_\varepsilon^+(r) &= \frac{\Theta_\varepsilon((r-r_s) - \varepsilon) \sqrt{(r-r_g)^2 + \varepsilon^2}}{r}, r \geq r_g, \\ \varepsilon &\in (0, 1]. \end{aligned} \tag{2.1.28}$$

Remark 2.1.19. Let us rewrite now the metric (2.1.24) (above horizon) in the form

$$\begin{aligned} (ds_{\epsilon}^{+2})_{\epsilon} &= -(h_{\epsilon}^{+}(r)dt^2)_{\epsilon} + \left([h_{\epsilon}^{+}(r)]^{-1} dr^2 \right)_{\epsilon} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= -(h_{\epsilon}^{+}(r))_{\epsilon} \left(\left[dt - [h_{\epsilon}^{+}(r)]^{-1} dr \right] \left[dt + [h_{\epsilon}^{+}(r)]^{-1} dr \right] \right)_{\epsilon} \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{2.1.29}$$

and define a new generalized Colombeau coordinates $((\tau_{\epsilon})_{\epsilon}, \bar{r}, \theta, \varphi)$, where $(\tau_{\epsilon}(t, r))_{\epsilon} \in \mathbf{G}(\mathbb{R}^2)$, by formula

$$\begin{aligned} (d\tau_{\epsilon}(t, r))_{\epsilon} &= dt + \left([h_{\epsilon}^{+}(r)]^{-1} dr \right)_{\epsilon}, \\ \bar{r} &= r. \end{aligned} \tag{2.1.30}$$

Remark 2.1.20. Notice that:

(i) Colombeau generalized coordinates (2.1.26) are the Colombeau extension of the canonical Eddington-Finkelstein coordinates (2.1.14) by Colombeau generalized function.

(ii) In contrast with canonical Eddington-Finkelstein coordinates (2.1.14) (see Remark 2.1.7), Colombeau generalized coordinates (2.1.26) holds at Schwarzschild horizon $r = r_g$ as at Schwarzschild horizon Colombeau generalized function $\left([h_{\epsilon}^{+}(r)]^{-1} \right)_{\epsilon}$ become well defined Colombeau generalized number $[h_{\epsilon}^{+}(r_g)]^{-1} \in \mathbb{R}$.

Rewriting now the metric (2.1.25) in terms of the Colombeau generalized coordinates $((\tau_{\epsilon})_{\epsilon}, \bar{r}, \theta, \varphi)$, it then above horizon takes the form

$$\begin{aligned} (ds_{\epsilon}^{+2})_{\epsilon} &= -\left((h_{\epsilon}^{+}(\bar{r}))_{\epsilon} \right) \left(\left[d\tau_{\epsilon} - 2[h_{\epsilon}^{+}(\bar{r})]^{-1} d\bar{r} \right] d\tau_{\epsilon} \right)_{\epsilon} + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= -\left((h_{\epsilon}^{+}(\bar{r}))_{\epsilon} \right) (d\tau_{\epsilon}^2)_{\epsilon} + 2d\bar{r}(d\tau_{\epsilon})_{\epsilon} + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \tag{2.1.31}$$

We rewrite now Colombeau metric (2.1.27) in the equivalent form

$$(ds_{\epsilon}^{+2})_{\epsilon} = -(h_{\epsilon}^{+}(r)dt^2)_{\epsilon} + 2drd\tau + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.1.32}$$

Colombeau metric (2.1.28) define the distributional Eddington-Finkelstein space-time

$$\mathbf{EF}_{\geq}^{+} \triangleq \left\{ (\tilde{\mathcal{S}}^2 \times \{\bar{r} \geq 2m\}) \times \tilde{\mathbb{R}}, g_{\mathbf{EF}_{\geq}^{+}} \right\} \tag{2.1.33}$$

above the Eddington-Finkelstein horizon $r = 2m$.

Remark 2.1.21. Notice that

$$\begin{aligned} (h_{\epsilon}^{+}(r))_{\epsilon} \Big|_{r=r_g} &= r_g^{-1}(\epsilon)_{\epsilon}, \left([h_{\epsilon}^{+}(r)]^{-1} \right)_{\epsilon} \Big|_{r=r_g} = r_g \cdot (\epsilon^{-1})_{\epsilon} \in \tilde{\mathbb{R}}, \\ (d\tau_{\epsilon})_{\epsilon} \Big|_{r=r_g} &= dt + \left((\epsilon^{-1})_{\epsilon} \right) \cdot r_g dr, \\ (d\tau_{\epsilon}^2)_{\epsilon} \Big|_{r=r_g} &= dt^2 + 2 \left((\epsilon^{-1})_{\epsilon} \right) \cdot r_g dt dr + \left((\epsilon^{-2})_{\epsilon} \right) r_g^2 dr^2, \\ \epsilon &\in (0, 1]. \end{aligned} \tag{2.1.34}$$

Of course at horizon $\left(h_\varepsilon^+(t, r_g)\right)_\varepsilon \approx 0$, because at horizon $h_0^+(t, r_g) = 0$, however it follows from (2.1.24) at horizon the quantities $\left(\left(h_\varepsilon^+(r_g)\right)_\varepsilon\right)\left(d\tau_\varepsilon^2(t, r_g)\right)_\varepsilon \approx \left(\left(\varepsilon^{-1}\right)_\varepsilon\right)r_g dr^2$ and $\left(d\tau_\varepsilon\right)_\varepsilon \approx \left(\left(\varepsilon^{-1}\right)_\varepsilon\right)r_g dr$ are infinite large Colombeau quantities, *i.e.*, the differential $\left(d\tau_\varepsilon\right)_\varepsilon$ is not classical but it is Colombeau differential.

Remark 2.1.22. Note that:

(i) under coordinate change (2.1.26) the distributional curvature scalars of the distributional Schwarzschild space-time given by metric (2.1.24), does not changes because these scalars depend only on variable $r = \bar{r}$,

(ii) in contrast with classical Eddington-Finkelstein space-time

$$\mathbf{EF}_\pm = \left(\mathbf{S}^2 \times \{r \geq 2m\} \cup \{0 < r \leq 2m\}\right) \times \mathbb{R}, g_{\mathbf{EF}_\pm}(r, \theta),$$

distributional Eddington-Finkelstein space-time has a gravitational singularity at horizon.

Remark 2.1.23. Note that for the case of the distributional space-time the relations (2.1.24) obviously takes the form

$$\left(\gamma_{\alpha\beta}(\varepsilon)\right)_\varepsilon = \left(-g_{\alpha\beta}(\varepsilon) + \frac{g_{0\alpha}(\varepsilon)g_{0\beta}(\varepsilon)}{g_{00}(\varepsilon)}\right)_\varepsilon \tag{2.1.35}$$

where (2.1.30) give the connection between the Colombeau metric of the distributional real space

$$\left(dl_\varepsilon^2\right)_\varepsilon = \left(\left(\gamma_{\alpha\beta}(\varepsilon)dx^\alpha dx^\beta\right)\right)_\varepsilon \tag{2.1.36}$$

and the Colombeau metric of the four-dimensional distributional space-time

$$\begin{aligned} \left(ds_\varepsilon^2\right)_\varepsilon &= \left(g_{\alpha\beta}(\varepsilon)dx^\alpha dx^\beta\right)_\varepsilon + 2\left(g_{0\alpha}(\varepsilon)dx^0 dx^\alpha\right)_\varepsilon \\ &+ \left(g_{00}(\varepsilon)(dx^0)^2\right)_\varepsilon. \end{aligned} \tag{2.1.37}$$

For distributional Eddington-Finkelstein metric (2.1.29) above horizon of the corresponding Colombeau metric of the distributional real space is

$$\begin{aligned} \left(dl_{\varepsilon, \mathbf{EF}_\pm^+}^2\right)_\varepsilon &= \left(\left(h_\varepsilon^+(r)\right)^{-1} dr^2\right)_\varepsilon \\ &+ r^2 \left[(d\theta)^2 + \sin^2 \theta (d\phi)^2\right]. \end{aligned} \tag{2.1.38}$$

Remark 2.1.24. Notice since the distributional Eddington-Finkelstein space-time (2.1.29) has a gravitational singularity (see Definition 1.1.1) at horizon, there is no contradiction mentioned above for the case of the regular classical Eddington-Finkelstein metric (2.1.15) and the corresponding singular metric (2.1.17), see Remark 2.1.15.

2.1.2 Distributional Kruskal-Szekeres Spacetime

Recall that the classical Kruskal-Szekeres coordinates are defined, from the classical Schwarzschild coordinates (t, r, θ, ϕ) , by replacing t and r by a new time coordinate T and a new spatial coordinate X :

$$\begin{aligned}
 r > r_g : T &= \left(\frac{r}{r_g} - 1\right)^{1/2} \exp\left(\frac{r}{r_g}\right) \sinh\left(\frac{t}{2r_g}\right), \\
 X &= \left(\frac{r}{r_g} - 1\right)^{1/2} \exp\left(\frac{r}{r_g}\right) \cosh\left(\frac{t}{2r_g}\right), \\
 0 < r < r_g : T &= \left(1 - \frac{r}{r_g}\right)^{1/2} \exp\left(\frac{r}{r_g}\right) \sinh\left(\frac{t}{2r_g}\right), \\
 X &= \left(1 - \frac{r}{r_g}\right)^{1/2} \exp\left(\frac{r}{r_g}\right) \cosh\left(\frac{t}{2r_g}\right).
 \end{aligned}
 \tag{2.1.39}$$

It follows that the Schwarzschild radius r_s in terms of Kruskal-Szekeres coordinates, is implicitly given by

$$T^2 - X^2 = \left(1 - \frac{r}{r_g}\right)^{1/2} \exp\left(\frac{r}{r_g}\right)
 \tag{2.1.40}$$

for both interior and exterior regions, i.e. $r \in \mathbb{R}_+ \setminus \{0\}$. In these new coordinates the metric of the Schwarzschild black hole manifold is given by

$$ds^2 = \frac{4r_g^3}{r} \exp\left(-\frac{r}{r_g}\right) \left[-dT^2 + dX^2\right] + r^2 d\Omega^2.
 \tag{2.1.41}$$

The location of the event horizon ($r_g = 2GM$) in these coordinates obviously is given by

$$T^2 - X^2 = 0 \Rightarrow T = \pm X.
 \tag{2.1.42}$$

Remark 2.1.25. Note that the metric (2.1.37) ofcourse is perfectly well defined and non-singular at the event horizon. The curvature singularity is located at $T^2 - X^2 = 1$. Under this property Kruskal-Szekeres spacetime in physical literature mistakenly considered as regular Lorentzian spacetime, except singular submanifold $\{(T, X) | T^2 - X^2 = 1\}$.

Remark 2.1.26. In contrast with Eddington-Finkelstein coordinates the classical Kruskal-Szekeres coordinates holds at Schwarzschild horizon, but however the differentials dT, dX of the functions $T(r, t), X(r, t)$ are singular at Schwarzschild horizon $r = r_g$ and therefore Kruskal-Szegeres spacetime cannot be considered as Schwarzschild spacetime in Kruskal-Szekeres coordinates (2.1.35)-(2.1.36).

Remark 2.1.27. In order to avoid these difficulties one can apply instead of the Kruskal-Szekeres coordinates (2.1.35)-(2.1.36) the following distributional Kruskal-Szekeres coordinates to Colombeau generalized metric (2.1.8)

$$r \geq r_g, \varepsilon \in (0, 1]:$$

$$\begin{aligned}
 (T_\varepsilon)_\varepsilon &= \left[\left(\Theta_\varepsilon \left((r - r_g) - \varepsilon \right) \right)_\varepsilon \right] \left[\left(\left(\frac{r}{r_g} - 1 + \varepsilon \right)^{1/2} \right)_\varepsilon \right] \exp\left(\frac{r}{r_g}\right) \sinh\left(\frac{t}{2r_g}\right), \\
 (X_\varepsilon)_\varepsilon &= \left[\left(\Theta_\varepsilon \left((r - r_g) - \varepsilon \right) \right)_\varepsilon \right] \left[\left(\left(\frac{r}{r_g} - 1 + \varepsilon \right)^{1/2} \right)_\varepsilon \right] \exp\left(\frac{r}{r_g}\right) \cosh\left(\frac{t}{2r_g}\right),
 \end{aligned}$$

$$0 < r < r_g :$$

$$\begin{aligned} (T_\varepsilon)_\varepsilon &= \left[\left(\Theta_\varepsilon \left((r - r_g) - \varepsilon \right) \right)_\varepsilon \right] \left[\left(\left(1 - \frac{r}{r_g} + \varepsilon \right)^{1/2} \right)_\varepsilon \right] \exp \left(\frac{r}{r_g} \right) \sinh \left(\frac{t}{2r_g} \right), \\ (X_\varepsilon)_\varepsilon &= \left[\left(\Theta_\varepsilon \left((r - r_g) - \varepsilon \right) \right)_\varepsilon \right] \left[\left(\left(1 - \frac{r}{r_g} + \varepsilon \right)^{1/2} \right)_\varepsilon \right] \exp \left(\frac{r}{r_g} \right) \cosh \left(\frac{t}{2r_g} \right). \end{aligned} \tag{2.1.43}$$

Therefore for both interior and exterior regions we get

$$(T_\varepsilon^2)_\varepsilon - (X_\varepsilon^2)_\varepsilon = \left[\left(\left(1 - \frac{r}{r_g} + \varepsilon \right)^{1/2} \right)_\varepsilon \right] \exp \left(\frac{r}{r_g} \right). \tag{2.1.44}$$

Remark 2.1.28. Note that in contrast with (2.1.37) at horizon $r = r_g$:

$$(T_\varepsilon^2)_\varepsilon - (X_\varepsilon^2)_\varepsilon = e \left[(\varepsilon^{1/2})_\varepsilon \right] \neq 0_{\mathbb{R}}. \tag{2.1.45}$$

In these new distributional coordinates the Colombeau metric (2.1.8) of the distributional Schwarzschild black hole manifold above horizon is given by formula

$$(ds_\varepsilon^2)_\varepsilon = \left(\frac{4r_g^3 \left(\Theta_\varepsilon \left((r - r_g) - \varepsilon \right) \right)_\varepsilon}{r} \exp \left(-\frac{r}{r_g} \right) \left[-dT_\varepsilon^2 + dX_\varepsilon^2 \right] \right)_\varepsilon + r^2 d\Omega^2. \tag{2.1.46}$$

Here $(\Theta_\varepsilon(u))_\varepsilon$ is the generalized Heaviside function given by Equation (2.1.3).

Remark 2.1.29. Note that in contrast with (2.1.36) Colombeau generalized metric (2.1.39) is non degenerate at horizon $r = r_g$ in Colombeau sense.

2.2. Distributional Schwarzschild Spacetime and Distributional Rindler Spacetime with Distributional Levi-Civita Connection. Generalized Einstein Equivalence Principle

2.2.1. Distributional Schwarzschild Spacetime with Distributional Levi-Civita Connection

Remark 2.2.1. Note that due to the degeneracy of the metric (2.1.11) at Schwarzschild horizon, the classical Levi-Civita connection on whole Schwarzschild spacetime is not available [18] [19] as classical Levi-Civita connection on Schwarzschild horizon becomes infinity

$$\Gamma_{11}^1(r) \Big|_{r=2m} = \lim_{r \rightarrow 2m} \frac{-m}{r(r-2m)} = -\infty, \quad \Gamma_{01}^0(r) \Big|_{r=2m} = \lim_{r \rightarrow 2m} \frac{m}{r(r-2m)} = \infty, \tag{2.2.1}$$

Remark 2.2.2. In order to avoid difficulties with classical Levi-Civita connection mentioned above in Remark 2.2.1, in papers [18] [19] we have applied the non smooth regularization via Schwarzschild horizon, see Remark 2.1.5 and Equation (2.1.6). Corresponding Colombeau distributional connections $(\Gamma_{kj}^{+l}(\varepsilon))_\varepsilon$ and $(\Gamma_{kj}^{-l}(\varepsilon))_\varepsilon$ above and below Schwarzschild horizon are [18] [19]:

$$\begin{aligned} (\Gamma_{kj}^{+l}(\varepsilon))_\varepsilon &= \frac{1}{2} \left((g_\varepsilon^{+lm}) \left[(g_\varepsilon^+)_{mk,j} + (g_\varepsilon^+)_{mj,k} - (g_\varepsilon^+)_{kj,m} \right] \right)_\varepsilon, \\ (\Gamma_{kj}^{-l}(\varepsilon))_\varepsilon &= \frac{1}{2} \left((g_\varepsilon^{-lm}) \left[(g_\varepsilon^-)_{mk,j} + (g_\varepsilon^-)_{mj,k} - (g_\varepsilon^-)_{kj,m} \right] \right)_\varepsilon. \end{aligned} \tag{2.2.2}$$

Obviously distributional connections $(\Gamma_{kj}^{+l}[h_\varepsilon^+])_\varepsilon, (\Gamma_{kj}^{-l}[h_\varepsilon^-])_\varepsilon$ coincides, in distributional sense, with the corresponding classical Levi-Civita connections on $\mathbb{R}^3 \setminus \{r = 2m\}$, since $(h_\varepsilon^+)_\varepsilon = h_0^+, (h_\varepsilon^-)_\varepsilon = h_0^-$ and $(g_\varepsilon^{+lm})_\varepsilon = g_0^{+lm}, (g_\varepsilon^{-lm})_\varepsilon = g_0^{-lm}$ there. Clearly, connections $\Gamma_{kj}^{+l}(\varepsilon), \Gamma_{kj}^{-l}(\varepsilon), \varepsilon \in (0, 1]$ in respect the regularized metric $g_\varepsilon^\pm, \varepsilon \in (0, 1], i.e., (g_\varepsilon^\pm)_{ij;k} = 0$. Proceeding in this manner, we obtain the nonstandard result [22] [23] see also **Appendix B**:

$$\begin{aligned} \left(\left[R_\varepsilon^+ \right]_1^1 \right)_\varepsilon &= \left(\left[R_\varepsilon^+ \right]_0^0 \right)_\varepsilon = -4\pi m \frac{\delta(r-2m)}{r^2}, \\ \left(\left[R_\varepsilon^- \right]_1^1 \right)_\varepsilon &= \left(\left[R_\varepsilon^- \right]_0^0 \right)_\varepsilon = 4\pi m \frac{\delta(r-2m)}{r^2}. \end{aligned} \tag{2.2.3}$$

Remark 2.2.3. As expected, the distributional Ricci tensor as well as the distributional Ricci scalar vanish identically on $\mathbb{R}^3 \setminus \{r = 2m\}$, since $\text{supp}(\delta(r-2m)) = \{r = 2m\}$. This result is in a good agreement with canonical result [24]-[30] on $\mathbb{R}^3 \setminus \{r = 2m\}$ since distributional connections (2.2.2) coincide with the corresponding classical Levi-Civita connections on $\mathbb{R}^3 \setminus \{r = 2m\}$ at least in distributional sense. For $(r_\varepsilon - 2m)_\varepsilon \approx_{\mathbb{R}} 0$ we obtain the nonstandard result:

$$\begin{aligned} (\mathbf{R}^{\pm\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}^\pm(r_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} \left(\frac{\varepsilon^4}{4m^4 (\varepsilon^2 + (r_\varepsilon - 2m)^2)^3} \right)_\varepsilon + \dots, \\ (\mathbf{R}^{\pm\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}^\pm(r_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} \left(\frac{\varepsilon^4}{4m^4 (\varepsilon^2 + (r_\varepsilon - 2m)^2)^3} \right)_\varepsilon + \dots, \end{aligned} \tag{2.2.3}$$

where $|(r_\varepsilon - 2m)_\varepsilon| \approx_{\mathbb{R}} 0$, see **Appendix C**. For $|(r_\varepsilon - 2m)_\varepsilon| \approx_{\mathbb{R}} (\varepsilon)_\varepsilon$, see **Appendix C**, Remark C.10, Equation (C22), we obtain [18] [19]:

$$\begin{aligned} (\mathbf{R}^{\pm\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}^\pm(r_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} O(1) \left(\frac{\varepsilon^4}{4m^4 [(r_\varepsilon - 2m)^2 + \varepsilon^2]} \right)_{\varepsilon \in (0, \eta]} + \dots, \\ (\mathbf{R}^{\pm\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}^\pm(r_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} O(1) \left(\frac{\varepsilon^4}{4m^4 [(r_\varepsilon - 2m)^2 + \varepsilon^2]} \right)_{\varepsilon \in (0, \eta]} + \dots, \end{aligned} \tag{2.2.4}$$

For $|r-2m| \in (0, \eta], \eta \ll 1$, see **Appendix C**, Remark C.10, Equation (C22), we obtain

$$\begin{aligned} (\mathbf{R}^{\pm\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}^\pm(r_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} O(1) \frac{1}{4m^4 (r-2m)_{|r-2m| \in (0, \eta]}^2} + \dots, \\ (\mathbf{R}^{\pm\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}^\pm(r_\varepsilon, \varepsilon))_\varepsilon &\approx_{\mathbb{R}} O(1) \frac{1}{4m^4 (r-2m)_{|r-2m| \in (0, \eta]}^2} + \dots. \end{aligned} \tag{2.2.4}$$

2.2.2. Distributional Rindler Space-Time with Distributional Levi-Civita Connection. Non-Regularity Conditions and Nonclassical Nature of the Rindler Space-Time

We remind now that 2D Rindler spacetime is a patch of Minkowski spacetime, see **Figure 7**. In 2D, the Rindler metric is

$$ds^2 = dR^2 - R^2 d\eta^2. \tag{2.2.5}$$

Remark 2.2.4. Due to the degeneracy of the metric (2.2.5) at Rindler horizon $R = 0$, the classical Levi-Civita connection is not available on whole \mathbb{R}^2 , e.g.,

$$\Gamma_{44}^1 = R, \Gamma_{14}^4 = \Gamma_{41}^4 = R^{-1}, \tag{2.2.6}$$

and all other components being zero.

Remark 2.2.5. We emphasize that Rindler space-time $\mathbb{R}^{1,1}$ is satisfied the same non-regularity conditions as Schwarzschild space-time Sch_{\gg} , see Remark 2.1.12. Let $\Gamma_{\hat{x}}$ be a contour located in Rindler space-time $\mathbb{R}^{1,1}$ and let \hat{x} be Rindler horizon as plotted in **Figure 8** and let $\Delta_{\Gamma} A_k$ be the change in a vector $A_k(x)$ after parallel displacement (as plotted in **Figure 2**) around closed contour Γ (see Definition 2.1.4) located in Rindler space-time $\mathbb{R}^{1,1}$ as plotted in **Figure 8**.

Remark 2.2.6. We emphasize that in physical literature the Rindler metric (2.2.5) mistakenly were considered as is just a part of the Minkowski space-time $\mathbb{R}^{1,1}$. Obviously by non-regularity conditions $\Delta_{\Gamma} A_k = \infty$ the geometry of Rindler space-time essentially ardinally different in comparizon with the geometry of Minkowski space-time $\mathbb{R}^{1,1}$ even if the Rindler horizon is excluded

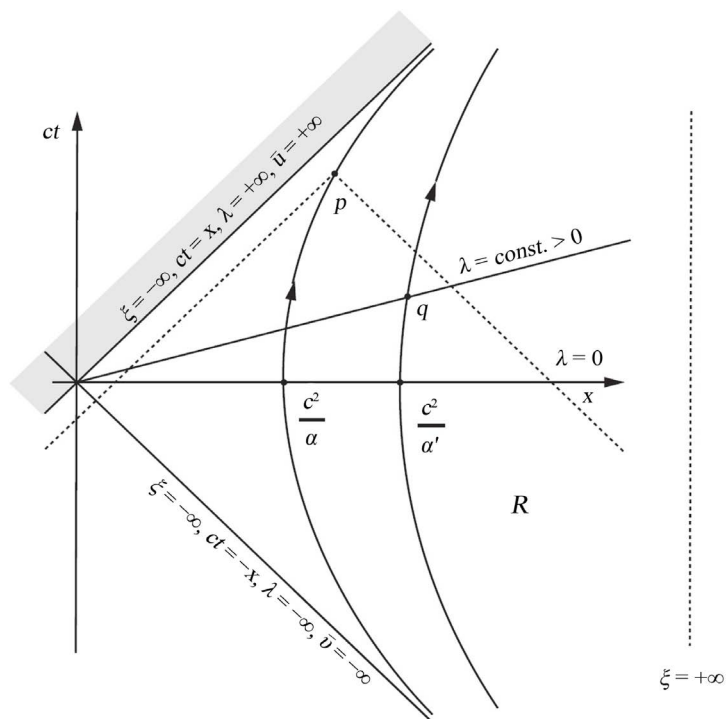


Figure 7. Hyperbolic motion in the right rindler wedge. $x^2 - c^2 t^2 = (c^2/a)^2$.

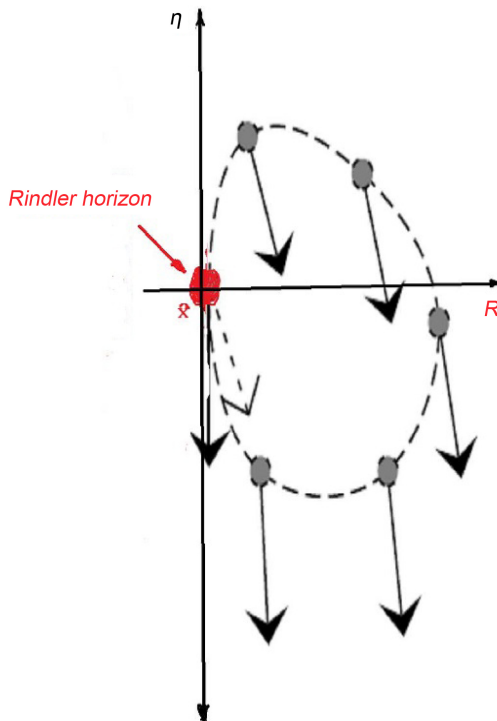


Figure 8. This is the right wedge, which covers one quarter of the Penrose diagram. Parallel displacement along a curve $\Gamma^{\hat{x}}$ in Rindler spacetime $\mathbb{R}^{1,1}$ such that $\Gamma^{\hat{x}} = \Gamma_{\hat{x}} \setminus \{\hat{x}\}$, then always $\Delta_{\Gamma} A_k = \infty$.

from the space $\mathbb{R}^{1,1}$.

Remark 2.2.7. Note that in order to avoid this difficultnes mentioned above (see Remark 2.2.4-2.2.5), the origin in classical consideration the Rindler horizon is always excluded from the space $\mathbb{R}^{3,1}$ and we are working on $\mathbb{R}^{3,1} = \mathbb{R}^{3,1} \setminus \{0\}$, and therefore for Einstein’s tensor

$$\mathbf{G}_i^k \triangleq \mathbf{R}_i^k - \frac{1}{2} \delta_i^k \mathbf{R}, \quad \mathbf{R} \triangleq \mathbf{R}_i^i \tag{2.2.7}$$

following Moller [24] we get

$$\mathbf{G}_2^2 = \mathbf{G}_3^3 = -\frac{1}{2g_{44}} \left[g_{44}'' - \frac{(g_{44}')^2}{2g_{44}} \right] = -\frac{1}{2R^2} \left[2 - \frac{(2R)'}{2R^2} \right] \equiv 0, \tag{2.2.8}$$

where the accents indicate differentiation with respect variable R , and all other components of \mathbf{G}_i^k vanish identically. Thus Rindler metrical tensor satisfy on $\mathbb{R}^{3,1} \setminus \{0\}$ the Einstein field equations

$$\mathbf{G}_i^k \triangleq \mathbf{R}_i^k - \frac{1}{2} \delta_i^k \mathbf{R} = 0. \tag{2.2.9}$$

Remark 2.2.8. By calculations mentioned above, from Mo’ller’s times until nowadays, Rindler metrical tensor was mistakenly considered in physical literature as an vacuum solution of the Einstein’s field equations,e.g.,solution for empty space,see Møller [23].

Remark 2.2.9. Note that Levi-Civita connection on the whole space $\mathbb{R}^{3,1}$ is available only in Colombeau sense under smooth regularization $R^2 \rightarrow R^2 + \varepsilon^2, \varepsilon \in (0,1]$ and therefore we forced to change metric (2.5) by Colombeau object

$$\begin{aligned} \left[\left(ds_\varepsilon^2 \right)_\varepsilon \right] &= dR^2 - \left[\left(R^2 + \varepsilon^2 \right)_\varepsilon \right] d\eta^2 = dR^2 - \left[\left(g_{44,\varepsilon} \right)_\varepsilon \right] d\eta^2 \\ \left[\left(g_{44,\varepsilon} \right)_\varepsilon \right] &= \left[\left(R^2 + \varepsilon^2 \right)_\varepsilon \right], \varepsilon \in (0,1]. \end{aligned} \tag{2.2.10}$$

Then for Einstein distributional tensor [18] [19] [20]:

$$\left(\mathbf{G}_{i,\varepsilon}^k \right)_\varepsilon \triangleq \left(\mathbf{R}_{i,\varepsilon}^k \right)_\varepsilon - \frac{1}{2} \delta_i^k \left(\mathbf{R}_\varepsilon \right)_\varepsilon, \left(\mathbf{R}_\varepsilon \right)_\varepsilon \triangleq \mathbf{R}_{i,\varepsilon}^i \tag{2.2.11}$$

we get

$$\left(\mathbf{G}_{2,\varepsilon}^2 (R) \right)_\varepsilon = \left(\mathbf{G}_{3,\varepsilon}^3 (R) \right)_\varepsilon = - \left(\frac{1}{2g_{44,\varepsilon}} \left[g_{44,\varepsilon}'' - \frac{\left(g_{44,\varepsilon}' \right)^2}{2g_{44,\varepsilon}} \right] \right)_\varepsilon - \left(\frac{\varepsilon^2}{\left(R^2 + \varepsilon^2 \right)^2} \right)_\varepsilon. \tag{2.2.12}$$

Thus,

$$\left[\left(\mathbf{G}_{2,\varepsilon}^2 (0) \right)_\varepsilon \right] = \left[\left(\mathbf{G}_{3,\varepsilon}^3 (0) \right)_\varepsilon \right] = \left[\left(\varepsilon^{-2} \right)_\varepsilon \right], \tag{2.2.13}$$

where $\left[\left(\varepsilon^{-2} \right)_\varepsilon \right] \in \tilde{\mathbb{R}}$ is infinite Colombeau generalized numbers, and therefore $\left(\mathbf{G}_{2,\varepsilon}^2 (R) \right)_\varepsilon$ and $\left(\mathbf{G}_{3,\varepsilon}^3 (R) \right)_\varepsilon$ is nontrivial Colombeau generalized functions and distributional Rindler metric tensor given by (2.2.12) that is non vacuum Colombeau solution of the Einstein field equations.

2.2.3. Generalized Einstein Equivalence Principle

We remind that originally Einstein’s gravity was formulated by using classical pseudo Riemannian geometry with classical Levi-Civita’s connection. In classical pseudo Riemannian geometry, the Levi-Civita connection is a specific connection on the tangent bundle of a manifold. More specifically, it is the torsion-free metric connection, *i.e.*, the torsion-free connection on the tangent bundle (an affine connection) preserving a given (pseudo-Riemannian) Riemannian metric. The fundamental theorem of classical Riemannian geometry states that there is a unique connection which satisfies these properties.

Remark 2.3.1. Note that classical Einstein “Equivalence Principle” asserts the equivalence between inertial and gravitational forces of acceleration. The classical Einstein equivalence principle is the heart and soul of gravitational theory, for it is possible to argue convincingly that if EEP is valid, then gravitation must be a “curved spacetime” phenomenon, in other words, gravity must be governed by a “metric theory of gravity”, whose postulates are:

- 1) Spacetime is endowed with a symmetric Lorentzian metric.
- 2) The trajectories of freely falling test bodies are geodesics of that metric.
- 3) In local freely falling reference frames, the non-gravitational laws of physics are those written in the language of special relativity.

In order to obtain appropriate generalization of EEP based on distributional Colombeau geometry [4] [5] [6] [7] we claim the following generalized

equivalence principle (GEEP):

- 1) Spacetime in general case is endowed with a symmetric distributional Lorentzian metric.
- 2) The trajectories of freely falling test bodies are geodesics of that distributional metric.
- 3) In local freely falling distributional reference frames, the non-gravitational laws of physics are those written in the language of special relativity.

3. Quantum Scalar Field in Curved Distributional Spacetime. Unruh Effect Revisited

3.1. Canonical Quantization in Curved Distributional Spacetime

In a recent work [19] the authors advocated the use De Witt-Schwinger approach [37] [38] [39] [40] in order to establish QFT in general ditributional curved spacetime. The vacuum energy density of free scalar quantum field Φ with a distributional background spacetime is considered successfully. It has been widely believed that, except in very extreme situations, the influence of gravity on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is false by showing that there exist well-behaved spacetime evolutions where the vacuum energy density of free quantum fields is forced, by the very same background distributional spacetime such as in BHs, to become dominant over any classical energy density component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on curved distributional spacetimes. In particular we obtain that the vacuum fluctuations $\langle \Phi^2 \rangle$ have a singular behavior on BHs horizon r_+ : $\langle \Phi^2(r) \rangle \sim |r - r_+|^{-2}$.

Much of formalism can be explained with Colombeau generalized scalar field [19]. The basic concepts and methods extend straightforwardly to distributional tensor and distributional spinor fields. To begin with let us take a spacetime of arbitrary dimension D , with a metric $g_{\mu\nu}$ of signature $(+ - \dots -)$. The action for the Colombeau generalized scalar field $(\varphi_\varepsilon)_\varepsilon \in \mathbf{G}(M)$ is

$$(S_\varepsilon)_\varepsilon = \left(\int_M d^D x \frac{1}{2} \sqrt{|g_\varepsilon|} \left(g_\varepsilon^{\mu\nu} \partial_\mu \varphi_\varepsilon \partial_\nu \varphi_\varepsilon \right) - (m^2 + \xi R_\varepsilon) \varphi_\varepsilon^2 \right)_\varepsilon. \quad (3.1.1)$$

Here ξ is a coupling constant (see [40] chapter 3). The corresponding equation of motion is

$$\left(\left[\square_{\varepsilon,x} + m^2 + \xi R_\varepsilon \right] \varphi_\varepsilon \right)_\varepsilon, \varepsilon \in (0,1]. \quad (3.1.2)$$

Here

$$\left(\square_{\varepsilon,x} \varphi_\varepsilon \right)_\varepsilon = \left(|g_\varepsilon|^{-1/2} \partial_\mu |g_\varepsilon|^{1/2} g_\varepsilon^{\mu\nu} \partial_\nu \varphi_\varepsilon \right)_\varepsilon. \quad (3.1.3)$$

With \hbar explicit, the mass m should be replaced by m/\hbar . Separating out a time coordinate x^0 , $x^\mu = (x^0, x^i), i=1,2,3$ we can write the action as

$$(S_\varepsilon)_\varepsilon = \left(\int dx^0 L_\varepsilon \right)_\varepsilon, (L_\varepsilon)_\varepsilon = \left(\int d^{D-1} x \mathcal{L}_\varepsilon \right)_\varepsilon. \quad (3.1.4)$$

The canonical momentum at a time x^0 is given by

$$(\pi_\varepsilon(\underline{x}))_\varepsilon = (\delta L_\varepsilon / \delta (\partial_0 \varphi_\varepsilon(\underline{x})))_\varepsilon = (|h_\varepsilon|^{1/2} n^\mu \partial_\mu \varphi_\varepsilon(\underline{x}))_\varepsilon, \quad (3.1.5)$$

where \underline{x} labels a point on a surface of constant x^0 , the x^0 argument of $(\varphi_\varepsilon)_\varepsilon$ is suppressed, n^μ is the unit normal to the surface, and $(|h_\varepsilon|)_\varepsilon$ is the determinant of the induced spatial metric $(h_{ij}(\varepsilon))_\varepsilon$. In order to quantize, the Colombeau generalized field $(\varphi_\varepsilon)_\varepsilon$ and its conjugate momentum $(\pi_\varepsilon(\underline{x}))_\varepsilon$ are now promoted to hermitian operators and required to satisfy the canonical commutation relation,

$$([\varphi_\varepsilon(\underline{x}), \pi_\varepsilon(\underline{y})])_\varepsilon = i\hbar \delta^{D-1}(\underline{x}, \underline{y}), \varepsilon \in (0, 1]. \quad (3.1.6)$$

Here $\int d^{D-1}y \delta^{D-1}(\underline{x}, \underline{y}) f(\underline{y}) = f(\underline{x})$ for any scalar function $f \in D(\mathbb{R}^3)$, without the use of a metric volume element. We form now a conserved bracket from two complex Colombeau solutions to the scalar wave Equation (3.1.2) by [19]:

$$(\langle \varphi_\varepsilon, \phi_\varepsilon \rangle)_\varepsilon = \left(\int_\Sigma d\Sigma_\mu j_\varepsilon^\mu \right)_\varepsilon, \varepsilon \in (0, 1], \quad (3.1.7)$$

where

$$(j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon = (i/\hbar) (|g_\varepsilon|^{1/2} g_\varepsilon^{\mu\nu} (\bar{\varphi}_\varepsilon \partial_\nu \phi_\varepsilon - \varphi_\varepsilon \partial_\nu \bar{\phi}_\varepsilon))_\varepsilon. \quad (3.1.8)$$

Using equation of motion Equation (3.1.2) one obtains corresponding Colombeau generalization of the canonical Green functions equations. In particular for the Colombeau distributional propagator

$$i(G_\varepsilon^\pm(x, x'))_\varepsilon = (\langle 0 | T(\varphi_\varepsilon^\pm(x) \varphi_\varepsilon^\pm(x')) | 0 \rangle)_\varepsilon, \varepsilon \in (0, 1], \quad (3.1.9)$$

one obtains directly

$$([\square_{\varepsilon, x} + m^2 + \xi \mathbf{R}^\pm(x, \varepsilon)] G_\varepsilon^\pm(x, x'))_\varepsilon = -([\square_{\varepsilon, x} + m^2 + \xi \mathbf{R}^\pm(x, \varepsilon)]^{-1/2})_\varepsilon \delta^n(x - x'). \quad (3.1.10)$$

We obtain now an adiabatic expansion of $(G_\varepsilon^\pm(x, x'))_\varepsilon$ [19]. Introducing Riemann normal coordinates y^μ for the point x , with origin at the point x' one obtains

$$\begin{aligned} (g_{\mu\nu}^\pm(x, \varepsilon))_\varepsilon &= \eta_{\mu\nu} + \frac{1}{3} [(\mathbf{R}_{\mu\alpha\nu\beta}^\pm(\varepsilon))_\varepsilon] y^\alpha y^\beta - \frac{1}{6} [(\mathbf{R}_{\mu\alpha\nu\beta;\gamma}^\pm(\varepsilon))_\varepsilon] y^\alpha y^\beta y^\gamma \\ &+ \left[\frac{1}{20} (\mathbf{R}_{\mu\alpha\nu\beta;\gamma\delta}^\pm(\varepsilon))_\varepsilon + \frac{2}{45} [(\mathbf{R}_{\alpha\mu\beta\lambda}^\pm(\varepsilon))_\varepsilon] (\mathbf{R}_{\gamma\nu\delta}^{\pm\lambda}(\varepsilon))_\varepsilon \right] y^\alpha y^\beta y^\gamma y^\delta + \dots \end{aligned} \quad (3.1.11)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor, and the coefficients are all evaluated at $y = 0$. Defining now

$$(\mathcal{L}_\varepsilon^\pm(x, x'))_\varepsilon = \left[\left((-g_{\mu\nu}^\pm(x, \varepsilon))^{1/4} \right)_\varepsilon (G_\varepsilon^\pm(x, x'))_\varepsilon \right] \quad (3.1.12)$$

and its Colombeau-Fourier transform $(\mathcal{L}_\varepsilon^\pm(k))_\varepsilon$ by

$$(\mathcal{L}_\varepsilon^\pm(x, x'))_\varepsilon = (2\pi)^{-n} \left(\int d^n k e^{-ik \cdot y} \mathcal{L}_\varepsilon^\pm(k) \right)_\varepsilon \quad (3.1.13)$$

where $k \cdot y = \eta^{\alpha\beta} k_\alpha y_\beta$, one can work in a sort of localized momentum space.

Expanding (3.1.10) in normal coordinates and converting to k -space, $(\mathcal{L}_\varepsilon^\pm(k))_\varepsilon$ can readily be solved by iteration to any adiabatic order. The result to adiabatic order four (*i.e.*, four derivatives of the metric) is

$$\begin{aligned}
 (\mathcal{L}_\varepsilon^\pm(k))_\varepsilon &= (k^2 - m^2)^{-1} - \left(\frac{1}{6} - \xi\right) (k^2 - m^2)^{-2} (\mathbf{R}^\pm(\varepsilon))_\varepsilon \\
 &\quad + \frac{i}{2} \left(\frac{1}{6} - \xi\right) \partial^\alpha (k^2 - m^2)^{-2} (\mathbf{R}_{;\alpha}^\pm(\varepsilon))_\varepsilon \\
 &\quad - \frac{1}{3} \left[(a_{\alpha\beta}^\pm(\varepsilon))_\varepsilon \right] \partial^\alpha \partial^\beta (k^2 - m^2)^{-2} \\
 &\quad + \left[\left(\frac{1}{6} - \xi\right)^2 (\mathbf{R}^{\pm 2}(\varepsilon))_\varepsilon + \frac{2}{3} (a_{\lambda}^{\pm\lambda}(\varepsilon))_\varepsilon \right] (k^2 - m^2)^{-3},
 \end{aligned}
 \tag{3.1.14}$$

where $\partial_\alpha = \partial/\partial k^\alpha$,

$$\begin{aligned}
 (a_{\alpha\beta}^\pm(\varepsilon))_\varepsilon &\asymp \left(\frac{1}{2} - \xi\right) (\mathbf{R}_{;\alpha\beta}^\pm(\varepsilon))_\varepsilon + \frac{1}{120} (\mathbf{R}_{;\alpha\beta}^\pm(\varepsilon))_\varepsilon - \frac{1}{140} (\mathbf{R}_{\alpha\beta;\lambda}^\pm(\varepsilon))_\varepsilon \\
 &\quad - \frac{1}{30} \left[(\mathbf{R}_\alpha^{\pm\lambda}(\varepsilon))_\varepsilon \right] (\mathbf{R}_{\lambda\beta}^\pm(\varepsilon))_\varepsilon + \frac{1}{60} \left[(\mathbf{R}_{\alpha\beta}^{\pm\kappa\lambda}(\varepsilon))_\varepsilon \right] (\mathbf{R}_{\kappa\lambda}^\pm(\varepsilon))_\varepsilon \\
 &\quad + \frac{1}{60} \left[(\mathbf{R}^{\pm\lambda\mu\kappa}_\alpha(\varepsilon))_\varepsilon \right] (\mathbf{R}_{\lambda\mu\kappa\beta}^\pm(\varepsilon))_\varepsilon,
 \end{aligned}
 \tag{3.1.15}$$

and we are using the symbol \asymp to indicate that this is an asymptotic expansion. One ensures that Equation (3.1.13) represents a time-ordered product by performing the k^0 integral along the appropriate contour in **Figure 9**. This is equivalent to replacing m^2 by $m^2 - i\epsilon$. Similarly, the adiabatic expansions of other Green functions can be obtained by using the other contours in **Figure 9**. Substituting Equation (3.1.14) into Equation (3.1.13) gives [19]

$$\begin{aligned}
 (\mathcal{L}_\varepsilon^\pm(x, x'))_\varepsilon &= (2\pi)^{-n} \times \left[\int d^n k e^{-iky} (k^2 - m^2)^{-1} \left[a_0^\pm(x, x'; \varepsilon) \right. \right. \\
 &\quad \left. \left. + a_1^\pm(x, x'; \varepsilon) \left(-\frac{\partial}{\partial m^2}\right) + a_2^\pm(x, x'; \varepsilon) \left(\frac{\partial}{\partial m^2}\right)^2 \right] \right]_\varepsilon,
 \end{aligned}
 \tag{3.1.16}$$

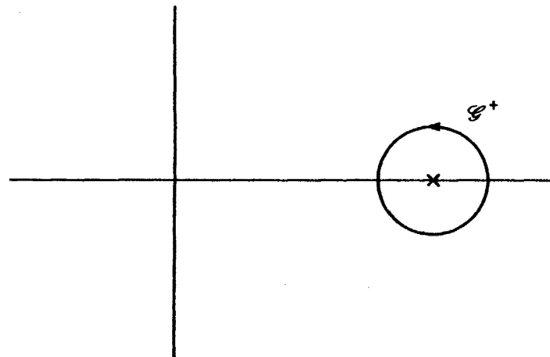


Figure 9. The contour in the complex k^0 plane \mathbb{C} to be used in the evaluation of the integral giving \mathcal{L}^+ . The cross indicates the pole at $k^0 = (|\mathbf{k}|^2 + m^2)^{1/2}$.

where $(a_0^\pm(x, x'; \varepsilon))_\varepsilon = 1$ and, to adiabatic order 4,

$$\begin{cases} (a_1^\pm(x, x'; \varepsilon))_\varepsilon = \left(\frac{1}{6} - \xi\right) (\mathbf{R}^\pm(\varepsilon))_\varepsilon - \frac{i}{2} \left(\frac{1}{6} - \xi\right) \left[(\mathbf{R}_{;\alpha}^\pm(\varepsilon))_\varepsilon \right] y^\alpha \\ \qquad \qquad \qquad - \frac{1}{3} \left[(a_{\alpha\beta}^\pm(\varepsilon))_\varepsilon \right] y^\alpha y^\beta \\ (a_2^\pm(x, x'; \varepsilon))_\varepsilon = \frac{1}{2} \left(\frac{1}{6} - \xi\right) (\mathbf{R}^{\pm 2}(\varepsilon))_\varepsilon + \frac{1}{3} (a_{\lambda}^{\pm\lambda}(\varepsilon))_\varepsilon \end{cases} \quad (3.1.17)$$

with all geometric quantities on the right-hand side of Equation (3.1.17) evaluated at x' .

In Equation (3.16), then the $d^n k$ integration may be interchanged with the ds integration, and performed explicitly to yield (dropping the $i\varepsilon$).

$$\begin{aligned} (\mathcal{L}_\varepsilon^\pm(x, x'))_\varepsilon &= -i(4\pi)^{-n/2} \left(\int_0^\infty ds (is)^{-n/2} \exp \left[-im^2 s + \frac{\sigma(x, x')}{2is} \right]_\varepsilon^\pm \mathcal{F}(x, x'; is) \right)_\varepsilon \\ \sigma(x, x') &= \frac{1}{2} y_\alpha y^\alpha. \end{aligned} \quad (3.1.18)$$

The function $\sigma(x, x')$ which is one-half of the square of the proper distance between x and x' , while the function $(\mathcal{F}_\varepsilon(x, x'; is))_\varepsilon$ has the following asymptotic adiabatic expansion

$$(\mathcal{F}_\varepsilon^\pm(x, x'; is))_\varepsilon \asymp (a_0^\pm(x, x'; \varepsilon))_\varepsilon + is (a_1^\pm(x, x'; \varepsilon))_\varepsilon + (is)^2 (a_2^\pm(x, x'; \varepsilon))_\varepsilon + \dots \quad (3.1.19)$$

Using Equation (3.1.12), Equation (3.1.18) gives a representation of $(G_\varepsilon^\pm(x, x'))_\varepsilon$:

$$\begin{aligned} (G_\varepsilon^\pm(x, x'))_\varepsilon &= -i(4\pi)_\varepsilon^{-n/2} \left(\left[(\Delta_\pm^{1/2}(x, x'; \varepsilon))_\varepsilon \right] \int_0^\infty ds (is)^{-n/2} \right. \\ &\quad \left. \times \exp \left[-im^2 s + \frac{\sigma(x, x')}{2is} \right]_\varepsilon (x, x'; is) \right) \end{aligned} \quad (3.1.20)$$

where $(\Delta_\pm(x, x'; \varepsilon))_\varepsilon$ is the distributional Van Vleck determinant

$$(\Delta_\pm(x, x'; \varepsilon))_\varepsilon = -\det [\partial_\mu \partial_\nu \sigma(x, x')] \left([g^\pm(x, \varepsilon) g^\pm(x', \varepsilon)]^{-1/2} \right)_\varepsilon. \quad (3.1.21)$$

In the normal coordinates about x' that we are currently using, $(\Delta_\pm(x, x'; \varepsilon))_\varepsilon$ reduces to $([-g^\pm(x, \varepsilon)]^{-1/2})_\varepsilon$. The full asymptotic expansion of $(\mathcal{F}_\varepsilon^\pm(x, x'; is))_\varepsilon$ to all adiabatic orders are

$$(\mathcal{F}_\varepsilon^\pm(x, x'; is))_\varepsilon \asymp \sum_{j=0}^\infty (is)^j (a_j^\pm(x, x'; \varepsilon))_\varepsilon \quad (3.1.22)$$

with $(a_0^\pm(x, x'; \varepsilon))_\varepsilon = 1$, the other $(a_j^\pm(x, x'; \varepsilon))_\varepsilon$ being given by canonical recursion relations which enable their adiabatic expansions to be obtained.

Remark 3.1.1. Note that the Expansions (3.1.19) and (3.1.22) are, however, only asymptotic approximations in the limit of large adiabatic parameter T .

If (3.1.22) is substituted into (3.1.20) the integral can be performed to give the adiabatic expansion of the Feynman propagator in coordinate space:

$$\begin{aligned} (G_\varepsilon^\pm(x, x'))_\varepsilon \asymp & -(4\pi i)^{-n/2} \left(\Delta_\pm^{1/2}(x, x'; \varepsilon)_{j=0}^\infty a_j^\pm(x, x'; \varepsilon) \left(-\frac{\partial}{\partial m^2} \right)^j \right. \\ & \left. \times \left[\left(-\frac{2m^2}{\sigma} \right)^{\frac{n-2}{4}} H_{(n-2)/2}^{(2)} \left(\left(2m^2 \sigma \right)^{\frac{1}{2}} \right) \right] \right)_\varepsilon \end{aligned} \tag{3.1.23}$$

which, strictly, a small imaginary part $i\varepsilon$ should be subtracted from σ .

Remark 3.1.2. Since we have not imposed global boundary conditions on the distributional Green function Colombeau solution of (3.1.10), the expansion (3.1.23) does not determine the particular vacuum state in (3.1.9). In particular, the “ $i\varepsilon$ ” in the expansion of $(G_\varepsilon^\pm(x, x'))_\varepsilon$ only ensures that (3.1.23) represents the expectation value, in some set of states, of a time-ordered product of fields. Under some circumstances the use of “ $i\varepsilon$ ” in the exact representation (3.1.20) may give additional information concerning the global nature of the states.

3.2. Effective Action for the Quantum Matter Fields in Curved Distributional Space-Time

As in classical case one can obtain Colombeau generalized quantity $(W_\varepsilon)_\varepsilon$, called the effective action for the quantum matter fields in curved distributional spacetime, which, when functionally differentiated, yields

$$\left(\frac{2}{(-g(\varepsilon))^{\frac{1}{2}}} \frac{\delta W_\varepsilon}{\delta g^{\mu\nu}(\varepsilon)} \right)_\varepsilon = (\langle \mathbf{T}_{\mu\nu}(\varepsilon) \rangle)_\varepsilon \tag{3.2.1}$$

Note that the generating functional

$$(Z_\varepsilon[\mathbf{J}_\varepsilon])_\varepsilon = (D[\varphi_\varepsilon] \exp\{iS_m(\varepsilon) + i\int \mathbf{J}_\varepsilon(x) \varphi_\varepsilon(x) d^n x\})_\varepsilon \tag{3.2.2}$$

was interpreted physically as the vacuum persistence amplitude $(\langle \mathbf{out}_\varepsilon, 0 | 0, \mathbf{in}_\varepsilon \rangle)_\varepsilon$. The presence of the external distributional current density $(\mathbf{J}_\varepsilon)_\varepsilon$ can cause the initial vacuum state $(|0, \mathbf{in}_\varepsilon\rangle)_\varepsilon$ to be unstable, *i.e.*, it can bring about the production of particles.

Following canonical calculation one obtains [19]

$$(Z_\varepsilon^\pm[0])_\varepsilon \propto \left(\left[\det(-G_\varepsilon^\pm(x, x')) \right]^{\frac{1}{2}} \right)_\varepsilon \tag{3.2.3}$$

where the proportionality constant is metric-independent and can be ignored. Thus we obtain

$$(W_\varepsilon^\pm)_\varepsilon = -i(\ln Z_\varepsilon^\pm[0])_\varepsilon = -\frac{i}{2} \left(\text{tr} \left[\ln(-\hat{G}_\varepsilon^\pm) \right] \right)_\varepsilon. \tag{3.2.4}$$

In (3.2.4) $(\hat{G}_\varepsilon^\pm)_\varepsilon$ is to be interpreted as an Colombeau generalized operator which acts on an linear space \mathfrak{V} of generalized vectors $|x, \varepsilon\rangle, \varepsilon \in (0, 1]$ normalized by

$$\left(\langle x, \varepsilon | x', \varepsilon \rangle\right)_\varepsilon = \delta(x - x') \left(\left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} \right)_\varepsilon \tag{3.2.5}$$

in such a way that

$$\left(G_\varepsilon^\pm(x, x')\right)_\varepsilon = \left(\langle x, \varepsilon | \hat{G}_\varepsilon^\pm | x', \varepsilon \rangle\right)_\varepsilon. \tag{3.2.6}$$

Remark 3.2.1. Note that the trace $(\text{tr}[\cdot])_\varepsilon$ of an Colombeau generalized operator $(\mathfrak{R}_\varepsilon)_\varepsilon$ which acts on a linear space \mathfrak{S} , is defined by

$$\left(\text{tr}[\mathfrak{R}_\varepsilon]\right)_\varepsilon = \left(\int d^n x \left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} \mathfrak{R}_{xx; \varepsilon} \right)_\varepsilon = \left(\int d^n x \left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} \langle x | \mathfrak{R}_{xx; \varepsilon} | x' \rangle \right)_\varepsilon. \tag{3.2.7}$$

Writing now the Colombeau generalized operator $(\hat{G}_\varepsilon^\pm)_\varepsilon$ as

$$\left(\hat{G}_\varepsilon^\pm\right)_\varepsilon = -\left(\mathcal{F}_\varepsilon^{\pm 1}\right)_\varepsilon = -i \left(\int_0^\infty ds \exp[-s \mathcal{F}_\varepsilon^\pm] \right)_\varepsilon, \tag{3.2.8}$$

by Equation (3.1.20) we obtain

$$\begin{aligned} \left(\langle x | \exp[-s \mathcal{F}_\varepsilon^\pm] | x' \rangle\right)_\varepsilon &= i(4\pi)^{-n/2} \left[\left(\Delta_\pm^{1/2}(x, x'; \varepsilon) \right)_\varepsilon \right] \\ &\times \exp \left[-im^2 s + \frac{\sigma(x, x')}{2is} \right] \mathcal{F}_\varepsilon^\pm(x, x'; is) (is)^{-n/2}. \end{aligned} \tag{3.2.9}$$

Proceeding in standard manner we get [19]

$$\left(W_\varepsilon^\pm\right)_\varepsilon = \frac{i}{2} \left(\int d^n x \left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} \right)_\varepsilon \left(\lim_{x \rightarrow x'} \int_{m^2}^\infty G_\varepsilon^\pm(x, x'; m^2) dm^2 \right)_\varepsilon. \tag{3.2.10}$$

Interchanging now the order of integration and taking the limit $x \rightarrow x'$ one obtains

$$\left(W_\varepsilon^\pm\right)_\varepsilon = \frac{i}{2} \left(\int_{m^2}^\infty dm^2 \int d^n x \left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} G_\varepsilon^\pm(x, x; m^2) \right)_\varepsilon. \tag{3.2.11}$$

Colombeau generalized quantity $(W_\varepsilon^\pm)_\varepsilon$ is called as the one-loop effective action. In the case of fermion effective actions, there would be a remaining trace over spinorial indices. From Equation (3.2.11) we may define an effective Lagrangian density $(L_{\varepsilon; \text{eff}}^\pm(x))_\varepsilon$ by

$$\left(W_\varepsilon^\pm\right)_\varepsilon = \left(\int d^n x \left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} L_{\varepsilon; \text{eff}}^\pm(x) \right)_\varepsilon \tag{3.2.12}$$

whence one get

$$\left(L_\varepsilon^\pm(x)\right)_\varepsilon = \left(\left[-g^\pm(x, \varepsilon) \right]^{-\frac{1}{2}} L_{\varepsilon; \text{eff}}^\pm(x) \right)_\varepsilon = \frac{i}{2} \left(\lim_{x \rightarrow x'} \int_{m^2}^\infty dm^2 G_\varepsilon^\pm(x, x'; m^2) \right)_\varepsilon. \tag{3.2.13}$$

3.3. Stress-Tensor Renormalization

Note that $(L_\varepsilon^\pm(x))_\varepsilon$ diverges at the lower end of the s integral because the $\sigma/2s$ damping factor in the exponent vanishes in the limit $x \rightarrow x'$. (Convergence at the upper end is guaranteed by the $-i\epsilon$ that is implicitly added

to m^2 in the De Witt-Schwinger representation of $(L_{\varepsilon}^{\pm}(x))_{\varepsilon}$. In four dimensions, the potentially divergent terms in the DeWitt-Schwinger expansion of $(L_{\varepsilon}^{\pm}(x))_{\varepsilon}$ are

$$(L_{\varepsilon;\text{div}}^{\pm}(x))_{\varepsilon} = -(32\pi^2)^{-1} \left(\lim_{x \rightarrow x'} \left[(\Delta_{\pm}^{1/2}(x, x'; \varepsilon))_{\varepsilon} \right] \int_0^{\infty} \frac{ds}{s^3} \exp \left[-im^2 s + \frac{\sigma(x, x')}{2is} \right] \right. \tag{3.3.1}$$

$$\left. \times \left[a_0^{\pm}(x, x'; \varepsilon) + isa_1^{\pm}(x, x'; \varepsilon) + (is)^2 a_2^{\pm}(x, x'; \varepsilon) \right] \right)_{\varepsilon}$$

where the coefficients a_0^{\pm} , a_1^{\pm} and a_2^{\pm} are given by Equation (3.1.17). The remaining terms in this asymptotic expansion, involving a_3^{\pm} and higher, are finite in the limit $x \rightarrow x'$.

Let us determine now the precise form of the geometrical $(L_{\varepsilon;\text{div}}^{\pm}(x))_{\varepsilon}$ terms, to compare them with the distributional generalization of the gravitational Lagrangian that appears in [19]. This is a delicate matter because (3.3.1) is, of course, infinite. What we require is to display the divergent terms in the form $\infty \times$ [geometrical object]. This can be done in a variety of ways. For example, in n dimensions, the asymptotic (adiabatic) expansion of $(L_{\varepsilon;\text{eff}}^{\pm}(x))_{\varepsilon}$ is

$$(L_{\varepsilon;\text{eff}}^{\pm}(x))_{\varepsilon} \asymp 2^{-1} (4\pi)^{-n/2} \left(\lim_{x \rightarrow x'} \left[(\Delta_{\pm}^{1/2}(x, x'; \varepsilon))_{\varepsilon} \right] \sum_{j=0}^{\infty} a_j(x, x'; \varepsilon) \right. \tag{3.3.2}$$

$$\left. \times \int_0^{\infty} ids (is)^{j-1-n/2} \exp \left[-im^2 s + \frac{\sigma(x, x')}{2is} \right] \right)_{\varepsilon}$$

of which the first $n/2+1$ terms are divergent as $\sigma \rightarrow 0$. If n is treated as a variable which can be analytically continued throughout the complex plane, then we may take the $x \rightarrow x'$ limit

$$(L_{\varepsilon;\text{eff}}^{\pm}(x))_{\varepsilon} \asymp 2^{-1} (4\pi)^{-n/2} \left(\sum_{j=0}^{\infty} a_j(x, \varepsilon) \int_0^{\infty} ids (is)^{j-1-n/2} \exp[-im^2 s] \right)_{\varepsilon} \tag{3.3.3}$$

$$= 2^{-1} (4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x, \varepsilon) (m^2)^{n/2-j} \Gamma\left(j - \frac{n}{2}\right), \quad a_j(x, \varepsilon) = a_j(x, x, \varepsilon).$$

From Equation (3.3.3) it follows we shall wish to retain the units of $L_{\varepsilon;\text{eff}}^{\pm}(x)$ as $(\text{length})^{-4}$, even when $n \neq 4$. It is therefore necessary to introduce an arbitrary mass scale μ and to rewrite Equation (3.3.3) as

$$(L_{\varepsilon;\text{eff}}^{\pm}(x))_{\varepsilon} \asymp 2^{-1} (4\pi)^{-n/2} \left(\frac{m}{\mu} \right)^{n-4} \left(\sum_{j=0}^{\infty} a_j(x, \varepsilon) (m^2)^{4-2j} \Gamma\left(j - \frac{n}{2}\right) \right)_{\varepsilon}. \tag{3.3.4}$$

If $n \rightarrow 4$, the first three terms of Equation (3.3.4) diverge because of poles in the Γ -functions:

$$\Gamma\left(-\frac{n}{4}\right) = \frac{4}{n(n-2)} \left(\frac{2}{4-n} - \gamma \right) + O(n-4),$$

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{4}{(2-n)} \left(\frac{2}{4-n} - \gamma \right) + O(n-4), \tag{3.3.5}$$

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{2}{4-n} - \gamma + O(n-4).$$

Denoting these first three terms by $(L_{\varepsilon;\text{div}}^{\pm}(x))_{\varepsilon}$, we have

$$(L_{\varepsilon;\text{div}}^{\pm}(x))_{\varepsilon} = (4\pi)^{-n/2} \left\{ \frac{1}{n-4} + \frac{1}{2} \left[\gamma + \ln \left(\frac{m^2}{\mu^2} \right) \right] \right\} \times \left(\left[\frac{4m^4 a_0(x;\varepsilon)}{n(n-2)} - \frac{2m^2 a_1(x;\varepsilon)}{n-2} + a_2(x;\varepsilon) \right] \right)_{\varepsilon}. \tag{3.3.6}$$

The functions $a_0(x;\varepsilon), a_1(x;\varepsilon)$ and $a_2(x;\varepsilon)$ are given by taking the coincidence limits of (3.1.17)

$$\begin{aligned} (a_0^{\pm}(x;\varepsilon))_{\varepsilon} &= 1, (a_1^{\pm}(x;\varepsilon))_{\varepsilon} = \left(\frac{1}{6} - \xi \right) (\mathbf{R}^{\pm}(\varepsilon))_{\varepsilon}, \\ (a_2^{\pm}(x;\varepsilon))_{\varepsilon} &= \frac{1}{180} (\mathbf{R}_{\alpha\beta\gamma\delta}^{\pm}(x,\varepsilon) \mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} \\ &\quad - \frac{1}{180} (\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon) \mathbf{R}_{\alpha\beta}^{\pm}(x,\varepsilon))_{\varepsilon} \\ &\quad - \frac{1}{6} \left(\frac{1}{5} - \xi \right) (\square_{\varepsilon,x} \mathbf{R}^{\pm}(x,\varepsilon))_{\varepsilon} \\ &\quad + \frac{1}{2} \left(\frac{1}{6} - \xi \right) (\mathbf{R}^{\pm 2}(x,\varepsilon))_{\varepsilon}. \end{aligned} \tag{3.3.7}$$

Finally one obtains [19]

$$(L_{\varepsilon;\text{ren}}^{\pm}(x))_{\varepsilon} \asymp -\frac{1}{64\pi^2} \left(\int_0^{\infty} \text{id}s \ln(is) \frac{\partial^3}{\partial(is)^3} \left[\mathcal{F}_{\varepsilon}^{\pm}(x,x;is) e^{-ism^2} \right] \right)_{\varepsilon}. \tag{3.3.8}$$

Remark 3.3.1. All the higher order ($j > 2$) terms in the DeWitt-Schwinger expansion of the effective Lagrangian (3.3.4) are infrared divergent at $n = 4$ as $m \rightarrow 0$, we can still use this expansion to yield the ultraviolet divergent terms arising from $j = 0, 1$ and 2 in the four-dimensional case. We may put $m = 0$ immediately in the $j = 0$ and 1 terms in the expansion, because they are of positive power for $n \sim 4$. These terms therefore vanish. The only nonvanishing potentially ultraviolet divergent term is therefore $j = 2$:

$$2^{-1} (4\pi)^{-n/2} \left(\frac{m}{\mu} \right)^{n-4} a_2(x,\varepsilon) \Gamma \left(2 - \frac{n}{2} \right), \tag{3.3.9}$$

which must be handled carefully. Substituting for $a_2(x)$ with $\xi = \xi(n)$ from (3.3.7), and rearranging terms, we may write the divergent term in the effective action arising from (3.3.9) as follows

$$\begin{aligned} (W_{\varepsilon;\text{div}}^{\pm})_{\varepsilon} &= 2^{-1} (4\pi)^{-n/2} \left(\frac{m}{\mu} \right)^{n-4} \Gamma \left(2 - \frac{n}{2} \right) \left(\int \text{d}^n x \left[-g^{\pm}(x,\varepsilon) \right]^{\frac{1}{2}} a_2(x,\varepsilon) \right)_{\varepsilon} \\ &= 2^{-1} (4\pi)^{-n/2} \left(\frac{m}{\mu} \right)^{n-4} \Gamma \left(2 - \frac{n}{2} \right) \\ &\quad \times \left(\int \text{d}^n x \left[-g^{\pm}(x,\varepsilon) \right]^{\frac{1}{2}} \left[\tilde{\alpha} F_{\varepsilon}^{\pm}(x) + \tilde{\beta} G_{\varepsilon}^{\pm}(x) \right] \right)_{\varepsilon} + O(n-4), \end{aligned} \tag{3.3.10}$$

where

$$\begin{aligned} (F_\varepsilon(x))_\varepsilon &= (\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x, \varepsilon) \mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x, \varepsilon))_\varepsilon - 2(\mathbf{R}^{\pm\alpha\beta}(x, \varepsilon) \mathbf{R}^{\pm}_{\alpha\beta}(x, \varepsilon))_\varepsilon \\ &\quad + \frac{1}{3}(\mathbf{R}^{\pm 2}(x, \varepsilon))_\varepsilon, \\ (G_\varepsilon^\pm(x))_\varepsilon &= (\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x, \varepsilon) \mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x, \varepsilon))_\varepsilon, \\ \tilde{\alpha} &= \frac{1}{120}, \tilde{\beta} = -\frac{1}{360}. \end{aligned} \tag{3.3.11}$$

Finally we obtain [19]

$$\begin{aligned} & \left(\langle T_\mu^\mu(x, \varepsilon) \rangle_{\text{ren}} \right)_\varepsilon \\ &= -(1/2880\pi^2) \left[\tilde{\alpha} \left(F_\varepsilon(x) - \frac{2}{3} \square_{\varepsilon, x} \mathbf{R}^\pm(x, \varepsilon) \right)_\varepsilon + \tilde{\beta} (G_\varepsilon^\pm(x))_\varepsilon \right] \\ &= -(1/2880\pi^2) \times \left[(\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x, \varepsilon) \mathbf{R}^{\pm\alpha\beta\gamma\delta}(x, \varepsilon))_\varepsilon \right. \\ &\quad \left. - (\mathbf{R}^{\pm}_{\alpha\beta}(x, \varepsilon) \mathbf{R}^{\pm\alpha\beta}(x, \varepsilon))_\varepsilon - (\square_{\varepsilon, x} \mathbf{R}^\pm(x, \varepsilon))_\varepsilon \right]. \end{aligned} \tag{3.3.12}$$

Therefore for the case of the distributional Schwarzschild spesetime using Equation (2.2.4) and Equation (3.3.12) $| (r_\varepsilon - 2m)_\varepsilon | \approx_{\mathbb{R}} (\varepsilon)_{\varepsilon \in (0, \eta]}, \eta \ll 1$, see **Appendix C**, Remark C.10, Equation (C22), we obtain

$$\begin{aligned} \left(\langle T_\mu^\mu(r, \varepsilon) \rangle_{\text{ren}} \right)_\varepsilon &\approx_{\mathbb{R}} -(2880 \cdot \pi^2)^{-1} \left[\left(\left[16^{-1} m^2 (r_\varepsilon - 2m)^2 + \varepsilon^2 \right]^{-1} \right)_\varepsilon + \dots \right] \\ &\approx_{\mathbb{R}} -O(1) (2880 \cdot 16 \cdot \pi^2)^{-1} m^{-2} (r_\varepsilon - 2m)^{-2}. \end{aligned} \tag{3.3.13}$$

Finally from Equation (3.3.13) for $|r - 2m| \in (0, \eta], \eta \ll 1$, see **Appendix C**, Remark C.10, Equation (C22), we obtain

$$\left(\langle T_\mu^\mu(r, \varepsilon) \rangle_{\text{ren}} \right)_{|r-2m| \in (0, \eta]} \approx_{\mathbb{R}} -O(1) (2880 \cdot 16 \cdot \pi^2)^{-1} m^{-2} (r - 2m)_{|r-2m| \in (0, \eta]}^{-2}. \tag{3.3.14}$$

Remark 3.3.2. Thus QFT in distributional curved spacetime predict that the infalling observer burns up at the BH horizon.

Remark 3.3.3. In order to avoid singularity at horizon $r = 2m$ in Equation (3.3.13) one have to apply the Loop Quantum Gravity approach [41]-[46]. The first one concerns the requirement of selfadjointness to the metric components. For instance, the classical quantity

$$g_{tx} = -\frac{(E^x)' K_\phi}{2\sqrt{E^x} \sqrt{1 + K_\phi^2 - \frac{2Gm}{\sqrt{E^x}}}}, \tag{3.3.14}$$

defined as an evolving constant (*i.e.* a Dirac observable), must correspond to a selfadjoint operator at the quantum level. Classically, K_ϕ and E^x are pure gauge, and g_{tx} is just a function of the observable m . In the interior of the horizon, if \hat{g}_{tx} is a selfadjoint operator, a necessary condition will be [41]-[46]

$$1 + K_\phi^2 - \frac{2Gm}{l_P \sqrt{k_j}} \geq 0. \tag{3.3.15}$$

At the singularity, i.e. $j = 1$, and owing to the bounded nature of $K_\varphi^2 < \infty$,

$$\sqrt{k_1} \geq \frac{2Gm}{l_P(1 + K_\varphi^2)} > 0. \tag{3.3.16}$$

Therefore, this argument strongly suggests that the classical singularity will be resolved at the quantum level since k_1 must be a non-vanishing integer.

Remark 3.3.4. Let $\langle T_\nu^\mu(r) \rangle_{\text{ren}}^H$ be $\langle 0^H | T_\nu^\mu(r) | 0^H \rangle_{\text{ren}}$, where $|0^H\rangle$ is the Hartle-Hawking vacuum state [37]. Notice that the main feature of the tensor $\hat{T}_\nu^\mu(r) \triangleq \langle T_\nu^\mu(r) \rangle_{\text{ren}}^H$ formally calculated in classical literature (see, for example, [37] chapter 11.3) is that its components are finite on the event horizon r_+ . An observer at rest at a point r close to the event horizon records the local energy density $\epsilon = -\hat{T}_t^t(r)$. This quantity remains finite as $r \rightarrow r_+$. On the other hand, the temperature measured by the such observer is

$$\Theta_{\text{loc}}(r) = \frac{k}{2\pi} \left(1 + \frac{r_+}{r} \right)^{-1/2}, \tag{3.3.17}$$

grows infinitely near the horizon [37]. The local temperature can be measured by using a two-level system as a thermometer. Transitions between levels are caused by the absorption and emission of quanta of the fields (photons). After a sufficiently long exposure, the probability for a system to occupy the upper level will be less than that for the lower level by a factor $\exp(-\Delta E / \Theta_{\text{loc}}(r))$, where ΔE is the energy difference between the levels. It is well known that the temperature in the vicinity of r_+ is $\Theta_{\text{loc}} \approx \theta_a = a / 2\pi$, where a is the observer's acceleration [37]; as $r \rightarrow r_+$, $\Theta_{\text{loc}}(r) \rightarrow \infty$. The radiation energy density ϵ in the neighborhood of such a point is [37]

$$\epsilon \approx \sigma (k/2\pi)^4 \ll \sigma \theta_a^4. \tag{3.3.18}$$

Therefore Stefan-Boltzmann law under formal calculation by using classical Schwarzschild geometry is evidently violated. Let us remind that the acceleration $a = \sqrt{a_i a^i}$ of free fall of a body which is initially at rest in the Schwarzschild reference frame is [37]

$$a(r) = \frac{m}{r^2 \left(1 - \frac{2m}{r} \right)^{1/2}}. \tag{3.3.19}$$

The acceleration points along the radius and is directed toward the center; as $r \rightarrow 2m$:

$$a(r) = \frac{\sqrt{2m}}{4m(r - 2m)^{1/2}}. \tag{3.3.20}$$

From Equation (3.3.20) and Equation (3.3.14) as $|r - 2m| \in (0, \eta], \eta \ll 1$, see **Appendix C**, Remark C.10, Equation (C22), we obtain:

$$\sigma \theta_a^4 = \frac{\sigma}{64m^2 (r - 2m)_{|r-2m| \in (0, \eta]}^2} \asymp - \left(\langle T_\mu^\mu(r, \epsilon) \rangle_{\text{ren}} \right)_{|r-2m| \in (0, \eta]}. \tag{3.3.21}$$

Therefore Stefan-Boltzmann law under rigorous calculation by using distributional Schwarzschild geometry evidently is not violated.

3.4. Unruh Effect Revisited

We remind now that a black holes have an approximate Rindler region near the Schwarzschild horizon. For the the distributional Schwarzschild solution (2.1.8) by coordinate transformation

$$r = 2m\left(1 + \left(\delta^2 + \varepsilon^2\right)_\varepsilon\right), \varepsilon \in (0, 1], \tag{3.4.1}$$

we obtain

$$\left(ds_\varepsilon^2\right)_\varepsilon = -\left(\left(\delta^2 + \varepsilon^2\right)_\varepsilon\right) dt^2 + 16m^2 d\delta^2 + 4m^2 d\Omega_2^2 + \dots \tag{3.4.2}$$

The (t, δ) piece of this metric (3.4.2) is Rindler space (we can rescale t , δ and ε to make it look exactly like (2.2.10). Thus from (3.3.13) using (3.4.1) we obtain directly for $\delta \asymp 0$

$$\left(\left\langle T_\mu^\mu(\delta, \varepsilon) \right\rangle_{\text{ren}}\right)_{\varepsilon=0} \asymp \delta^{-4}. \tag{3.4.3}$$

Therefore, sufficiently strongly accelerated observer burns up near the Rindler horizon. Thus, Polchinski’s account is not a violation of the Einstein equivalence principle.

Remark 3.4.1. Note that by using Equation (A.8) and Equation (A.9) (see **Appendix A**) one obtains Equation (3.4.3) directly from distributionel Möller metric (1.2.13) and distributionel Rindler metric (2.2.10).

$$\left(\left\langle T_\mu^\mu(x, \varepsilon) \right\rangle_{\text{ren}}\right)_\varepsilon \asymp \frac{-4g^4 O(1)}{\left[\left(a + gx\right)^2 + \left(\varepsilon^2\right)_\varepsilon\right]^2}. \tag{3.4.4}$$

The Unruh effect is the prediction that an accelerating observer will observe blackbody radiation where an inertial observer would observe none. The Unruh effect was first described by Stephen Fulling in 1973, Paul Davies in 1975 and W. G. Unruh in 1976 [47]. The Unruh temperature, derived by William Unruh in 1976, is the effective temperature experienced by a uniformly accelerating detector in a vacuum field. It is given by [47]:

$$T = \frac{\hbar g}{2\pi c \kappa_B}, \tag{3.4.5}$$

where g is the local acceleration, κ_B is the Boltzmann constant, \hbar is the reduced Planck constant, and c is the speed of light. Thus, for example, a proper acceleration of 2.47×10^{20} m/sec² corresponds approximately to a temperature of 1 K. Notice that for a proper acceleration of 2.47×10^{20} m/sec² the event horizon very close to observer by distance $|x_{hor}| \cong 1/2748 \cdot 2m^{-1} = 3.6387 \times 10^{-4}$ m. It is currently not clear whether the Unruh effect has actually been observed, since the claimed observations are disputed. There is also some doubt about whether the Unruh effect implies the existence of Unruh radiation. Although Unruh’s prediction that an accelerating detector would see a thermal bath is not

controversial, the interpretation of the transitions in the detector in the nonaccelerating frame is. It is widely, although not universally, believed that each transition in the detector is accompanied by the emission of a particle, and that this particle will propagate to infinity and be seen as Unruh radiation. The existence of Unruh radiation is not universally accepted. Some claim that it has already been observed [48], while others claim that it is not emitted at all [49]. While the skeptics accept that an accelerating object thermalizes at the Unruh temperature, they do not believe that this leads to the emission of photons, arguing that the emission and absorption rates of the accelerating particle are balanced. By the Einstein equivalence principle Stefan-Boltzmann law holds near the Möller horizon. Therefore by Equation (3.4.4) and Stefan-Boltzmann law the temperature measured by the observer located near the Möller horizon is

$$T \approx g(1 + gx)^{-1}. \quad (3.4.6)$$

Thus observer with a proper acceleration of 2.47×10^{20} m/sec² burns up near the Möller horizon.

4. Conclusion

On a Riemannian or a semi-Riemannian manifold, the metric determines invariants like the Levi-Civita connection and the Riemann curvature. If the metric becomes degenerate (as in singular semi-Riemannian geometry), these constructions no longer work, because they are based on the inverse of the metric, and on the related operations like the contraction between covariant indices. In order to avoid these difficulties distributional geometry by using Colombeau generalized functions [3]-[10]. In authors papers [18] [19] appropriate generalization of classical GR based on Colombeau generalized functions is proposed.

Such generalization of classical GR based on appropriate generalization of the Einstein equivalence principle (GEEP) is mentioned above in subsection 2.3. Using Rindler distributional geometry Unruh effect revisited. We pointed out that GEEP avoid the contradiction which was mentioned by Z. Merali in paper [47], and therefore Polchinski's account [1] doesn't violate the Einstein equivalence principle.

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Appendix

Appendix A1

Let us introduce now Colombeau generalized metric which has the form

$$\begin{aligned} (ds_\varepsilon^2)_\varepsilon = & -\left(A_\varepsilon(r)(dx^0)^2\right)_\varepsilon - 2\left(D_\varepsilon(r)dx^0dr\right)_\varepsilon + \left((B_\varepsilon(r)+C_\varepsilon(r))(dr)^2\right)_\varepsilon \\ & + \left(B_\varepsilon(r)r^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right]\right)_\varepsilon, r \in \mathbb{R}. \end{aligned} \tag{A1.1}$$

The Colombeau scalars $(\mathbf{R}(r, \varepsilon))_\varepsilon, (\mathbf{R}^{\mu\nu}(r, \varepsilon)\mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon$ and $(\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon))_\varepsilon$, in terms of Colombeau generalized functions $(A_\varepsilon(r))_\varepsilon, (B_\varepsilon(r))_\varepsilon, (C_\varepsilon(r))_\varepsilon, (D_\varepsilon(r))_\varepsilon$ are expressed as

$$\begin{aligned} (\mathbf{R}(r, \varepsilon))_\varepsilon = & \left(\frac{A_\varepsilon}{\Delta_\varepsilon} \left[2 \left(-2 \frac{A'_\varepsilon}{A_\varepsilon} - 3 \frac{B'_\varepsilon}{B_\varepsilon} + \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right) + \frac{2}{r^2} \frac{A_\varepsilon C_\varepsilon + D_\varepsilon^2}{A_\varepsilon B_\varepsilon} - \frac{A''_\varepsilon}{A_\varepsilon} \right. \right. \\ & \left. \left. - 2 \frac{B''_\varepsilon}{B_\varepsilon} + \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - 2 \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \left(\frac{1}{2} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right] \right)_\varepsilon, \end{aligned}$$

$$\begin{aligned} (\mathbf{R}^{\mu\nu}(r, \varepsilon)\mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon = & \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left(\frac{1}{2} \frac{A''_\varepsilon}{A_\varepsilon} - \frac{1}{4} \frac{A'_\varepsilon \Delta'_\varepsilon}{A_\varepsilon \Delta_\varepsilon} + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{1}{r} \frac{A'_\varepsilon}{A_\varepsilon} \right)^2 \right)_\varepsilon \\ & + 2 \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{r} \left(\frac{1}{2} \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} - \frac{A'_\varepsilon}{A_\varepsilon} - 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) + \frac{1}{r^2} \frac{A_\varepsilon C_\varepsilon + D_\varepsilon^2}{A_\varepsilon B_\varepsilon} \right. \right. \\ & \left. \left. - \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} - \frac{1}{2} \frac{B''_\varepsilon}{B_\varepsilon} + \frac{1}{4} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} \right]^2 \right)_\varepsilon + \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{2} \frac{A''_\varepsilon}{A_\varepsilon} - \frac{1}{4} \frac{A'_\varepsilon \Delta'_\varepsilon}{A_\varepsilon \Delta_\varepsilon} \right. \right. \\ & \left. \left. + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{B''_\varepsilon}{B_\varepsilon} - \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - \frac{1}{2} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} + \frac{1}{r} \left(\frac{A'_\varepsilon}{A_\varepsilon} - \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} + 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) \right]^2 \right)_\varepsilon, \end{aligned}$$

$$\begin{aligned} (\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon))_\varepsilon = & \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left(\frac{A''_\varepsilon}{A_\varepsilon} - \frac{1}{2} \frac{A'_\varepsilon \Delta'_\varepsilon}{A_\varepsilon \Delta_\varepsilon} \right)^2 + 2 \frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left(\frac{1}{r} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} \right)^2 \right)_\varepsilon \\ & + 4 \frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{r} \frac{B'_\varepsilon}{B_\varepsilon} - \frac{1}{r^2} \frac{A_\varepsilon C_\varepsilon + D_\varepsilon^2}{A_\varepsilon B_\varepsilon} + \frac{1}{4} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 \right]^2 \\ & + 2 \frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{r} \left(\frac{A'_\varepsilon}{A_\varepsilon} + 2 \frac{B'_\varepsilon}{B_\varepsilon} - \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right) + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{B''_\varepsilon}{B_\varepsilon} - \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - \frac{1}{2} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} \right]^2 \right)_\varepsilon. \end{aligned}$$

$$\Delta_\varepsilon = A_\varepsilon (B_\varepsilon + C_\varepsilon) + D_\varepsilon^2 \tag{A1.2}$$

Remark A1.1. Note that the Colombeau scalars $(\mathbf{R}(r, \varepsilon))_\varepsilon, (\mathbf{R}^{\mu\nu}(r, \varepsilon)\mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon$ and $(\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon))_\varepsilon$ can be extended on Colombeau generalized numbers $\mathbf{r} = [(r_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}$ as corresponding generalized point value (see Definition 1.5.4) by formulas:

$$\begin{aligned}
 & \text{(i)} \left[\left(\mathbf{R}(\mathbf{r}, \varepsilon) \right)_{\varepsilon} \right] \triangleq \left[\left(\mathbf{R}(r, \varepsilon) \right)_{\varepsilon} \right], \\
 & \text{(ii)} \left[\left(\mathbf{R}^{\mu\nu}(\mathbf{r}, \varepsilon) \mathbf{R}_{\mu\nu}(\mathbf{r}, \varepsilon) \right)_{\varepsilon} \right] \triangleq \left[\left(\mathbf{R}^{\mu\nu}(r, \varepsilon) \mathbf{R}_{\mu\nu}(r, \varepsilon) \right)_{\varepsilon} \right] \\
 & \text{(iii)} \left[\left(\mathbf{R}^{\rho\sigma\mu\nu}(\mathbf{r}, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(\mathbf{r}, \varepsilon) \right)_{\varepsilon} \right] \triangleq \left[\left(\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon) \right)_{\varepsilon} \right].
 \end{aligned} \tag{A1.3}$$

The distributional Möller’s metric is

$$\begin{aligned}
 \left(d_{\varepsilon} s^2 \right)_{\varepsilon} &= - \left(A_{\varepsilon}(x) dt^2 \right)_{\varepsilon} + dx^2 + dy^2 + dz^2, \\
 A_{\varepsilon}(x) &= \left[(a + gx)^2 + \varepsilon^2 \right], \varepsilon \in (0, 1].
 \end{aligned} \tag{A1.4}$$

In order to apply Equation (A1.2) directly we chose now $g = g(\theta, \varphi)$, where angles θ, φ correspond to spherical coordinates: $x = r \sin \theta \cos \varphi$
 $y = r \sin \theta \sin \varphi, z = r \cos \theta$. In spherical coordinates we get

$$\begin{aligned}
 \left(d_{\varepsilon} s^2 \right)_{\varepsilon} &= - \left(A_{\varepsilon}(r\eta(\theta, \varphi)) dt^2 \right)_{\varepsilon} + dr^2 + r^2 d\Omega^2, \\
 d\Omega^2 &= d\theta^2 + \sin \theta d\varphi^2, \eta(\theta, \varphi) = \sin \theta \cos \varphi, \\
 A_{\varepsilon}(r\eta) &= \left[(a + g\eta r)^2 + \varepsilon^2 \right], \varepsilon \in (0, 1], \\
 A'_{\varepsilon}(r\eta) &= 2g\eta(a + g\eta r), A''_{\varepsilon}(r) = 2g^2\eta^2, \\
 \eta &= \eta(\theta, \varphi).
 \end{aligned} \tag{A1.5}$$

We choose now in the Equation (A1.2): $B_{\varepsilon}(r) = 1, C_{\varepsilon}(r) = 0$, and $g\eta = g^* = const, \sin \theta \cos \varphi \neq 0$ and rewrite Equation (A1.5) in the following equivalent form

$$\begin{aligned}
 \left(d_{\varepsilon} s^2 \right)_{\varepsilon} &= - \left(A_{\varepsilon}(r) dt^2 \right)_{\varepsilon} + dr^2 + r^2 d\Omega^2, \\
 A_{\varepsilon}(r) &= (a + gr)^2 + \varepsilon^2.
 \end{aligned} \tag{A1.6}$$

Note that

$$\Delta_{\varepsilon} = A_{\varepsilon}(B_{\varepsilon} + C_{\varepsilon}) = A_{\varepsilon}. \tag{A1.7}$$

From Equations (A1.5)-(A1.7) by Equation (A1.2) we get

$$\begin{aligned}
 & \left(\mathbf{R}(r, \varepsilon) \right)_{\varepsilon} \\
 &= \left(\frac{A_{\varepsilon}}{\Delta_{\varepsilon}} \left[\frac{2}{r} \left(-2 \frac{A'_{\varepsilon}}{A_{\varepsilon}} - 3 \frac{B'_{\varepsilon}}{B_{\varepsilon}} + \frac{\Delta'_{\varepsilon}}{\Delta_{\varepsilon}} \right) + \frac{2}{r^2} \frac{A_{\varepsilon} C_{\varepsilon} + D_{\varepsilon}^2}{A_{\varepsilon} B_{\varepsilon}} - \frac{A''_{\varepsilon}}{A_{\varepsilon}} \right. \right. \\
 & \quad \left. \left. - 2 \frac{B''_{\varepsilon}}{B_{\varepsilon}} + \frac{1}{2} \left(\frac{B'_{\varepsilon}}{B_{\varepsilon}} \right)^2 - 2 \frac{A'_{\varepsilon} B'_{\varepsilon}}{A_{\varepsilon} B_{\varepsilon}} + \left(\frac{1}{2} \frac{A'_{\varepsilon}}{A_{\varepsilon}} + \frac{B'_{\varepsilon}}{B_{\varepsilon}} \right) \frac{\Delta'_{\varepsilon}}{\Delta_{\varepsilon}} \right] \right)_{\varepsilon} \\
 &= \left(- \frac{2}{r} \frac{A'_{\varepsilon}}{A_{\varepsilon}} - \frac{A''_{\varepsilon}}{A_{\varepsilon}} + \frac{1}{2} \frac{A_{\varepsilon}{}'^2}{A_{\varepsilon}^2} \right)_{\varepsilon} \\
 &= - \left(\frac{4g(a + gr)}{r \left[(a + gr)^2 + \varepsilon^2 \right]} - \frac{2g^2}{(a + gr)^2 + \varepsilon^2} + \frac{2\tilde{g}^2(a + gr)^2}{\left[(a + gr)^2 + \varepsilon^2 \right]^2} \right)_{\varepsilon} \\
 &= \left(2\tilde{g}^2 \frac{(a + gr)^2 + \varepsilon^2 - \varepsilon^2}{\left[(a + gr)^2 + \varepsilon^2 \right]^2} - \frac{4g(a + gr)}{r \left[(a + gr)^2 + \varepsilon^2 \right]} - \frac{2\tilde{g}^2}{(a + \tilde{g}r)^2 + \varepsilon^2} \right)_{\varepsilon}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{2\tilde{g}^2}{(a + \tilde{g}r)^2 + \varepsilon^2} + \frac{-2\tilde{g}^2\varepsilon^2}{[(a + \tilde{g}r)^2 + \varepsilon^2]^2} - \frac{4\tilde{g}(a + \tilde{g}r)}{r[(a + \tilde{g}r)^2 + \varepsilon^2]} - \frac{2\tilde{g}^2}{(a + \tilde{g}r)^2 + \varepsilon^2} \right)_{\varepsilon} \\
 &= \left(\frac{-2\tilde{g}^2\varepsilon^2}{[(a + \tilde{g}r)^2 + \varepsilon^2]^2} - \frac{4\tilde{g}(a + \tilde{g}r)}{r[(a + \tilde{g}r)^2 + \varepsilon^2]} \right)_{\varepsilon}.
 \end{aligned}
 \tag{A1.8}$$

From Equation (A1.8) in the limit $a + g^*r_{\varepsilon} \rightarrow 0$ we get

$$(\mathbf{R}(r_{\varepsilon}, \varepsilon))_{\varepsilon} \approx_{\mathbb{R}} \left(\frac{-2g^{*2}\varepsilon^2}{[(a + g^*r_{\varepsilon})^2 + \varepsilon^2]^2} \right)_{\varepsilon}, \quad [(a + g^*r_{\varepsilon})_{\varepsilon}] \approx_{\mathbb{R}} 0.
 \tag{A1.9}$$

Remark A1.2. Note that: (1) Equation (1.2.14) in a nice agriment with Equation (A1.9), see Remark 1.2.2-Remark 1.2.4. (2) For $r = (r_{\varepsilon})_{\varepsilon}$ located beyond horizon, i.e. $a + g^*r \not\approx_{\mathbb{R}} 0$ one obtains classical result

$$(\mathbf{R}(r, \varepsilon))_{\varepsilon} \approx_{\mathbb{R}} O(g^{*2})(\varepsilon^2)_{\varepsilon} \approx_{\mathbb{R}} 0,
 \tag{A1.10}$$

see Definition 1.5.2. (i). (3) At horizon $r_{\varepsilon} = r_{hor} : a + g^*r_{hor} = 0$ from Equation (A1.9) one obtains nonclassical result

$$(\mathbf{R}(r_{hor}, \varepsilon))_{\varepsilon} \approx_{\mathbb{R}} O(g^{*2})(\varepsilon^{-2})_{\varepsilon} \approx_{\mathbb{R}} \infty,
 \tag{A1.11}$$

see Definition 1.5.2. (ii).

Remark A1.3. Let $[(a + g^*r_{\varepsilon})_{\varepsilon}] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (A1.9) we obtain

$$[(\mathbf{R}(r_{\varepsilon}, \varepsilon))_{\varepsilon}] \approx_{\mathbb{R}} \left[\left(\frac{-2g^{*2}\varepsilon^2}{[(a + g^*r_{\varepsilon})^2 + \varepsilon^2]^2} \right)_{\varepsilon} \right].
 \tag{A1.12}$$

From Equations (A1.5)-(A1.7) by formulae (A1.2) we get

$$\begin{aligned}
 &(\mathbf{R}^{\mu\nu}(r, \varepsilon)\mathbf{R}_{\mu\nu}(r, \varepsilon))_{\varepsilon} \\
 &= \left(\frac{A_{\varepsilon}^2}{\Delta_{\varepsilon}^2} \left(\frac{1}{2} \frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4} \frac{A_{\varepsilon}'\Delta_{\varepsilon}'}{A\Delta_{\varepsilon}} + \frac{1}{2} \frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}} + \frac{1}{r} \frac{A_{\varepsilon}'}{A_{\varepsilon}} \right)^2 \right)_{\varepsilon} \\
 &+ 2 \left(\frac{A_{\varepsilon}^2}{\Delta_{\varepsilon}^2} \left[\frac{1}{r} \left(\frac{1}{2} \frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}} - \frac{A_{\varepsilon}'}{A_{\varepsilon}} - 2 \frac{B_{\varepsilon}'}{B_{\varepsilon}} \right) + \frac{1}{r^2} \frac{A_{\varepsilon}C_{\varepsilon} + D_{\varepsilon}^2}{A_{\varepsilon}B_{\varepsilon}} \right. \right. \\
 &- \left. \left. \frac{1}{2} \frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}} - \frac{1}{2} \frac{B_{\varepsilon}''}{B_{\varepsilon}} + \frac{1}{4} \frac{B_{\varepsilon}'\Delta_{\varepsilon}'}{B_{\varepsilon}\Delta_{\varepsilon}} \right]^2 \right)_{\varepsilon} + \left(\frac{A_{\varepsilon}^2}{\Delta_{\varepsilon}^2} \left[\frac{1}{2} \frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4} \frac{A_{\varepsilon}'\Delta_{\varepsilon}'}{A_{\varepsilon}\Delta_{\varepsilon}} \right. \right. \\
 &+ \left. \left. \frac{1}{2} \frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}} + \frac{B_{\varepsilon}''}{B_{\varepsilon}} - \frac{1}{2} \left(\frac{B_{\varepsilon}'}{B_{\varepsilon}} \right)^2 - \frac{1}{2} \frac{B_{\varepsilon}'\Delta_{\varepsilon}'}{B_{\varepsilon}\Delta_{\varepsilon}} + \frac{1}{r} \left(\frac{A_{\varepsilon}'}{A_{\varepsilon}} - \frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}} + 2 \frac{B_{\varepsilon}'}{B_{\varepsilon}} \right) \right]^2 \right)_{\varepsilon}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\left(\frac{1}{2} \frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4} \frac{A'_\epsilon \Delta'_\epsilon}{A_\epsilon \Delta_\epsilon} + \frac{1}{r} \frac{A'_\epsilon}{A_\epsilon} \right)^2 \right)_\epsilon + 2 \left(\left[\frac{1}{r} \left(\frac{1}{2} \frac{\Delta'_\epsilon}{\Delta_\epsilon} - \frac{A'_\epsilon}{A_\epsilon} \right) - \frac{1}{r^2} \right]^2 \right)_\epsilon \\
 &+ \left(\left[\frac{1}{2} \frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4} \frac{A'_\epsilon \Delta'_\epsilon}{A_\epsilon \Delta_\epsilon} + \frac{1}{r} \left(\frac{A'_\epsilon}{A_\epsilon} - \frac{\Delta'_\epsilon}{\Delta_\epsilon} \right) \right]^2 \right)_\epsilon \\
 &= \left(\left[\frac{1}{2} \frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4} \frac{A'^2_\epsilon}{A_\epsilon^2} + \frac{1}{r} \frac{A'_\epsilon}{A_\epsilon} \right]^2 \right)_\epsilon + 2 \left(\left[-\frac{1}{2r} \frac{A'_\epsilon}{A_\epsilon} \right]^2 \right)_\epsilon \\
 &+ \left(\left[\frac{1}{2} \frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4} \frac{A'^2_\epsilon}{A_\epsilon^2} \right]^2 \right)_\epsilon \\
 &= \left(\left[\frac{1}{2} \frac{2\tilde{g}^2}{(a+\tilde{g}r)^2 + \epsilon^2} - \frac{1}{4} \frac{4\tilde{g}^2(a+\tilde{g}r)^2}{[(a+\tilde{g}r)^2 + \epsilon^2]^2} + \frac{1}{r} \frac{2\tilde{g}(a+\tilde{g}r)}{[(a+\tilde{g}r)^2 + \epsilon^2]} \right]^2 \right)_\epsilon \\
 &+ 2 \left(\left[-\frac{1}{2r} \frac{2\tilde{g}(a+\tilde{g}r)}{(a+\tilde{g}r)^2 + \epsilon^2} - \frac{1}{r^2} \right]^2 \right)_\epsilon \\
 &+ \left(\left[\frac{1}{2} \frac{2\tilde{g}^2}{(a+\tilde{g}r)^2 + \epsilon^2} - \frac{1}{4} \frac{4g^2(a+\tilde{g}r)^2}{[(a+\tilde{g}r)^2 + \epsilon^2]^2} \right]^2 \right)_\epsilon.
 \end{aligned}
 \tag{A1.13}$$

From Equation (A1.13) in the limit $a + g^* r_\epsilon \rightarrow 0$ we get

$$\left(\mathbf{R}^{\mu\nu}(r_\epsilon, \epsilon) \mathbf{R}_{\mu\nu}(r_\epsilon, \epsilon) \right)_\epsilon \approx_{\mathbb{R}} \left(\frac{4g^{*4} \mathcal{O}(1) \epsilon^4}{[(a + g^* r_\epsilon)^2 + \epsilon^2]^4} \right)_\epsilon, (a + g^* r_\epsilon)_\epsilon \approx_{\mathbb{R}} 0. \tag{A1.14}$$

Remark A1.4. At horizon $r_\epsilon = r_{hor} : a + g^* r_{hor} = 0$ from Equation (A1.14) one obtains nonclassical result

$$\left(\mathbf{R}^{\mu\nu}(r_{hor}, \epsilon) \mathbf{R}_{\mu\nu}(r_{hor}, \epsilon) \right)_\epsilon \approx_{\mathbb{R}} \mathcal{O}(g^{*2})(\epsilon^{-4})_\epsilon \approx_{\mathbb{R}} \infty, \tag{A1.15}$$

see Definition 1.5.2. (ii).

Remark A1.5. Let $\left[(a + g^* r_\epsilon)_\epsilon \right] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (A1.14) we obtain

$$\left[\left(\mathbf{R}^{\mu\nu}(r_\epsilon, \epsilon) \mathbf{R}_{\mu\nu}(r_\epsilon, \epsilon) \right)_\epsilon \right] \approx_{\mathbb{R}} \left[\left(\frac{4g^{*4} \mathcal{O}(1) \epsilon^4}{[(a + g^* r_\epsilon)^2 + \epsilon^2]^4} \right)_\epsilon \right]. \tag{A1.16}$$

From Equation (A1.4)-Equation (A1.6) by formulae (A1.2) we get

$$\begin{aligned}
 &\left(\mathbf{R}^{\rho\sigma\mu\nu}(r, \epsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \epsilon) \right)_\epsilon \\
 &= \left(\frac{A_\epsilon^2}{\Delta_\epsilon^2} \left(\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{2} \frac{A'_\epsilon \Delta'_\epsilon}{A_\epsilon \Delta_\epsilon} \right)^2 \right)_\epsilon + 2 \frac{A_\epsilon^2}{\Delta_\epsilon^2} \left(\frac{1}{r} \frac{A'_\epsilon}{A_\epsilon} + \frac{1}{2} \frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} \right)^2
 \end{aligned}$$

$$\begin{aligned}
 &+ 4 \frac{A_\epsilon^2}{\Delta_\epsilon^2} \left[\frac{1}{r} \frac{B'_\epsilon}{B_\epsilon} - \frac{1}{r^2} \frac{A_\epsilon C_\epsilon + D_\epsilon^2}{A_\epsilon B_\epsilon} + \frac{1}{4} \left(\frac{B'_\epsilon}{B_\epsilon} \right)^2 \right]^2 \\
 &+ 2 \frac{A_\epsilon^2}{\Delta_\epsilon^2} \left[\frac{1}{r} \left(\frac{A'_\epsilon}{A_\epsilon} + 2 \frac{B'_\epsilon}{B_\epsilon} - \frac{\Delta'_\epsilon}{\Delta_\epsilon} \right) + \frac{1}{2} \frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{B''_\epsilon}{B_\epsilon} - \frac{1}{2} \left(\frac{B'_\epsilon}{B_\epsilon} \right)^2 - \frac{1}{2} \frac{B'_\epsilon \Delta'_\epsilon}{B_\epsilon \Delta_\epsilon} \right]^2 \Bigg)_\epsilon \\
 &= \left(\left(\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{2} \frac{A'^2_\epsilon}{A_\epsilon^2} \right)^2 \right)_\epsilon + 2 \frac{1}{r} \left(\frac{A'^2_\epsilon}{A_\epsilon^2} \right)_\epsilon \\
 &= \left(\left(\frac{2\tilde{g}^2}{(a+g^*r)^2 + \epsilon^2} - \frac{2\tilde{g}^2(a+\tilde{g}r)^2}{[(a+g^*r)^2 + \epsilon^2]^2} \right)^2 \right)_\epsilon + \frac{1}{r} \left(\frac{8g^{*2}(a+g^*r)^2}{[(a+g^*r)^2 + \epsilon^2]^2} \right)_\epsilon \\
 &= \left(\left(\frac{2g^{*2}}{(a+g^*r)^2 + \epsilon^2} - \frac{2g^{*2}[(a+g^*r)^2 + \epsilon^2 - \epsilon^2]}{[(a+g^*r)^2 + \epsilon^2]^2} \right)^2 \right)_\epsilon + \frac{1}{r} \left(\frac{8g^{*2}(a+g^*r)^2}{[(a+g^*r)^2 + \epsilon^2]^2} \right)_\epsilon \\
 &= \left(\left(\frac{2\tilde{g}^2}{(a+\tilde{g}r)^2 + \epsilon^2} - \frac{2\tilde{g}^2[(a+\tilde{g}r)^2 + \epsilon^2]}{[(a+\tilde{g}r)^2 + \epsilon^2]^2} + \frac{2\tilde{g}^2\epsilon^2}{[(a+\tilde{g}r)^2 + \epsilon^2]^2} \right)^2 \right)_\epsilon \\
 &+ \frac{1}{r} \left(\frac{8\tilde{g}^2(a+\tilde{g}r)^2}{[(a+\tilde{g}r)^2 + \epsilon^2]^2} \right)_\epsilon \\
 &= \left(\frac{4\tilde{g}^4\epsilon^4}{[(a+\tilde{g}r)^2 + \epsilon^2]^4} \right)_\epsilon + \frac{1}{r} \left(\frac{8\tilde{g}^2(a+\tilde{g}r)^2}{[(a+\tilde{g}r)^2 + \epsilon^2]^2} \right)_\epsilon
 \end{aligned}
 \tag{A1.17}$$

In the limit $a + g^*r_\epsilon \rightarrow 0$ from (A1.12) we get

$$\begin{aligned}
 (\mathbf{R}^{\rho\sigma\mu\nu}(r, \epsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \epsilon))_\epsilon &\approx_{\mathbb{R}} \left(\frac{4g^{*4}\epsilon^4}{[(a+g^*r)^2 + \epsilon^2]^4} \right)_\epsilon, \\
 (a + g^*r_\epsilon)_\epsilon &\approx_{\mathbb{R}} 0.
 \end{aligned}
 \tag{A1.18}$$

Remark A1.6. At horizon $r_\epsilon = r_{hor} : a + \tilde{g}r_{hor} = 0$ from Equation (A1.18) one obtains nonclassical result

$$(\mathbf{R}^{\rho\sigma\mu\nu}(r, \epsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \epsilon))_\epsilon \approx_{\mathbb{R}} O(g^{*2})(\epsilon^{-4})_\epsilon \approx_{\mathbb{R}} \infty,
 \tag{A1.19}$$

see Definition 1.5.2. (ii).

Remark A1.7. Let $[(a + g^*r_\epsilon)_\epsilon] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (A1.18) we obtain

$$\left[\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \right] \approx_{\mathbb{R}} \left[\left(\frac{4g^{*4}\varepsilon^4}{\left[(a + g^*r_\varepsilon)^2 + \varepsilon^2 \right]^4} \right)_\varepsilon \right]. \quad (\text{A1.20})$$

Remark A1.8. We assume now there exist a fundamental generalized length $(l_\varepsilon)_\varepsilon$

$$\begin{aligned} (l_\varepsilon)_{\varepsilon \in (0, \eta]} &= b \times (\varepsilon)_{\varepsilon \in (0, \eta]}, \quad \eta \ll 1, \\ (l_\varepsilon)_{\varepsilon \in (\eta, 1]} &= a \end{aligned} \quad (\text{A1.21})$$

such that $\left| (a + g^*r_\varepsilon)_\varepsilon \right| \geq (l_\varepsilon)_\varepsilon = b \times (\varepsilon)_\varepsilon, b \in \mathbb{R}$. It mean there exist a thickness $th_{hor} = (l_\varepsilon)_\varepsilon$ of horizon. We introduce a norm $\|th_{hor}\|$ of a thickness th_{hor} by formula

$$\|th_{hor}\| = \sup_{\varepsilon \in (0, \eta]} |l_\varepsilon| = \eta, \quad (\text{A1.22})$$

where parameter η is a classical thickness of horizon.

By using (A1.21) we get the estimate

$$\begin{aligned} & \left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} \left(\frac{4g^{*4}\varepsilon^4}{\left[(a + g^*r)^2 + \varepsilon^2 \right]^4} \right)_{\varepsilon \in (0, \eta]} \\ &= \left(\frac{4g^{*4}}{\left[(a + g^*r)^2 + \varepsilon^2 \right]^2} \right)_{\varepsilon \in (0, \eta]} \times \left(\frac{\varepsilon^2}{\left[(a + g^*r)^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta]} \\ & \times \left(\frac{\varepsilon^2}{\left[(a + g^*r)^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta]} \quad (\text{A1.23}) \\ & \leq \frac{1}{\left[1 + b^2 \right]^2} \left(\frac{4g^{*4}}{\left[(a + g^*r)^2 + \varepsilon^2 \right]^2} \right)_{\varepsilon \in (0, \eta]} \\ & = \frac{1}{\left[1 + b^2 \right]^2} \left(\frac{4g^{*4}}{\left[(a + g^*r)^4 \right]^2} \right)_{|a + g^*r| \in (0, \eta]} \end{aligned}$$

Appendix A2

Let us consider now distributional Colombeau metric given by Equation (1.3.30) with $c = 1$

$$(ds_\varepsilon^2)_\varepsilon = - \frac{\left((r - \bar{\rho})^2 + \varepsilon^2 \right)_\varepsilon}{(r + \bar{\rho})^2} dt^2 + \left(1 + \frac{\bar{\rho}}{r} \right)^4 \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi) \right], \quad (\text{A2.1})$$

where $\varepsilon \in (0, 1]$, $\bar{\rho} = r_s/4$, r_s is a schwarzschild radius.

We choose now $D_\varepsilon(r) = C_\varepsilon(r) = 0$, and rewrite Equation (A2.1) in the following equivalent form

$$\begin{aligned}
 (d_\varepsilon s^2)_\varepsilon &= -(A_\varepsilon(r) dt^2)_\varepsilon + B(r) [dr^2 + r^2 d\Omega^2], \\
 A_\varepsilon(r) &= \frac{(r - \bar{\rho})^2 + \varepsilon^2}{(r + \bar{\rho})^2}, B(r) = \left(1 + \frac{\bar{\rho}}{r}\right)^4, \\
 \Delta_\varepsilon &= A_\varepsilon B_\varepsilon, \frac{A_\varepsilon}{\Delta_\varepsilon} = \frac{1}{B_\varepsilon}, \Delta'_\varepsilon = A'_\varepsilon B_\varepsilon + A_\varepsilon B'_\varepsilon, \\
 \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} &= \frac{A'_\varepsilon B_\varepsilon + A_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} = \frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon}, \\
 \frac{A'_\varepsilon}{A_\varepsilon} &= -\frac{2(\varepsilon^2 + 2\bar{\rho}^2 - 2r\bar{\rho})}{(r + \bar{\rho})((r - \bar{\rho})^2 + \varepsilon^2)}, \\
 \left(\frac{A'_\varepsilon}{A_\varepsilon}\right)^2 &= \frac{4(\varepsilon^2 + 2\bar{\rho}^2 - 2r\bar{\rho})^2}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)^2} = \frac{4\varepsilon^4 + 16\bar{\rho}(\bar{\rho} - r) + 16\bar{\rho}^2(r - \bar{\rho})^2}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)^2} \\
 &= \frac{4\varepsilon^4 + 16\bar{\rho}(\bar{\rho} - r)}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)^2} + \frac{16\bar{\rho}^2[(r - \bar{\rho})^2 + \varepsilon^2 - \varepsilon^2]}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)^2} \\
 &= \frac{4\varepsilon^4 + 16\bar{\rho}(\bar{\rho} - r)}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)^2} + \frac{16\bar{\rho}^2}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)} \\
 &\quad - \frac{16\bar{\rho}^2\varepsilon^2}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)^2}, \\
 \frac{A''_\varepsilon}{A_\varepsilon} &= \frac{2(3\varepsilon^2 + 8\bar{\rho}^2 - 4r\bar{\rho})}{(r + \bar{\rho})^2((r - \bar{\rho})^2 + \varepsilon^2)}.
 \end{aligned}
 \tag{A2.2}$$

We assume now that $r \asymp \bar{\rho}$, then from Equation (A2.2) we obtain

$$\begin{aligned}
 \frac{A'_\varepsilon}{A_\varepsilon} &\asymp -\frac{2\varepsilon^2}{\bar{\rho}((r - \bar{\rho})^2 + \varepsilon^2)} \\
 \left(\frac{A'_\varepsilon}{A_\varepsilon}\right)^2 &\asymp \frac{\varepsilon^4}{\bar{\rho}^2((r - \bar{\rho})^2 + \varepsilon^2)^2} + \frac{4}{[(r - \bar{\rho})^2 + \varepsilon^2]} - \frac{4\varepsilon^2}{((r - \bar{\rho})^2 + \varepsilon^2)^2}, \\
 \frac{A''_\varepsilon}{A_\varepsilon} &\asymp \frac{(3\varepsilon^2 + 4\bar{\rho}^2)}{2\bar{\rho}^2((r - \bar{\rho})^2 + \varepsilon^2)} \asymp \frac{2}{(r - \bar{\rho})^2 + \varepsilon^2}, \\
 -\frac{A''_\varepsilon}{A_\varepsilon} + \frac{1}{2}\left(\frac{A'_\varepsilon}{A_\varepsilon}\right)^2 &\asymp -\frac{2}{(r - \bar{\rho})^2 + \varepsilon^2} + \frac{2}{(r - \bar{\rho})^2 + \varepsilon^2} - \frac{2\varepsilon^2}{[(r - \bar{\rho})^2 + \varepsilon^2]^2} \\
 &= -\frac{2\varepsilon^2}{[(r - \bar{\rho})^2 + \varepsilon^2]^2}.
 \end{aligned}
 \tag{A2.3}$$

From Equation (A2.3) by formulae (A1.2) we get

$$\begin{aligned}
 (\mathbf{R}(r, \varepsilon))_\varepsilon &= \left(\frac{A_\varepsilon}{\Delta_\varepsilon} \left[\frac{2}{r} \left(-2 \frac{A'_\varepsilon}{A_\varepsilon} - 3 \frac{B'_\varepsilon}{B_\varepsilon} + \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right) - \frac{A''_\varepsilon}{A_\varepsilon} - 2 \frac{B''_\varepsilon}{B_\varepsilon} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - 2 \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \left(\frac{1}{2} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right] \right)_\varepsilon \\
 &= \left(\frac{1}{B_\varepsilon} \left[\frac{2}{r} \left(-\frac{A'_\varepsilon}{A_\varepsilon} - 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) - \frac{A''_\varepsilon}{A_\varepsilon} - 2 \frac{B''_\varepsilon}{B_\varepsilon} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - 2 \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \left(\frac{1}{2} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \right] \right)_\varepsilon \\
 &= \left(\frac{1}{B_\varepsilon} \left[\frac{2}{r} \left(-\frac{A'_\varepsilon}{A_\varepsilon} - 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) - \frac{A''_\varepsilon}{A_\varepsilon} - 2 \frac{B''_\varepsilon}{B_\varepsilon} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \frac{(A'_\varepsilon)^2}{A_\varepsilon^2} + \frac{3}{2} \frac{(B'_\varepsilon)^2}{B_\varepsilon^2} - \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} \right] \right)_\varepsilon.
 \end{aligned}
 \tag{A2.4}$$

From Equation (A2.4) in the limit $r_\varepsilon \rightarrow \bar{\rho}$ by formulae (A2.3) we get

$$(\mathbf{R}(r_\varepsilon, \varepsilon))_\varepsilon \approx_{\mathbb{R}} - \left(\frac{2\varepsilon^2}{[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2]^2} \right)_\varepsilon.
 \tag{A2.5}$$

Remark A2.1. Note that: (1) Equation (A2.5) in a nice agriment with Equation (A1.9). For $r = (r_\varepsilon)_\varepsilon$ located beyond horizon, *i.e.* $r - \bar{\rho} \not\approx_{\mathbb{R}} 0$ one obtains classical result

$$(\mathbf{R}(r_\varepsilon, \varepsilon))_\varepsilon \approx_{\mathbb{R}} O(1)(\varepsilon^2)_\varepsilon \approx_{\mathbb{R}} 0,
 \tag{A2.6}$$

see Definition 1.5.2. (i). (3) At horizon $r_\varepsilon = r_{hor} : r_{hor} - \bar{\rho} = 0$ from Equation (A2.5) one obtains nonclassical result

$$(\mathbf{R}(r_{hor}, \varepsilon))_\varepsilon \approx_{\mathbb{R}} O(1)(\varepsilon^{-2})_\varepsilon \approx_{\mathbb{R}} \infty,
 \tag{A2.7}$$

see Definition 1.5.2. (ii).

Remark A2.2. Let $[(r_\varepsilon - \bar{\rho})_\varepsilon] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (A2.5) we obtain

$$[(\mathbf{R}(r_\varepsilon, \varepsilon))_\varepsilon] \approx_{\mathbb{R}} \left[\left(-\frac{2\varepsilon^2}{[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2]^2} \right)_\varepsilon \right].
 \tag{A2.8}$$

From Equation (A2.3) by formulae (A1.2) we get

$$\begin{aligned}
 &(\mathbf{R}^{\mu\nu}(r, \varepsilon)\mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon \\
 &= \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left(\frac{1}{2} \frac{A''_\varepsilon}{A_\varepsilon} - \frac{1}{4} \frac{A'_\varepsilon \Delta'_\varepsilon}{A_\varepsilon \Delta_\varepsilon} + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{1}{r} \frac{A'_\varepsilon}{A_\varepsilon} \right)^2 \right)_\varepsilon \\
 &\quad + 2 \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{r} \left(\frac{1}{2} \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} - \frac{A'_\varepsilon}{A_\varepsilon} - 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) - \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} - \frac{1}{2} \frac{B''_\varepsilon}{B_\varepsilon} + \frac{1}{4} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} \right]^2 \right)_\varepsilon
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{2} \frac{A_\varepsilon''}{A_\varepsilon} - \frac{1}{4} \frac{A'_\varepsilon \Delta'_\varepsilon}{A_\varepsilon \Delta_\varepsilon} + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{B_\varepsilon''}{B_\varepsilon} - \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 \right. \right. \\
 & \left. \left. - \frac{1}{2} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} + \frac{1}{r} \left(\frac{A'_\varepsilon}{A_\varepsilon} - \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} + 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) \right]^2 \right)_\varepsilon \\
 & = \left(\frac{1}{B_\varepsilon^2} \left(\frac{1}{2} \frac{A_\varepsilon''}{A_\varepsilon} - \frac{1}{4} \frac{A'_\varepsilon}{A_\varepsilon} \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{1}{r} \frac{A'_\varepsilon}{A_\varepsilon} \right)^2 \right)_\varepsilon \\
 & + 2 \left(\frac{1}{B_\varepsilon^2} \left[\frac{1}{r} \left(\frac{1}{2} \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) - \frac{A'_\varepsilon}{A_\varepsilon} - 2 \frac{B'_\varepsilon}{B_\varepsilon} \right) \right. \right. \\
 & \left. \left. - \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} - \frac{1}{2} \frac{B_\varepsilon''}{B_\varepsilon} + \frac{1}{4} \frac{B'_\varepsilon}{B_\varepsilon} \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \right]^2 \right)_\varepsilon \\
 & + \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{2} \frac{A_\varepsilon''}{A_\varepsilon} - \frac{1}{4} \frac{A'_\varepsilon}{A_\varepsilon} \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} \right. \right. \\
 & \left. \left. + \frac{B_\varepsilon''}{B_\varepsilon} - \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - \frac{1}{2} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} + \frac{1}{r} \frac{B'_\varepsilon}{B_\varepsilon} \right]^2 \right)_\varepsilon.
 \end{aligned}
 \tag{A2.9}$$

From Equation (A2.9) in the limit $r_\varepsilon \rightarrow \bar{\rho}$ by formulae (A2.3) we get

$$\left(\mathbf{R}^{\mu\nu}(r, \varepsilon) \mathbf{R}_{\mu\nu}(r, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} \left(\frac{\varepsilon^4}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right)_\varepsilon.
 \tag{A2.10}$$

Remark A2.3. Note that: (1) For $r = (r_\varepsilon)_\varepsilon$ located beyond horizon, i.e. $r - \bar{\rho} \not\approx_{\mathbb{R}} 0$ one obtains classical result

$$\left(\mathbf{R}^{\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} O(1)(\varepsilon^4)_\varepsilon \approx_{\mathbb{R}} 0,
 \tag{A2.11}$$

see Definition 1.5.2. (i). (2) At horizon $r_\varepsilon = r_{hor} : r_{hor} - \bar{\rho} = 0$ from Equation (A2.10) one obtains nonclassical result

$$\left(\mathbf{R}^{\mu\nu}(r_{hor}, \varepsilon) \mathbf{R}_{\mu\nu}(r_{hor}, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} O(1)(\varepsilon^{-4})_\varepsilon \approx_{\mathbb{R}} \infty,
 \tag{A2.12}$$

see Definition 1.5.2. (ii).

Remark A2.4. Let $\left[(r_\varepsilon - \bar{\rho}) \right]_\varepsilon \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (A2.10) we obtain

$$\left[\left(\mathbf{R}^{\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \right] \approx_{\mathbb{R}} \left[\left(\frac{\varepsilon^4}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right)_\varepsilon \right].
 \tag{A2.13}$$

From Equation (A2.3) by formulae (A1.2) we get

$$\begin{aligned}
 & \left(\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon) \right)_\varepsilon \\
 & = \left(\frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left(\frac{A_\varepsilon''}{A_\varepsilon} - \frac{1}{2} \frac{A'_\varepsilon \Delta'_\varepsilon}{A_\varepsilon \Delta_\varepsilon} \right)^2 + 2 \frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left(\frac{1}{r} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} \right)^2 \right)_\varepsilon
 \end{aligned}$$

$$\begin{aligned}
 & + 4 \frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{r} \frac{B'_\varepsilon}{B_\varepsilon} + \frac{1}{4} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 \right]^2 + 2 \frac{A_\varepsilon^2}{\Delta_\varepsilon^2} \left[\frac{1}{r} \left(\frac{A'_\varepsilon}{A_\varepsilon} + 2 \frac{B'_\varepsilon}{B_\varepsilon} - \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right) \right. \\
 & \left. + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{B''_\varepsilon}{B_\varepsilon} - \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - \frac{1}{2} \frac{B'_\varepsilon \Delta'_\varepsilon}{B_\varepsilon \Delta_\varepsilon} \right]^2 \Bigg)_\varepsilon \\
 & = \left(\frac{1}{B_\varepsilon^2} \left(\frac{A''_\varepsilon}{A_\varepsilon} - \frac{1}{2} \frac{A'_\varepsilon}{A_\varepsilon} \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \right) \right)^2 + 2 \frac{1}{B_\varepsilon^2} \left(\frac{1}{r} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} \right)^2 \\
 & + 4 \frac{1}{B_\varepsilon^2} \left[\frac{1}{r} \frac{B'_\varepsilon}{B_\varepsilon} + \frac{1}{4} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 \right]^2 + 2 \frac{1}{B_\varepsilon^2} \left[\frac{1}{r} \left(\frac{A'_\varepsilon}{A_\varepsilon} + 2 \frac{B'_\varepsilon}{B_\varepsilon} - \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \right) \right. \\
 & \left. + \frac{1}{2} \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \frac{B''_\varepsilon}{B_\varepsilon} - \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - \frac{1}{2} \frac{B'_\varepsilon}{B_\varepsilon} \left(\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \right]^2 \Bigg)_\varepsilon.
 \end{aligned} \tag{A2.14}$$

From Equation (A2.14) in the limit $r_\varepsilon \rightarrow \bar{\rho}, \varepsilon \in (0,1]$ by formulae (A2.3) we get

$$\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} K(\bar{\rho}) + \left[\frac{\varepsilon^4}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right]_\varepsilon, \tag{A2.15}$$

where $K(r)$ is a Kretschman scalar: $K(r) = 3 \cdot 4^{13} r^6 r_s^2 (4r + r_s)^{-12}$.

Remark A2.5. Note that: (1) For $r = (r_\varepsilon)_\varepsilon$ located beyond horizon, i.e. $r - \bar{\rho} \not\approx_{\mathbb{R}} 0$ one obtains classical result

$$\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} \left(K(r_\varepsilon) \right)_\varepsilon, \tag{A2.16}$$

see Definition 1.5.2. (i). (2) At horizon $r_\varepsilon = r_{hor} : r_{hor} - \bar{\rho} = 0, \varepsilon \in (0,1]$ from Equation (A2.15) one obtains nonclassical result

$$\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} K(\bar{\rho}) + O(1)(\varepsilon^{-4})_\varepsilon \approx_{\mathbb{R}} \infty, \tag{A2.17}$$

see Definition 1.5.2. (ii).

Remark A2.6. Let $\left[(r_\varepsilon - \bar{\rho})_\varepsilon \right] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (A2.15) we obtain

$$\left[\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \right] \approx_{\mathbb{R}} K(\bar{\rho}) + \left[\left[\frac{\varepsilon^4}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right]_\varepsilon \right]. \tag{A2.18}$$

Remark A2.7. We assume now there exist a fundamental generalized length $(l_\varepsilon)_\varepsilon$

$$\begin{aligned}
 (l_\varepsilon)_{\varepsilon \in (0,\eta]} &= a(\varepsilon)_{\varepsilon \in (0,\eta]}, \eta \ll 1, \\
 (l_\varepsilon)_{\varepsilon \in (\eta,1]} &= a,
 \end{aligned} \tag{A2.19}$$

such that $\left| (r_\varepsilon - \bar{\rho})_\varepsilon \right| \geq (l_\varepsilon)_\varepsilon = a(\varepsilon)_\varepsilon$. It meant there exist a thickness $th_{hor} = (l_\varepsilon)_\varepsilon$ of BH horizon. We introduce a norm $\|th_{hor}\|$ of a thickness th_{hor} by formula

$$\|th_{hor}\| = \sup_{\varepsilon \in (0, \eta)} |l_\varepsilon| = \eta, \tag{A2.20}$$

where parameter η is a classical thickness of BH horizon.

By using (A2.19) we get the estimate

$$\begin{aligned} & \left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \\ & \approx_{\mathbb{R}} K(\bar{\rho}) + \left(\frac{\varepsilon^4}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^4} \right)_{\varepsilon \in (0, \eta)} \\ & \approx_{\mathbb{R}} \left(\frac{1}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^2} \right)_{\varepsilon \in (0, \eta)} \times \left(\frac{\varepsilon^2}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta)} \\ & \quad \times \left(\frac{\varepsilon^2}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta)} \\ & \leq K(\bar{\rho}) + \left(\frac{1}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^2} \right)_{\varepsilon \in (0, \eta)} \left(\left(\frac{\varepsilon^2}{\left[a^2 \varepsilon^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta)} \right)^2 \\ & = K(\bar{\rho}) + \frac{1}{\left[a^2 + 1 \right]^2} \left(\frac{1}{\left[(r_\varepsilon - \bar{\rho})^2 + \varepsilon^2 \right]^2} \right)_{\varepsilon \in (0, \eta)} \\ & \leq K(\bar{\rho}) + \frac{1}{\left[a^2 + 1 \right]^2} \left(\frac{1}{(r - \bar{\rho})^4} \right)_{|r - \bar{\rho}| \in (0, \eta)} \end{aligned} \tag{A2.21}$$

Appendix B

We calculate now the distributional curvature at Schwarzschild horizon. In the usual Schwarzschild coordinates $(t, r > 0, \theta, \phi), r \neq 2m$ the metric is

$$\begin{cases} ds^2 = h(r) dt^2 - h(r)^{-1} dr^2 + r^2 d\Omega^2, \\ h(r) = -1 + \frac{2m}{r}. \end{cases} \tag{B.1}$$

Metric takes the form above horizon $r > 2m$ and below horizon $r < 2m$ correspondingly

$$\begin{cases} \text{above horizon } r > 2m : \\ ds^{+2} = h^+(r) dt^2 - [h^+(r)]^{-1} dr^2 + r^2 d\Omega^2, \\ h^+(r) = -1 + \frac{2m}{r} = -\frac{r-2m}{r} \\ \text{below horizon } r < 2m : \\ ds^{-2} = h^-(r) dt^2 - h^-(r)^{-1} dr^2 + r^2 d\Omega^2, \\ h^-(r) = -1 + \frac{2m}{r} = \frac{2m-r}{r} \end{cases} \tag{B.2}$$

Remark B.1. Following the above discussion we consider the metric coefficients $h^+(r)$, $[h^+(r)]^{-1}h^-(r)$, and $[h^-(r)]^{-1}$ as an element of $D'(\mathbb{R}^3)$ and embed it into $\mathbf{D}(\tilde{\mathbb{R}}^3)$ by replacement above horizon $r \geq 2m$ and below horizon $r \leq 2m$ correspondingly

$$\begin{aligned} r \geq 2m : r - 2m &\mapsto \sqrt{(r - 2m)^2 + \epsilon^2}, \\ r \leq 2m : 2m - r &\mapsto \sqrt{(2m - r)^2 + \epsilon^2}. \end{aligned} \tag{B.3}$$

Note that, accordingly, we have fixed the differentiable structure of the manifold: the Cartesian coordinates associated with the spherical Schwarzschild coordinates in (B.1) are extended through the origin. We have above $r \geq 2m$ (below ($r \leq 2m$)) horizon

$$h(r) = \begin{cases} -\frac{r-2m}{r} & \text{if } r \geq 2m \\ 0 & \text{if } r \leq 2m \end{cases} \mapsto (h_\epsilon^+(r))_\epsilon = \begin{pmatrix} -\sqrt{(r-2m)^2 + \epsilon^2} \\ r \end{pmatrix}_\epsilon,$$

where $(h_\epsilon^+(r))_\epsilon \in \mathbf{G}(\mathbb{R}^3, B^+(2m, R))$, $B^+(2m, R) = \{x \in \mathbb{R}^3 \mid 2m \leq \|x\| \leq R\}$.

$$h^{-1}(r) = \begin{cases} -\frac{r}{r-2m}, & r > 2m \\ \infty, & r = 2m \end{cases} \mapsto (h_\epsilon^+)^{-1}(r) = h^-(r) = \begin{cases} -\frac{r-2m}{r} & \text{if } r \leq 2m \\ 0 & \text{if } r \geq 2m \end{cases}$$

$$\mapsto h_\epsilon^-(r) = \begin{pmatrix} \sqrt{(2m-r)^2 + \epsilon^2} \\ r \end{pmatrix}_\epsilon \in \mathbf{G}(\mathbb{R}^3, B^-(0, 2m)),$$

where $B^-(0, 2m) = \{x \in \mathbb{R}^3 \mid 0 < \|x\| \leq 2m\}$

$$\begin{cases} -\frac{r}{r-2m}, & r < 2m \\ \infty, & r = 2m \end{cases} \mapsto (h_\epsilon^-)^{-1}(r) = \begin{pmatrix} r \\ \sqrt{(r-2m)^2 + \epsilon^2} \end{pmatrix}_\epsilon \in \mathbf{G}(\mathbb{R}^3, B^-(0, 2m))$$

(B.4)

Inserting (B.4) into (B.2) we obtain a generalized object modeling the singular Schwarzschild metric above (below) horizon, *i.e.*,

$$\begin{aligned} (ds_\epsilon^{+2})_\epsilon &= (h_\epsilon^+(r) dt^2)_\epsilon - ([h_\epsilon^+(r)]^{-1} dr^2)_\epsilon + r^2 d\Omega^2, \\ (ds_\epsilon^{-2})_\epsilon &= (h_\epsilon^-(r) dt^2)_\epsilon - ([h_\epsilon^-(r)]^{-1} dr^2)_\epsilon + r^2 d\Omega^2 \end{aligned} \tag{B.5}$$

The generalized Ricci tensor above horizon $[\mathbf{R}^+]_\alpha^\beta$ may now be calculated componentwise using the classical formulae

$$\begin{aligned} ([\mathbf{R}^+]_{00})_\epsilon &= ([\mathbf{R}^+]_{11})_\epsilon = \frac{1}{2} \left((h_\epsilon^{+''})_\epsilon + \frac{2}{r} (h_\epsilon^{+'})_\epsilon \right) \\ ([\mathbf{R}^+]_{22})_\epsilon &= ([\mathbf{R}^+]_{33})_\epsilon = \frac{(h_\epsilon^{+'})_\epsilon}{r} + \frac{1 + (h_\epsilon^+)_\epsilon}{r^2}. \end{aligned} \tag{B.6}$$

From (B.4) by differentiation we obtain

$$h_\epsilon^{+'}(r) = -\frac{r-2m}{r[(r-2m)^2 + \epsilon^2]^{1/2}} + \frac{[(r-2m)^2 + \epsilon^2]^{1/2}}{r^2},$$

$$\begin{aligned}
 & r \left(h_{\epsilon}^{+'} \right)_{\epsilon} + 1 + \left(h_{\epsilon}^{+} \right)_{\epsilon} \\
 &= r \left\{ -\frac{r-2m}{r \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^2} \right\} + 1 - \frac{\sqrt{(r-2m)^2 + \epsilon^2}}{r} \\
 &= -\frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r} + 1 - \frac{\sqrt{(r-2m)^2 + \epsilon^2}}{r} \\
 &= -\frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + 1. \\
 & h_{\epsilon}''(r) = -\left(\frac{r-2m}{r \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} \right)' + \left(\frac{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^2} \right)' \\
 &= -\frac{1}{r \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{(r-2m)^2}{r \left[(r-2m)^2 + \epsilon^2 \right]^{3/2}} + \frac{r-2m}{r^2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} \\
 &\quad + \frac{r-2m}{r^2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} - \frac{2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^3}. \\
 & r^2 \left(h_{\epsilon}^{+''} \right)_{\epsilon} + 2r \left(h_{\epsilon}^{+'} \right)_{\epsilon} \\
 &= r^2 \left\{ -\frac{1}{r \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{(r-2m)^2}{r \left[(r-2m)^2 + \epsilon^2 \right]^{3/2}} + \frac{r-2m}{r^2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} \right. \\
 &\quad \left. + \frac{r-2m}{r^2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} - \frac{2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^3} \right\} \\
 &\quad + 2r \left\{ -\frac{r-2m}{r \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^2} \right\} \\
 &= -\frac{r}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2 + \epsilon^2 \right]^{3/2}} + \frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} \\
 &\quad + \frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} - \frac{2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r} \\
 &\quad - \frac{2(r-2m)}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r} \\
 &= -\frac{r}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2 + \epsilon^2 \right]^{3/2}}.
 \end{aligned}$$

(B.7)

angular components of the Ricci tensor (using the abbreviation

$$\tilde{\Phi}(r) = \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \Phi(\bar{x}) \tag{B.8}$$

and let $\Phi(\bar{x})$ be the function $\Phi(\bar{x}) \in S_{2m}(\mathbb{R}^3, B^+(2m, R_0))$, where by $S_{2m}(\mathbb{R}^3, B^+(2m, R_0))$ we denote the class of the functions $\Phi(\bar{x})$ with compact support such that:

- (i) $\text{supp}(\Phi(\bar{x})) \subset B^+(2m, R_0) \subset \{\bar{x} \mid R_0 \geq \|\bar{x}\| \geq 2m\}$ 2) $\tilde{\Phi}(r) \in C^\infty(\mathbb{R})$.

Then for any function $\Phi(\bar{x}) \in S_{2m}(\mathbb{R}^3, B^+(2m, R_0))$ we get:

$$\begin{aligned} & \int_K \left(\left[\mathbf{R}_\epsilon^+ \right]_2 \right)_\epsilon \Phi(\bar{x}) d^3x \\ &= \int_K \left(\left[\mathbf{R}_\epsilon^+ \right]_3 \right)_\epsilon \Phi(\bar{x}) d^3x \\ &= \int_{2m}^R \left(r \left(h_\epsilon^{+'} \right)_\epsilon + 1 + \left(h_\epsilon^+ \right)_\epsilon \right) \tilde{\Phi}(r) dr \\ &= \int_{2m}^R \left\{ \frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int_{2m}^R \tilde{\Phi}(r) dr. \end{aligned} \tag{B.9}$$

By replacement $r-2m = u$, from (B.9) we obtain

$$\begin{aligned} & \int_K \left(\left[\mathbf{R}_\epsilon^+ \right]_2 \right)_\epsilon \Phi(x) d^3x \\ &= \int_K \left(\left[\mathbf{R}_\epsilon^+ \right]_3 \right)_\epsilon \Phi(x) d^3x \\ &= - \int_0^{R-2m} \frac{u \tilde{\Phi}(u+2m) du}{(u^2 + \epsilon^2)^{1/2}} + \int_0^{R-2m} \tilde{\Phi}(u+2m) du. \end{aligned} \tag{B.10}$$

By replacement $u = \epsilon \eta$, from (B.10) we obtain the expression

$$\begin{aligned} \mathbf{I}_3^+(\epsilon) &= \int_K \left(\left[\mathbf{R}_\epsilon^+ \right]_3 \right)_\epsilon \Phi(x) d^3x = \mathbf{I}_2^+(\epsilon) = \int_K \left(\left[\mathbf{R}_\epsilon^+ \right]_2 \right)_\epsilon \Phi(\bar{x}) d^3x \\ &= -\epsilon \times \left(\int_0^\epsilon \frac{\eta \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{(\eta^2 + 1)^{1/2}} - \int_0^\epsilon \tilde{\Phi}(\epsilon \eta + 2m) d\eta \right). \end{aligned} \tag{B.11}$$

From Equation (B.11) we get

$$\begin{aligned} \mathbf{I}_3^+(\epsilon) &= \mathbf{I}_2^+(\epsilon) = -\epsilon \frac{\tilde{\Phi}(2m)}{0!} \int_0^\epsilon \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} - 1 \right] d\eta \\ &\quad - \frac{\epsilon^2}{1!} \int_0^\epsilon \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta \\ &= -\epsilon \tilde{\Phi}(2m) \left[\sqrt{\left(\frac{R-2m}{\epsilon} \right)^2 + 1} - 1 - \frac{R-2m}{\epsilon} \right] \\ &\quad - \frac{\epsilon^2}{1} \int_0^\epsilon \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta, \end{aligned} \tag{B.12}$$

where we have expressed the function $\tilde{\Phi}(\epsilon\eta + 2m)$ as

$$\tilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1} \frac{\Phi^{(l)}(2m)}{l!} (\epsilon\eta)^l + \frac{1}{n!} (\epsilon\eta)^n \Phi^{(n)}(\xi), \tag{B.13}$$

$$\xi \triangleq \theta\epsilon\eta + 2m, \quad 1 > \theta > 0, \quad n = 1$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l \tilde{\Phi} / d\xi^l$.

Equations (B.12)-(3.13) give

$$\lim_{\epsilon \rightarrow 0} \mathbf{I}_3^+(\epsilon) = \lim_{\epsilon \rightarrow 0} \mathbf{I}_2^+(\epsilon) = \lim_{\epsilon \rightarrow 0} \left\{ -\epsilon \tilde{\Phi}(2m) \left[\sqrt{\left(\frac{R-2m}{\epsilon}\right)^2 + 1} - 1 - \frac{R-2m}{\epsilon} \right] \right. \\ \left. + \lim_{\epsilon \rightarrow 0} \left\{ -\frac{\epsilon^2}{1} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta \right\} \right\} = 0. \tag{B.14}$$

Since $S'_{2m}(B^+(2m, R)) \subset D'(\mathbb{R}^3)$, where $B^+(2m, R) = \{x \in \mathbb{R}^3 \mid 2m \leq \|x\| \leq R\}$ from Equation (B.14) we get:

$$w\text{-}\lim_{\epsilon \rightarrow 0} [\mathbf{R}_\epsilon^+]_3 = \lim_{\epsilon \rightarrow 0} \mathbf{I}_3^+(\epsilon) = 0, \tag{B.15}$$

$$w\text{-}\lim_{\epsilon \rightarrow 0} [\mathbf{R}_\epsilon^+]_2 = \lim_{\epsilon \rightarrow 0} \mathbf{I}_2^+(\epsilon) = 0.$$

For $([\mathbf{R}_\epsilon^+]_1)_\epsilon, ([\mathbf{R}_\epsilon^+]_0)_\epsilon$ we get:

$$2 \int_K ([\mathbf{R}_\epsilon^+]_1)_\epsilon \Phi(x) d^3x \\ = 2 \int_K ([\mathbf{R}_\epsilon^+]_0)_\epsilon \Phi(x) d^3x \\ = \int_{2m}^R \left(r^2 (h_\epsilon^{+''})_\epsilon + 2r (h_\epsilon^{+'})_\epsilon \right) \tilde{\Phi}(r) dr \tag{B.16}$$

$$= \int_{2m}^R \left\{ \frac{r}{[(r-2m)^2 + \epsilon^2]^{1/2}} + \frac{r(r-2m)^2}{[(r-2m)^2 + \epsilon^2]^{3/2}} \right\} \tilde{\Phi}(r) dr.$$

where use is made of the relation

$$\lim_{s \rightarrow \infty} \left[\int_0^s \frac{\eta^2 d\eta}{(\eta^2 + 1)^{3/2}} - \int_0^s \frac{d\eta}{(u^2 + 1)^{1/2}} \right] = -1 \tag{B.17}$$

Finally we obtain

$$w\text{-}\lim_{\epsilon \rightarrow 0} [\mathbf{R}_\epsilon^+]_1 = w\text{-}\lim_{\epsilon \rightarrow 0} [\mathbf{R}_\epsilon^+]_0 = -m\tilde{\Phi}(2m). \tag{B.18}$$

The Colombeau generalized Ricci tensor below horizon $[\mathbf{R}_\epsilon^-]_\alpha = [\mathbf{R}_\epsilon^-]_\alpha^\beta$ may now be calculated componentwise using the classical formulae

$$\left\{ \begin{aligned} ([\mathbf{R}_\epsilon^-]_0)_\epsilon &= ([\mathbf{R}_\epsilon^-]_1)_\epsilon = \frac{1}{2} \left((h_\epsilon^{-''})_\epsilon + \frac{2}{r} (h_\epsilon^{-'})_\epsilon \right), \\ ([\mathbf{R}_\epsilon^-]_2)_\epsilon &= ([\mathbf{R}_\epsilon^-]_3)_\epsilon = \frac{(h_\epsilon^{-'})_\epsilon}{r} + \frac{1 + (h_\epsilon^-)_\epsilon}{r^2}. \end{aligned} \right. \tag{B.19}$$

From (B.4) we obtain

$$\begin{aligned}
 h_\epsilon^-(r) &= -\frac{r-2m}{r} \mapsto h_\epsilon^-(r) = \left(\frac{\sqrt{(2m-r)^2 + \epsilon^2}}{r} \right) = -h_\epsilon^+(r), \quad r < 2m. \\
 h_\epsilon^{-'}(r) &= -h_\epsilon^{+'}(r) = \frac{r-2m}{r \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} - \frac{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^2}, \\
 r \left(h_\epsilon^{-'} \right)_\epsilon + 1 + \left(h_\epsilon^- \right)_\epsilon &= -r \left(h_\epsilon^{+'} \right)_\epsilon + 1 - \left(h_\epsilon^+ \right)_\epsilon = \frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + 1. \\
 h_\epsilon^{-''}(r) &= -h_\epsilon^{+''}(r) = -\frac{r-2m}{r^2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} + \frac{2 \left[(r-2m)^2 + \epsilon^2 \right]^{1/2}}{r^3}. \\
 r^2 \left(h_\epsilon^{-''} \right)_\epsilon + 2r \left(h_\epsilon^{-'} \right)_\epsilon &= -r^2 \left(h_\epsilon^{+''} \right)_\epsilon - 2r \left(h_\epsilon^{+'} \right)_\epsilon \\
 &= \frac{r}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2 + \epsilon^2 \right]^{3/2}}.
 \end{aligned} \tag{B.20}$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation $\tilde{\Phi}(r) = \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \Phi(\vec{x})$ and let $\Phi(\vec{x})$ be the function $\Phi(\vec{x}) \in S_{2m}^-(\mathbb{R}^3, B^-(0, 2m))$, where by $S_{2m}^-(\mathbb{R}^3, B^-(0, 2m))$ we denote the class of the functions $\Phi(\vec{x})$ with compact support $K \subset B^-(0, 2m)$, $B^-(0, 2m) = \{ \vec{x} \mid 0 \leq \|\vec{x}\| \leq 2m \}$ such that:

(i) $\text{supp}(\Phi(\vec{x})) \subset \{ \vec{x} \mid 0 \leq \|\vec{x}\| \leq 2m \}$ (ii) $\tilde{\Phi}(r) \in C^\infty(\mathbb{R})$.

Then for any function $\Phi(\vec{x}) \in S_{2m}^-(\mathbb{R}^3, B^-(0, 2m))$ we get

$$\begin{aligned}
 &\int_K \left(\left[\mathbf{R}_\epsilon^- \right]_2 \right)_\epsilon \Phi(\vec{x}) d^3x \\
 &= \int_K \left(\left[\mathbf{R}_\epsilon^- \right]_3 \right)_\epsilon \Phi(\vec{x}) d^3x \\
 &= \int_0^{2m} \left(r \left(h_\epsilon^{-'} \right)_\epsilon + 1 + \left(h_\epsilon^- \right)_\epsilon \right) \tilde{\Phi}(r) dr \\
 &= \int_0^{2m} \left\{ \frac{r-2m}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int_0^{2m} \tilde{\Phi}(r) dr.
 \end{aligned} \tag{B.21}$$

By replacement $r-2m = u$, from Equation (B.21) we obtain

$$\begin{aligned}
 &\int_K \left(\left[\mathbf{R}_\epsilon^- \right]_2 \right)_\epsilon \Phi(x) d^3x \\
 &= \int_K \left(\left[\mathbf{R}_\epsilon^- \right]_3 \right)_\epsilon \Phi(x) d^3x \\
 &= \int_{-2m}^0 \frac{u \tilde{\Phi}(u+2m) du}{(u^2 + \epsilon^2)^{1/2}} + \int_{-2m}^0 \tilde{\Phi}(u+2m) du.
 \end{aligned} \tag{B.22}$$

By replacement $u = \epsilon\eta$, from (B.22) we obtain

$$\begin{aligned} \mathbf{I}_3^-(\epsilon) &= \int_K \left(\left[\mathbf{R}_\epsilon^- \right]_3^3 \right) \Phi(x) d^3x = \mathbf{I}_2^-(\epsilon) = \int_K \left(\left[\mathbf{R}_\epsilon^- \right]_2^2 \right) \Phi(\bar{x}) d^3x \\ &= \epsilon \times \left(\int_{\frac{-2m}{\epsilon}}^0 \frac{\eta \tilde{\Phi}(\epsilon\eta + 2m) d\eta}{(\eta^2 + 1)^{1/2}} + \int_{\frac{-2m}{\epsilon}}^0 \tilde{\Phi}(\epsilon\eta + 2m) d\eta \right), \end{aligned} \tag{B.23}$$

which is calculated to give

$$\begin{aligned} \mathbf{I}_3^-(\epsilon) &= \mathbf{I}_2^-(\epsilon) = \epsilon \frac{\tilde{\Phi}(2m)}{0!} \int_{\frac{-2m}{\epsilon}}^0 \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} + 1 \right] d\eta \\ &\quad + \frac{\epsilon^2}{1!} \int_{\frac{-2m}{\epsilon}}^0 \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta \\ &= \epsilon \tilde{\Phi}(2m) \left[1 - \sqrt{\left(\frac{2m}{\epsilon} \right)^2 + 1} + \frac{2m}{\epsilon} \right] \\ &\quad + \frac{\epsilon^2}{1} \int_{\frac{-2m}{\epsilon}}^0 \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta, \end{aligned} \tag{B.24}$$

where we have expressed the function $\tilde{\Phi}(\epsilon\eta + 2m)$ as

$$\begin{cases} \tilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1} \frac{\Phi^{(l)}(2m)}{l!} (\epsilon\eta)^l + \frac{1}{n!} (\epsilon\eta)^n \Phi^{(n)}(\xi), \\ \xi \triangleq \theta\epsilon\eta + 2m, \quad 1 > \theta > 0, \quad n = 1 \end{cases} \tag{B.25}$$

with $\tilde{\Phi}^{(l)} \triangleq d^l \tilde{\Phi} / d\eta^l$. Equation (B.25) gives

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathbf{I}_3^-(\epsilon) &= \lim_{\epsilon \rightarrow 0} \mathbf{I}_2^-(\epsilon) = \lim_{\epsilon \rightarrow 0} \left\{ \epsilon \tilde{\Phi}(2m) \left[1 - \sqrt{\left(\frac{2m}{\epsilon} \right)^2 + 1} + \frac{2m}{\epsilon} \right] \right\} \\ &\quad + \lim_{\epsilon \rightarrow 0} \left\{ \frac{\epsilon^2}{2} \int_{\frac{-2m}{\epsilon}}^0 \left[\frac{\eta}{(\eta^2 + 1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta \right\} = 0. \end{aligned} \tag{B.26}$$

Since $S'_{2m}(B^-(0, 2m)) \subset D'(\mathbb{R}^3)$, where $B^-(0, 2m) = \{x \in \mathbb{R}^3 \mid 0 \leq \|x\| \leq 2m\}$ from Equation (B.26) we obtain

$$\begin{cases} w - \lim_{\epsilon \rightarrow 0} \left[\mathbf{R}_\epsilon^- \right]_3^3 = \lim_{\epsilon \rightarrow 0} \mathbf{I}_3^-(\epsilon) = 0. \\ w - \lim_{\epsilon \rightarrow 0} \left[\mathbf{R}_\epsilon^- \right]_2^2 = \lim_{\epsilon \rightarrow 0} \mathbf{I}_2^-(\epsilon) = 0. \end{cases} \tag{B.27}$$

For $\left(\left[\mathbf{R}_\epsilon^- \right]_1^1 \right)_\epsilon, \left(\left[\mathbf{R}_\epsilon^- \right]_0^0 \right)_\epsilon$ we get:

$$\begin{aligned} &2 \int_K \left(\left[\mathbf{R}_\epsilon^- \right]_1^1 \right)_\epsilon \Phi(x) d^3x \\ &= 2 \int_K \left(\left[\mathbf{R}_\epsilon^- \right]_0^0 \right)_\epsilon \Phi(x) d^3x \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{2m} \left(r^2 \left(h_{\epsilon}^{-r} \right)_{\epsilon} + 2r \left(h_{\epsilon}^{-r} \right)_{\epsilon} \right) \tilde{\Phi}(r) dr \\
 &= \int_0^{2m} \left\{ \frac{r}{\left[(r-2m)^2 + \epsilon^2 \right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2 + \epsilon^2 \right]^{3/2}} \right\} \tilde{\Phi}(r) dr.
 \end{aligned} \tag{B.28}$$

By replacement $r - 2m = u$, from (B.28) we obtain

$$\begin{aligned}
 I_1^+(\epsilon) &= 2 \left(\left[\mathbf{R}_{\epsilon}^- \right]_1^1 \right)_{\epsilon} \Phi(x) d^3x = I_2^+(\epsilon) = 2 \left(\left[\mathbf{R}_{\epsilon}^- \right]_0^0 \right)_{\epsilon} \Phi(x) d^3x \\
 &= \int_0^{2m} \left(r^2 \left(h_{\epsilon}^{-r} \right)_{\epsilon} + 2r \left(h_{\epsilon}^{-r} \right)_{\epsilon} \right) \tilde{\Phi}(r) dr \\
 &= \int_{-2m}^0 \left\{ \frac{u+2m}{\left(u^2 + \epsilon^2 \right)^{1/2}} - \frac{u^2(u+2m)}{\left(u^2 + \epsilon^2 \right)^{3/2}} \right\} \tilde{\Phi}(u+2m) du.
 \end{aligned} \tag{B.29}$$

By replacement $u = \epsilon \eta$, from (B.29) we obtain

$$\begin{aligned}
 &2 \int_K \left(\left[\mathbf{R}_{\epsilon}^- \right]_1^1 \right)_{\epsilon} \Phi(x) d^3x \\
 &= 2 \int_K \left(\left[\mathbf{R}_{\epsilon}^- \right]_0^0 \right)_{\epsilon} \Phi(x) d^3x \\
 &= \int_{\frac{-2m}{\epsilon}}^0 \left(r^2 \left(h_{\epsilon}^{-r} \right)_{\epsilon} + 2r \left(h_{\epsilon}^{-r} \right)_{\epsilon} \right) \tilde{\Phi}(r) dr \\
 &= \epsilon \int_{\frac{-2m}{\epsilon}}^0 \left\{ \frac{\epsilon \eta + 2m}{\left(\epsilon^2 \eta^2 + \epsilon^2 \right)^{1/2}} - \frac{\epsilon^2 \eta^2 (\epsilon \eta + 2m)}{\left(\epsilon^2 \eta^2 + \epsilon^2 \right)^{3/2}} \right\} \tilde{\Phi}(\epsilon \eta + 2m) d\eta \\
 &= \int_{\frac{-2m}{\epsilon}}^0 \frac{\epsilon^2 \eta \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\epsilon^2 \eta^2 + \epsilon^2 \right)^{1/2}} + 2m \int_{\frac{-2m}{\epsilon}}^0 \frac{\epsilon \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\epsilon^2 \eta^2 + \epsilon^2 \right)^{1/2}} \\
 &\quad - \int_{\frac{-2m}{\epsilon}}^0 \frac{\epsilon^4 \eta^3 \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\epsilon^2 \eta^2 + \epsilon^2 \right)^{3/2}} - 2m \int_{\frac{-2m}{\epsilon}}^0 \frac{\epsilon^3 \eta^2 \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\epsilon^2 \eta^2 + \epsilon^2 \right)^{3/2}} \\
 &= \mathcal{E} \int_{\frac{-2m}{\epsilon}}^0 \frac{\eta \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\eta^2 + 1 \right)^{1/2}} - \int_{\frac{-2m}{\epsilon}}^0 \frac{\eta^3 \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\eta^2 + 1 \right)^{3/2}} \\
 &\quad + 2m \left[\int_{\frac{-2m}{\epsilon}}^0 \frac{\tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\eta^2 + 1 \right)^{1/2}} - \int_{\frac{-2m}{\epsilon}}^0 \frac{\eta^2 \tilde{\Phi}(\epsilon \eta + 2m) d\eta}{\left(\eta^2 + 1 \right)^{3/2}} \right].
 \end{aligned} \tag{B.30}$$

which is calculated to give

$$\begin{aligned}
 \mathbf{I}_0^-(\epsilon) = \mathbf{I}_1^-(\epsilon) &= 2m \frac{\tilde{\Phi}(2m)}{0!} e^{\epsilon} \int_{\frac{-2m}{\epsilon}}^0 \left[\frac{1}{\left(\eta^2 + 1 \right)^{1/2}} - \frac{\eta^2}{\left(\eta^2 + 1 \right)^{3/2}} \right] d\eta \\
 &\quad + \frac{\epsilon}{1!} \int_0^{\frac{-2m}{\epsilon}} \tilde{\Phi}^{(1)}(\xi) \left[\frac{1}{\left(\eta^2 + 1 \right)^{1/2}} - \frac{\eta^2}{\left(\eta^2 + 1 \right)^{3/2}} \right] \eta d\eta + O(\epsilon^2).
 \end{aligned} \tag{B.31}$$

where we have expressed the function $\tilde{\Phi}(\epsilon\eta + 2m)$ as

$$\tilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1} \frac{\Phi^{\alpha\beta(l)}(2m)}{l!} (\epsilon\eta)^l + \frac{1}{n!} (\epsilon\eta)^n \Phi^{\alpha\beta(n)}(\xi), \tag{B.32}$$

$$\xi \triangleq \theta\epsilon\eta + 2m, \quad 1 > \theta > 0, \quad n = 1$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l \tilde{\Phi} / d\xi^l$. Equation (B.32) gives

$$\lim_{\epsilon \rightarrow 0} \mathbf{I}_0^-(\epsilon) = \lim_{\epsilon \rightarrow 0} \mathbf{I}_1^-(\epsilon) = 2m \lim_{\epsilon \rightarrow 0} \left\{ \frac{\tilde{\Phi}(2m)}{0!} \int_{-\frac{2m}{\epsilon}}^0 \left[\frac{1}{(\eta^2 + 1)^{1/2}} - \frac{\eta^2}{(\eta^2 + 1)^{3/2}} \right] d\eta \right\} \tag{B.33}$$

$$= 2m \tilde{\Phi}(2m) \lim_{s \rightarrow 0} \left[\int_{-s}^0 \frac{d\eta}{(\eta^2 + 1)^{1/2}} - \int_{-s}^0 \frac{\eta^2 d\eta}{(\eta^2 + 1)^{3/2}} \right] = 2m \tilde{\Phi}(2m).$$

where use is made of the relation

$$\lim_{s \rightarrow \infty} \left[\int_{-s}^0 \frac{d\eta}{(u^2 + 1)^{1/2}} - \int_{-s}^0 \frac{\eta^2 d\eta}{(\eta^2 + 1)^{3/2}} \right] = 1. \tag{B.34}$$

Thus

$$w - \lim_{\epsilon \rightarrow 0} \left[\mathbf{R}_\epsilon^- \right]_1^1 = w - \lim_{\epsilon \rightarrow 0} \left[\mathbf{R}_\epsilon^- \right]_0^0 = m \tilde{\Phi}(2m). \tag{B.35}$$

Appendix C

We calculate now the distributional Colombeau scalars

$(\mathbf{R}(r, \epsilon))_\epsilon, (\mathbf{R}^{\mu\nu}(r, \epsilon) \mathbf{R}_{\mu\nu}(r, \epsilon))_\epsilon$ and $(\mathbf{R}^{\rho\sigma\mu\nu}(r, \epsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \epsilon))_\epsilon$, in terms of Colombeau generalized functions $(A_\epsilon(r))_\epsilon, (B_\epsilon(r))_\epsilon, (C_\epsilon(r))_\epsilon, (D_\epsilon(r))_\epsilon$ is given above in **Appendix B** at Schwarzschild horizon. We choose now

$$B_\epsilon(r) = 1, C_\epsilon(r) = -1 + A_\epsilon^{-1}(r), D_\epsilon(r) = 0, \tag{C.1}$$

and rewrite Equation (A.1) in the following equivalent form

$$(ds_\epsilon^2)_\epsilon = - (A_\epsilon(r) dt^2)_\epsilon + (A_\epsilon^{-1}(r) dr^2)_\epsilon + r^2 d\Omega^2, \tag{C.2}$$

where $A_\epsilon(r)$ is given above by using Equations (B.2)-(B.4). Thus we obtain

$$A_\epsilon(r) = -r^{-1} \sqrt{(r - 2m)^2 + \epsilon^2}, \quad \Delta_\epsilon = A_\epsilon(B_\epsilon + C_\epsilon) + D_\epsilon^2 = 1, \quad \Delta'_\epsilon = 0,$$

$$A'_\epsilon(r) = \frac{-2m(r - 2m)}{r^2 \sqrt{(r - 2m)^2 + \epsilon^2}},$$

$$A''_\epsilon(r) = \frac{2m(-16m^3 + 24m^2 r - 12mr^2 - 4m\epsilon^2 + 2r^3 + r\epsilon^2)}{r^3 [(r - 2m)^2 + \epsilon^2]^{3/2}} \tag{C.3}$$

$$= \frac{4m(r - 2m)^3 + (r - 4m)\epsilon^2}{r^3 [(r - 2m)^2 + \epsilon^2]^{3/2}}.$$

From Equation (A.2) and Equation (C.3) we obtain

$$\begin{aligned}
 (\mathbf{R}(r, \varepsilon))_\varepsilon &= \left(\frac{A_\varepsilon}{\Delta_\varepsilon} \left[\frac{2}{r} \left(-2 \frac{A'_\varepsilon}{A_\varepsilon} - 3 \frac{B'_\varepsilon}{B_\varepsilon} + \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right) + \frac{2}{r^2} \frac{A_\varepsilon C_\varepsilon + D_\varepsilon^2}{A_\varepsilon B_\varepsilon} - \frac{A''_\varepsilon}{A_\varepsilon} - 2 \frac{B''_\varepsilon}{B_\varepsilon} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \left(\frac{B'_\varepsilon}{B_\varepsilon} \right)^2 - 2 \frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \left(\frac{1}{2} \frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon} \right) \frac{\Delta'_\varepsilon}{\Delta_\varepsilon} \right] \right)_\varepsilon \\
 &= \left(A_\varepsilon \left[\frac{2}{r} \left(-2 \frac{A'_\varepsilon}{A_\varepsilon} \right) - \frac{2A_\varepsilon(1-A_\varepsilon^{-1})}{r^2} - \frac{A''_\varepsilon}{A_\varepsilon} \right] \right)_\varepsilon \\
 &= \left(-\frac{4A'_\varepsilon}{r} - \frac{2A_\varepsilon}{r^2} + \frac{2}{r^2} - A''_\varepsilon \right) \\
 &= \left(\frac{8m(r-2m)}{r^3 \sqrt{(r-2m)^2 + \varepsilon^2}} \right)_\varepsilon + 2r^{-3} \left(\sqrt{(r-2m)^2 + \varepsilon^2} \right)_\varepsilon - \frac{2}{r^2} \\
 &\quad - \left(\frac{2m(-16m^3 + 24m^2r - 12mr^2 - 4m\varepsilon^2 + 2r^3 + r\varepsilon^2)}{r^3 [(r-2m)^2 + \varepsilon^2]^{3/2}} \right)_\varepsilon.
 \end{aligned} \tag{C.4}$$

Finally we obtain the following expression for the distributional Colombeau scalar $(\mathbf{R}(r, \varepsilon))_\varepsilon$

$$\begin{aligned}
 (\mathbf{R}(r, \varepsilon))_\varepsilon &= \left(\frac{8m(r-2m)}{r^3 \sqrt{(r-2m)^2 + \varepsilon^2}} \right)_\varepsilon + 2r^{-3} \left(\sqrt{(r-2m)^2 + \varepsilon^2} \right)_\varepsilon \\
 &\quad - \frac{2}{r^2} - \left(\frac{4m(r-2m)^3 + (r-4m)\varepsilon^2}{r^3 [(r-2m)^2 + \varepsilon^2]^{3/2}} \right)_\varepsilon.
 \end{aligned} \tag{C.5}$$

Remark C.1. Note that from Equation (C.5) follows that: $r \neq 2m$ $(\mathbf{R}(r, \varepsilon))_\varepsilon \approx_{\mathbb{R}} 0$, see

Definition 1.5.2. (i).

We assume now that $(r_\varepsilon)_\varepsilon \approx_{\mathbb{R}} 2m$ and therefore from Equation (C.5) we obtain

$$(\mathbf{R}(r_\varepsilon, \varepsilon))_\varepsilon \approx_{\mathbb{R}} \left(\frac{4m^2 \varepsilon^2}{8m^3 [(r_\varepsilon - 2m)^2 + \varepsilon^2]^{3/2}} \right)_\varepsilon. \tag{C.6}$$

Remark C.2. Note that from Equation (C.6) at horizon $r = 2m$ follows that:

$$(\mathbf{R}(r, \varepsilon))_\varepsilon = \left(\frac{4m^2 \varepsilon^2}{8m^3 [\varepsilon^2]^{3/2}} \right)_\varepsilon = (4m)^{-1} (\varepsilon^{-1})_\varepsilon \approx_{\mathbb{R}} \infty, \tag{C.7}$$

see Definition 1.5.2. (ii).

Remark C.3. Note that from Equation (C.5) follows that:

$$w\text{-}\lim_{\varepsilon \rightarrow 0} \mathbf{R}(r, \varepsilon) \sim \delta(r - 2m). \tag{C.8}$$

Remark C.4. Let $[(r_\varepsilon - 2m)_\varepsilon] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (C.6) we obtain

$$[(\mathbf{R}(r_\varepsilon, \varepsilon))_\varepsilon] \approx_{\mathbb{R}} \left[\left(\frac{4m^2 \varepsilon^2}{8m^3 [(r_\varepsilon - 2m)^2 + \varepsilon^2]^{3/2}} \right) \right]. \tag{C.9}$$

From Equation (A.2) and Equation (C.3) we obtain

$$\begin{aligned} & (\mathbf{R}^{\mu\nu}(r, \varepsilon) \mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon \\ &= +2 \left[\left(\frac{1}{r} A'_\varepsilon + \frac{-A_\varepsilon + 1}{r^2} \right)^2 \right]_\varepsilon + 2 \left[\left(\frac{1}{2} A''_\varepsilon + \frac{1}{r} A'_\varepsilon \right)^2 \right]_\varepsilon \\ &= 2 \left[\left(\frac{1}{r^3} \sqrt{\varepsilon^2 + (r - 2m)^2} + \frac{1}{r^2} \right) + 2 \frac{m}{r^3} \frac{r - 2m}{\sqrt{\varepsilon^2 + (r - 2m)^2}} \right]^2_\varepsilon \\ &+ 2 \left[\left(\frac{4m(r - 2m)^3 + (r - 4m)\varepsilon^2}{r^3 [(r - 2m)^2 + \varepsilon^2]^{3/2}} - 2 \frac{m}{r^3} \frac{r - 2m}{\sqrt{\varepsilon^2 + (r - 2m)^2}} \right)^2 \right]_\varepsilon. \end{aligned} \tag{C.10}$$

Remark C.5. Note that from Equation (C.10) follows that:

$$\begin{aligned} r \neq_{\mathbb{R}} 2m &\Rightarrow (\mathbf{R}^{\mu\nu}(r, \varepsilon) \mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon \approx_{\mathbb{R}} K(r), \\ K(r) &= 12 \frac{r_s^2}{r_s^6}, r_s = 2m \end{aligned} \tag{C.11}$$

see Definition 1.5.2. (i).

We assume now that $(r_\varepsilon)_\varepsilon \approx_{\mathbb{R}} 2m$ and therefore from Equation (C.10) we obtain

$$(\mathbf{R}^{\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}(r_\varepsilon, \varepsilon))_\varepsilon \approx_{\mathbb{R}} K(r_s) + \left(\frac{\varepsilon^4}{4m^4 [\varepsilon^2 + (r_\varepsilon - 2m)^2]^3} \right)_\varepsilon \tag{C.12}$$

Remark C.6. Note that from Equation (C.10) at horizon $r = 2m$ follows that:

$$(\mathbf{R}^{\mu\nu}(r, \varepsilon) \mathbf{R}_{\mu\nu}(r, \varepsilon))_\varepsilon = \left(\frac{1}{4m^4 \varepsilon^2} \right)_\varepsilon \approx_{\mathbb{R}} \infty, \tag{C.13}$$

see Definition 1.5.2. (ii).

Remark C.7. Let $[(r_\varepsilon - 2m)_\varepsilon] \approx_{\mathbb{R}} 0$, then from Equation (A1.3) and Equation (C.12) we obtain

$$[(\mathbf{R}^{\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\mu\nu}(r_\varepsilon, \varepsilon))_\varepsilon] \approx_{\mathbb{R}} K(r_s) + \left[\left(\frac{\varepsilon^4}{4m^4 (\varepsilon^2 + (r_\varepsilon - 2m)^2)^3} \right) \right]_\varepsilon \tag{C.14}$$

From Equation (A.2) and Equation (C.3) we obtain

$$\begin{aligned}
 & \left(\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon) \right)_\varepsilon \\
 &= \left((A'_\varepsilon)^2 + 2 \left(\frac{A'_\varepsilon}{r} \right)^2 \right)_\varepsilon + 4 \left(\left[\frac{1}{r^2} (1 - A'_\varepsilon) \right]^2 \right)_\varepsilon + 2 \left(\left[\frac{A'_\varepsilon}{r} \right]^2 \right)_\varepsilon \\
 &= \left(A'^2_\varepsilon + 4 \frac{A'^2_\varepsilon}{r^2} \right)_\varepsilon + 4 \left(\frac{1}{r^4} (1 - A'_\varepsilon)^2 \right)_\varepsilon \tag{C.15} \\
 &= \left(\left[\frac{4m(r-2m)^3 + (r-4m)\varepsilon^2}{r^3 [(r-2m)^2 + \varepsilon^2]^{3/2}} \right]^2 \right)_\varepsilon - \frac{8m^2(r-2m)^2}{r^6 [(r-2m)^2 + \varepsilon^2]} \\
 &+ \frac{4}{r^4} \left(1 + r^{-1} \sqrt{(r-2m)^2 + \varepsilon^2} \right)^2.
 \end{aligned}$$

Remark C.8. Note that from Equation (C.15) follows that:

$$r \not\approx_{\mathbb{R}} 2m \Rightarrow \left(\mathbf{R}^{\rho\sigma\mu\nu}(r, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} K(r), \tag{C.16}$$

see Definition 1.5.2. (i).

We assume now that $(r_\varepsilon)_\varepsilon \approx_{\mathbb{R}} 2m$ and therefore from Equation (C.10) we obtain

$$\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \approx_{\mathbb{R}} K(r_\varepsilon) + \left(\frac{\varepsilon^4}{4m^4 (\varepsilon^2 + (r_\varepsilon - 2m)^2)^3} \right)_\varepsilon \tag{C.17}$$

Remark C.9. Let $[(r_\varepsilon - 2m)_\varepsilon] \approx_{\mathbb{R}} 0$ then from Equation (A1.3) and Equation (C.12) we obtain

$$\left[\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \right] = K(r_\varepsilon) + \left(\frac{\varepsilon^4}{4m^4 (\varepsilon^2 + (r_\varepsilon - 2m)^2)^3} \right)_\varepsilon \tag{C.18}$$

Remark C.10. Note that from Equation (C.15) at horizon $r = 2m$ follows that:

$$\left[\left(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon, \varepsilon) \right)_\varepsilon \right] \approx_{\mathbb{R}} \infty, \tag{C.19}$$

see Definition 1.5.2. (ii).

Remark C.11. We assume now there exist a fundamental generalized length $(l_\varepsilon)_\varepsilon$

$$\begin{aligned}
 (l_\varepsilon)_{\varepsilon \in (0, \eta]} &= a(\varepsilon)_{\varepsilon \in (0, \eta]}, \eta \ll 1, \\
 (l_\varepsilon)_{\varepsilon \in (0, \eta]} &= a,
 \end{aligned} \tag{C.20}$$

such that $|(r_\varepsilon - \bar{\rho})_\varepsilon| \geq (l_\varepsilon)_\varepsilon = a(\varepsilon)_\varepsilon$ It meant there exist a thickness $th_{hor} = (l_\varepsilon)_\varepsilon$ of BH horizon. We introduce a norm $\|th_{hor}\|$ of a thickness th_{hor} by formula

$$\|th_{hor}\| = \sup_{\varepsilon \in (0, \eta]} |l_\varepsilon| = \eta \tag{C.21}$$

where parameter η is a classical thickness of BH horizon.

By using (C.20) we get the estimate

$$\begin{aligned}
 & \left[\left(\mathbf{R}^{\rho\sigma\mu\nu} (r_\varepsilon, \varepsilon) \mathbf{R}_{\rho\sigma\mu\nu} (r_\varepsilon, \varepsilon) \right)_\varepsilon \right] \approx_{\mathbb{R}} K(r_s) + \left[\left(\frac{\varepsilon^4}{4m^4 \left((r_\varepsilon - 2m)^2 + \varepsilon^2 \right)^3} \right)_\varepsilon \right]_{\varepsilon \in (0, \eta]} \\
 & = K(r_s) + \left[\left(\frac{1}{4m^4 \left[(r_\varepsilon - 2m)^2 + \varepsilon^2 \right]} \right)_\varepsilon \right]_{\varepsilon \in (0, \eta]} \\
 & \quad \times \left(\frac{\varepsilon^2}{\left[(r_\varepsilon - 2m)^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta]} \times \left(\frac{\varepsilon^2}{\left[(r_\varepsilon - 2m)^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta]} \tag{C.22} \\
 & \leq K(r_s) + \frac{1}{4m^4 [a^2 + 1]^2} \left(\frac{1}{4m^4 \left[(r_\varepsilon - 2m)^2 + \varepsilon^2 \right]} \right)_{\varepsilon \in (0, \eta]} \\
 & \leq K(r_s) + \frac{1}{8m^4 [a^2 + 1]^2} \left(\frac{1}{(r - 2m)^2} \right)_{r-2m \in (0, \eta]}
 \end{aligned}$$