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Iterative Solution of Mesh Constrained Optimal Control Problems with Two-Level Mesh Approximations of Parabolic State Equation

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Abstract

We consider a linear-quadratical optimal control problem of a system governed by parabolic equation with distributed in right-hand side control and control and state constraints. We construct a mesh approximation of this problem using different two-level approximations of the state equation, ADI and fractional steps approximations in time among others. Iterative solution methods are investigated for all constructed approximations of the optimal control problem. Their implementation can be carried out in parallel manner.

Keywords

Parabolic Optimal Control, State Constraints, Finite Difference Method, Constrained Saddle Point Problem, Iterative Method

1. Introduction

Optimal control of time-dependent production processes plays an important role in many real world applications. State constraints in optimal control of systems governed by partial differential equations have to be often included in the mathematical models. For instance, in continuous casting process a need to prevent the cracks in a slab and the solidification at a wrong place leads to the bounds on the temperature variable. Similar demands arise in the processes of crystal growth and cooling of glass melts (see articles [1] [2] [3] [4] [5] and bibliography therein). The introduction of pointwise state constraints yields adjoint variables and multipliers which only admit low regularity complicating both theoretical analysis and the constructing appropriate numerical methods.

There are few results in the area of numerical solution methods for the constrained parabolic optimal control problems. One of the approaches to overcome the difficulty connected with low regularity of the solutions is using Lavrentiev regularization. This approach has been used in [6] [7]. In [8] [9] [10] [11] a priori error estimates for space-time discretizations of linear-quadratic parabolic optimal control problems have been obtained for problems.

The implementations of the iterative methods for parabolic optimal control problems include the solution of the parabolic equation and corresponding adjoint parabolic equation at each iteration and this is the most time consuming part of the algorithms if applying the implicit (backward Euler) approximation of parabolic state equation. On the other hand, easily implementable explicit (forward Euler) approximation of a parabolic equation with a constant step in time requests extremely restrictive constraint for this step. Using explicit approximations of the parabolic equations with special series of non-uniform time steps allows partially avoid this deficiency. Such kind approximation is well-known for the differential equations [12] and they demonstrate the advantage in time of calculations in relation to the implicit schemes. Similar approximation have been used for the continuous casting problem [13] and recently applied to a parabolic state constrained problem [14]. One more approach to construct effective algorithm for parabolic optimal control problem is to use parareal approximation of the state equation [15] [16].

In this article we continue the investigations of [14] [16] [17] [18] [19] [20] on the iterative solution methods for the constrained saddle point problems with applications to optimal control problems. In the cited papers parabolic optimal control problems have been solved by using either backward or forward Euler approximating schemes for the state equation.

The main purpose of this article is to generalize these results for the case when any two-level scheme is used for the approximation of the parabolic state equation, including different splitting (locally one-dimensional) schemes.

To simplify the exposition we restrict ourselves to consider a problem in unit square with distributed control and observation, and to use finite difference schemes to approximate the state equation, while the most of the results can be extended for the case of lowest order finite element method, for a control in Neumann boundary condition, for final observation etc.

2. Formulation and Approximations of the Problem

Consider homogeneous Dirichlet initial-boundary value problem

$$\frac{\partial y}{\partial t} - \Delta y = u \text{ in } Q_T;\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T];\ y(x,0) = 0 \tag{1}$$

in the cylinder $Q_T = \Omega \times (0,T]$, $\Omega = (0,1)^2$, with lateral surface $\Sigma = \partial\Omega \times (0,T]$. We call $y(x,t)$ and $u(x,t)$ as state and control functions. It is well-known that for any $u \in L_2(Q_T)$ there exists a unique weak solution of problem (1) such that $y \in W(0,T) = L^\infty(0,T;H_0^1(\Omega)) \cap H^1(0,T,L^2(\Omega))$, and the following inequality takes place ([21] [22] [23]):

$$\sup_{0\le t\le T} \| y(t)\|_{H_0^1(\Omega))} + \| \frac{\partial y}{\partial t} \|_{L_2(Q_T)} \le c \| u \|_{L_2(Q_T)} . \tag{2}$$

Define objective function

$$J(y,u) = \frac{1}{2}\int_{Q_T} (y(x,t) - y_d(x,t))^2 dxdt + \frac{1}{2}\int_{Q_T} u^2 dxdt$$

with a given functions $y_d \in L_2(Q_T)$, and the sets of constraints for state and control:

$$U_{ad} = \{u \in L_2(Q_T) : | u(x,t) | \le u_{\max} \ a.e.\ (x,t) \in Q_T\},$$
$$Y_{ad} = \{y \in W(0,T) : y_{\min} \le y(x,t) \le y_{\max} \ a.e.\ (x,t) \in Q_T\},$$

Above $u_{\max} > 0$ and $-\infty \le y_{\min} < y_{\max} \le \infty$. We solve the following optimal control problem:

$$\min_{(y,u)\in K} J(y,u),$$
$$K = \{(y,u) \in Y_{ad} \times U_{ad} : \text{state equation (1) is satisfied}\}. \tag{3}$$

Problem (3) has a unique solution (cf., e.g. [14]).

We construct the finite-difference approximations of problem (3) using a uniform in x and t mesh $\omega_x \times \omega_t$ in $\overline{Q}_T$.

Let ω_x be the uniform mesh of the meshsize h on $\overline{\Omega}$, $\partial\omega_x = \omega_x \cap \partial\Omega$ be the set of the boundary nodes while $\omega_x^0 = \omega_x \setminus \partial\omega_x$. By V_h^0 we denote the space of mesh functions which are defined on ω_x and vanish in the boundary nodes $\partial\omega_x$, let $\dim V_h^0 = N_x$. The mesh on the time segment we denote by $\omega_t = \{t_j = j\tau, j = 0,1,\ldots,N_t; N_t\tau = T\}$.

We will use the notations $y,u,\ldots$ for the mesh functions from V_h^0 and for the vectors of their nodal values as well. We also don't distinguish the linear operators from V_h^0 to V_h^0 and corresponding them $N_x \times N_x$ matrices acting on the vectors of nodal values of mesh functions from V_h^0.

By $y_j = y(x,t_j) \in \mathrm{R}^{N_x}$ we denote the values on a time level $t_j = j\tau \in \omega_t$ of a mesh function of x and t, and by $\|\cdot\|_x$ -euclidian norm in the space R^{N_x}. We use the following notations: $N = N_t N_x$, $(.,.)$ and $\|\cdot\|$ are the inner product and the norm in R^N. Let E_x be $N_x \times N_x$ unit matrix while E be the unit matrix in $\mathrm{R}^{N\times N}$.

We denote by $\Lambda_1 : V_h^0 \to V_h^0$ the linear operator such that

$$\Lambda_1 y(x) = h^{-2}(2y(x_1,x_2) - y(x_1+h,x_2) - y(x_1-h,x_2) \text{ for } x \in \omega_x^0$$

and similarly for $\Lambda_2 y(x)$. Obviously, $\Lambda = \Lambda_1 + \Lambda_2$ is the mesh Laplasian with homogeneous Dirichlet boundary conditions. Recall that we use the same notations for corresponding matrices. Matrices Λ_i are symmetric, commute and positive definite with spectrum in a segment $[\xi_0,\xi_1]$, where ξ_1 is of order h^{-2}, while $\xi_0 > 0$ is bounded below by a constant which doesn't depend on h.

Let us introduce $N_x \times N_x$ matrices U, D and R, where U is a symmetric and positive matrix, and approximate state problem (1) by two-level finite difference scheme

$$U\frac{y_j - y_{j-1}}{\tau} + Dy_{j-1} = Ru_j,\ j = 1,2,\ldots,N_t,\ \ y_0 = 0. \tag{4}$$

Below we give several examples of two-level finite difference scheme (4).

1) Finite difference scheme with weights:

$$\frac{y_j - y_{j-1}}{\tau} + \Lambda(\sigma y_j + (1-\sigma)y_{j-1}) = u_j,\ j = 1,2,\ldots,N_t,\ \ y_0 = 0 \tag{5}$$

with $\sigma \in [0,1]$ can be written in the form (4) with $U = E_x + \sigma\tau\Lambda$, $D = \Lambda$ and $R = E_x$. Recall that (5) contains forward Euler ($\sigma = 0$), backward Euler ($\sigma = 1$) and Crank-Nicolson ($\sigma = 1/2$) schemes.

Scheme (5) is unconditionally stable for $\sigma \ge 1/2$ and stable in the case $0 \le \sigma < 1/2$ if $\tau \le 2(\xi_{\max}(1-2\sigma))^{-1}$, where $\xi_{\max} = 2\xi_1$ is the maximal eigenvalue of Λ. The stability estimate is:

$$\sum_{j=1}^{N_t} \| y_j \|_x^2 \le \sum_{j=1}^{N_t} \| u_j \|_x^2 . \tag{6}$$

2) Two-level scheme with factorized preconditioner:

$$\begin{gathered}(E_x + \sigma\tau\Lambda_1)(E_x + \sigma\tau\Lambda_2)\frac{y_j - y_{j-1}}{\tau} + \Lambda y_{j-1} = u_j,\\ j = 1,2,\ldots,N_t,\ y_0 = 0.\end{gathered} \tag{7}$$

Mesh scheme (7) is unconditionally stable for $\sigma \ge 0.5$ with stability estimate (6).

3) Fractional steps scheme:

$$\begin{gathered}0.5\frac{z_1 - y_{j-1}}{\tau} + \Lambda_1 z_1 = 0.5u_j,\\ 0.5\frac{z_2 - y_{j-1}}{\tau} + \Lambda_2 z_2 = 0.5u_j,\\ y_j = 0.5z_1 + 0.5z_2,\ j = 1,2,\ldots,N_t,\ \ y_0 = 0.\end{gathered} \tag{8}$$

System (8) can be written in form (4) with

$$U = (E_x + 2\tau\Lambda_1)(E_x + 2\tau\Lambda_2),\ D = \Lambda + 4\tau\Lambda_1\Lambda_2,\ R = E_x + \tau\Lambda.$$

Scheme (8) is unconditionally stable with stability estimate (6).

Let us further use notation $y_d : \omega_t \to V_h^0$ for an V_h^0-approximation of the function y_d. Define a mesh goal function and the sets of the constraints:

$$J_h(y,u) = \frac{1}{2}\sum_{j=1}^{N_t} \tau \| y_j - y_{dj} \|_x^2 + \frac{1}{2}\sum_{j=1}^{N_t} \tau \| u_j \|_x^2, \tag{9}$$

$$\begin{gathered}U_{ad}^h = \{u \in \mathrm{R}^N : | u_i^j | \le \bar{u}\ \forall x \in \omega_x, \forall t \in \omega_t\},\\ Y^h = \{y \in \mathrm{R}^N : y_{\min} \le y_j \le y_{\max}, \forall x \in \omega_x, \forall t \in \omega_t\}.\end{gathered}$$

The mesh optimal control problem reads as follows:

$$\begin{gathered}\min_{(y,u)\in K_h} J_h(y,u),\\ K_h = \{(y,u) \in Y_{ad}^h \times U_{ad}^h :\ (4) \text{ is satisfied}\}.\end{gathered} \tag{10}$$

Problem (10) has a unique solution because the set K_h is a convex compact and the quadratical function $J_h(y,u)$ is continuous and strictly convex on K_h.

3. Saddle Point Problem and Iterative Solution Method

Let us rewrite problem (10) in a "vector-matrix" form. Let the matrices $L, Q \in \mathrm{R}^{N\times N}$ be defined by the equalities:

$$(Ly)_j = \{U\frac{y_1}{\tau} \text{ for } j=1,\ U\frac{y_j - y_{j-1}}{\tau} + Dy_{j-1} \text{ for } j=2,\ldots,N_t\},$$
$$(Qu)_j = \{Ru_j \text{ for } j=1,\ldots,N_t\}.$$

Denote by ψ and φ the indicator functions of the sets Y_0^h and U_{ad}^h, respectively. We obtain the following algebraic form of mesh optimal control problem (10):

$$\min_{Ly=u}\{I(y,u) = \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2 + \psi(y) + \varphi(u)\}. \tag{11}$$

Below we suppose that for a two-level finite difference scheme (4) the following assumption is valid:

$$\exists \text{ const } C_{stab} : \|L^{-1}Q\| \le C_{stab}. \tag{12}$$

The stability estimate (6) approve that for all cited above particular cases of (4) assumption (12) is true (with constant $C_{stab} = 1$).

Remark 1 *More well-known finite difference schemes can be written in form* (4) *and satisfy assumption* (12): *different kinds of ADI schemes proposed in* [24], [25] *and* "*sequential*" *variant of fractional steps scheme* [26] [27] etc. *Also a more general variant of scheme* (8) (*see* [28]) *with positive weights* α_1, α_2, *such that* $\alpha_1 + \alpha_2 = 1$ *instead of* $\alpha_i = 1/2$ *as in* (8) *can be considered.*

We construct Lagrange function for problem (11):

$$\mathrm{L}(y,u,\lambda) = I(y,u) + \psi(y) + \varphi(u) + (\lambda, Ly - Qu).$$

A saddle point of this Lagrangian satisfies the following system:

$$\begin{pmatrix} E & 0 & L^T \\ 0 & E & -Q^T \\ L & -Q & 0 \end{pmatrix}\begin{pmatrix} y \\ u \\ \lambda \end{pmatrix} + \begin{pmatrix} \partial\psi(y) \\ \partial\varphi(u) \\ 0 \end{pmatrix} \ni \begin{pmatrix} y_d \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

where $\partial\psi(y)$ and $\partial\varphi(u)$ are the subdifferentials of the corresponding functions.

With the notations $z = (y,u)^T$, $f = (y_d, 0, 0)^T$, $\Psi(z) = \psi(y) + \varphi(u)$ and

$$\mathrm{A} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},\ B = \begin{pmatrix} L & -Q \end{pmatrix}$$

(11) becomes a particular case of minimization problem

$$\min_{Bz=0}\{\frac{1}{2}(Az,z) - (f,z) + \Psi(z)\}, \tag{14}$$

while (13)-a particular case of saddle point problem

$$\begin{pmatrix} \mathrm{A} & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} z \\ \eta \end{pmatrix} + \begin{pmatrix} \partial\Psi(z) \\ 0 \end{pmatrix} \ni \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{15}$$

We will use the following results from the article [17]:

Proposition 1 *Let*

$$\text{matrix } \mathrm{A} \in \mathrm{R}^{m\times m} \text{ is symmetric and positive definite;} \tag{16}$$

$$\text{matrix } B \in \mathrm{R}^{s\times m} \text{ has a full column rank;} \tag{17}$$

$$\Psi : \mathrm{R}^m \to \overline{\mathrm{R}} \text{ is a convex, proper and lower semicontinuous function;} \tag{18}$$

$$\{z \in \mathrm{R}^m : Bz = 0\} \cap \operatorname{int}\operatorname{dom}\Psi \neq \varnothing. \tag{19}$$

Then

1) Problem (15) has a non-empty set of the solutions $X = \{(z,\eta)\}$, where z is unique solution of (14).

2) Uzawa-type iterative method

$$\begin{aligned} &\mathrm{A}z^{k+1} + \partial\Psi(z^{k+1}) \ni B^T\eta^k + f, \\ &\frac{1}{\rho}D(\eta^{k+1} - \eta^k) + Bz^{k+1} = 0,\ \rho > 0 \end{aligned} \tag{20}$$

with a symmetric preconditioner D satisfying the inequality

$$(D\eta,\eta)\frac{(1+\varepsilon)\rho}{2}(\mathrm{A}^{-1}B^T\eta, B^T\eta)\ \ \forall \eta \in \mathrm{R}^s,\ \varepsilon > 0, \tag{21}$$

converges for any initial guess η^0: $(z^k,\eta^k) \to (z^*,\eta^*) \in X$ for $k \to \infty$.

Theorem 1 *Problem* (13) *has a solution* (y,u,λ) *with unique pair* (y,u), *which coincides with the solution of problem* (11).

Proof. Obviously, all assumptions (16)-(18) and (19) of proposition 1 are satisfied for problem (13). In particular, vector $(y_0,u_0) = (0,0)$ satisfies assumption (19).

Below we construct for problem (13) the easily implementable preconditioner D which is spectrally equivalent to $B\mathrm{A}^{-1}B^T$ with constants independent on τ and *h*. As this preconditioner we take $D = LL^T$. Then method (20) for problem (13) with this preconditioner reads as follows:

$$\begin{aligned} &y^{k+1} + \partial\psi(y^{k+1}) \ni y_d - L^T\lambda^k, \\ &u^{k+1} + \partial\varphi(u^{k+1}) \ni Q^T\lambda^k, \\ &LL^T\frac{\lambda^{k+1} - \lambda^k}{\rho} = Ly^{k+1} - Qu^{k+1}. \end{aligned} \tag{22}$$

Theorem 2 *Method* (22) *converges for any initial guess* λ^0 *if*

$$0 < \rho < \frac{2}{1 + C_{stab}^2}. \tag{23}$$

Proof. The matrix $B\mathrm{A}^{-1}B^T$ is spectrally equivalent to $D = LL^T$, namely,

$$LL^T B\mathrm{A}^{-1}B^T(1 + C_{stab}^2)LL^T, \tag{24}$$

where constant C_{stab} is defined in (12). In fact, by direct calculation we find $B\mathrm{A}^{-1}B^T = LL^T + QQ^T \geq LL^T$. Further, since $\| L^{-1}Q \| \leq C_{stab}$, then $\mathrm{P}Q^T L^{-T}\mathrm{P} \leq C_{stab}$. This inequality is equivalent to

$$QQ^T \leq C_{stab}^2 LL^T, \tag{25}$$

whence the result.

In virtue of inequality (24) the convergence condition (21) of proposition 1 is true for the parameter ρ from (23). ■

In the case of optimal control problem without state constraints we can estimate a rate of convergence for iterative method (22) and give an optimal iterative parameter ρ. Namely, the following statement holds:

Theorem 3 *Let* $\partial\psi = 0$. *Then there exists a unique solution* (y,u,λ) *of saddle point problem* (13), *and for theoretically optimal iterative parameter* $\rho_0 = \dfrac{1}{C_{stab}}$ *the following estimate for the rate of convergence of method* (22) *is valid*:

$$\| L^T(\lambda^{k+1}-\lambda)\| \le \left(1-\frac{1}{C_{stab}}\right)^{1/2} \| L^T(\lambda^k-\lambda)\|, \; k=0,1,\ldots \tag{26}$$

Proof. The uniqueness of (y,u) is proved in theorem 1. The uniqueness of λ in the case $\partial\psi = 0$ follows from the equation $y + L^T\lambda = y_d$.

Vector λ is the solution of the equation

$$LL^T\lambda + Q\circ(E+\partial\varphi)^{-1}\circ Q^T\lambda = Ly_d, \tag{27}$$

while iterative method (22) can be written in the form

$$LL^T\frac{\lambda^{k+1}-\lambda^k}{\rho} + LL^T\lambda^k + Q\circ(E+\partial\varphi)^{-1}\circ Q^T\lambda^k = Ly_d. \tag{28}$$

It is well-known (cf., e.g. [29]) that the operator $(E+\partial\varphi)^{-1}$ is co-coercive:

$$((E+\partial\varphi)^{-1}\lambda-(E+\partial\varphi)^{-1}\mu, \lambda-\mu) \ge \|(E+\partial\varphi)^{-1}\lambda-(E+\partial\varphi)^{-1}\mu\|^2$$

Because of this and (25) the operator $P = E + (L^{-1}Q)\circ(E+\partial\varphi)^{-1}\circ(Q^TL^{-T})$ satisfies the following properties (strong monotonicity and Lipshitz-continuity):

$$(P(\lambda)-P(\mu),\lambda-\mu) \ge \|\lambda-\mu\|^2,$$

$$(P(\lambda)-P(\mu),\eta) \le C_{stab}(P(\lambda)-P(\mu),\lambda-\mu)^{1/2}\|\eta\|$$

The rest of the prove is quite standard (cf. [18]). Namely, with the notations $\eta^k = L^T\lambda^k$, $\eta = L^T\lambda$ and $z^k = \eta^k - \eta$ we have the equation

$$\frac{z^{k+1}-z^k}{\rho} + P(\eta^k) - P(\eta) = 0.$$

We multiply this equation by $2\rho z^{k+1}$ and obtain the equality

$$\|z^{k+1}\|^2 - \|z^k\|^2 + \|z^{k+1}-z^k\|^2 + 2\rho(P(\eta^k)-P(\eta), z^{k+1}) = 0,$$

Due to the properties of P the following estimate holds:

$$\begin{aligned} 2\rho(P(\eta^k)-P(\eta),z^{k+1}) &= 2\rho(P(\eta^k)-P(\eta),z^k) + 2\rho(P(\eta^k)-P(\eta),z^{k+1}-z^k) \\ &\ge (2\rho-\rho^2C_{stab}^2)(P(\eta^k)-P(\eta),z^k) - \|z^{k+1}-z^k\|^2. \end{aligned}$$

Substituting this estimate in the previous equality we get

$$\|z^{k+1}\|^2 \le (1-\rho(2-\rho C_{stab}^2))\|z^k\|^2,$$

whence $\| z^k \| \to 0$ if $0 < \rho < \frac{2}{C_{stab}^2}$ and rate of convergence (26) is true for optimal parameter $\rho_0 = \frac{1}{C_{stab}^2}$. ■

Remark 2 *Due to the equalities* $y = y_d - L^T \lambda$, $y^k = y_d - L^T \lambda^{k-1}$, *we have the following estimate for the rate of convergence of* y^k:

$$\| y^k - y \| \le (1 - \frac{1}{C_{stab}})^{1/2} \| y^{k-1} - y \| \le (1 - \frac{1}{C_{stab}})^{(k-1)/2} \| y^1 - y \|$$
$$= (1 - \frac{1}{C_{stab}})^{(k-1)/2} \| L^T (\lambda^0 - \lambda) \| \quad k = 1, 2, \ldots$$

For the sequence of control vectors $\{u^k\}$ the estimate is as follows:

$$\| u^k - u \| = \| (E + \partial\varphi)^{-1} (\lambda^{k-1}) - (E + \partial\varphi)^{-1} (\lambda) \| \le \| L^{-T} \| \| L^T (\lambda^{k-1} - \lambda) \|$$
$$\le \| L^{-T} \| (1 - \frac{1}{C_{stab}})^{(k-1)/2} \| L^T (\lambda^0 - \lambda) \| \quad k = 1, 2, \ldots$$

Remark 3 *For the state constrained problems there are no estimates for rate of convergence for iterative method* (22). *In this case instead of* (27) *we have the equation*

$$L \circ (E + \partial\psi) \circ L^T \lambda + Q \circ (E + \partial\varphi)^{-1} \circ Q^T \lambda = L y_d,$$

which operator is only co-coercive. The convergence of the iterative methods for such kind of equations have been investigated in [18].

On the other hand, numerous calculations show that in this general case the choice of the iterative parameter $\rho; \rho_0$ as in theorem 3 is practical and seems to be close to optimal one.

When implementing method (22) one has to solve the inclusions with respect y^{k+1} and u^{k+1}, and a system of linear equations with matrix LL^T. Concerning solving the inclusions we underline that the matrices and the operators in them have diagonal form, so, their solving reduces to easy pointwise projection.

In turn, solving equation with the matrix LL^T is equivalent to solving direct (with L) and adjoint (with L^T) parabolic mesh schemes. In the case of mesh schemes with factorized preconditioner (7) or fractional steps scheme (8) approximating state equation their solving reduces to solving set of non-coupled "one-dimensional" mesh problems-systems of linear algebraic equations with tridiagonal matrices $E_x + \text{const}\, \tau \Lambda_i$, $i = 1, 2$. Obviously, these systems can be solved by a well-known direct method and parallel.

4. Variants and Generalizations

The effectiveness of the implementation of iterative method (22) is based on two main properties:

- Preconditioner has factorized form $D = LL^T$ and is spectrally equivalent to "main" matrix of the problem;
- Equations with the matrices L and L^T as in (7) and (8) can be easily implementable.

The first property is ensured by the inequality (12): $\| L^{-1}Q \| \le C_{stab}$. Just this inequality allows us to prove spectral equivalency of the matrix $B\mathrm{A}^{-1}B^T = LL^T + QQ^T$ and $D = LL^T$ with the constants independent on mesh parameters τ and h. In turn, this inequality is nothing but a stability estimate for a corresponding two-level approximation of a parabolic state equation. Numerous classes of stable two-level finite difference schemes for the parabolic equations can be found in [27].

The second property-easy solution of the equations with matrices L, L^T in (7) and (8)-is the consequence of their "local one-dimensional" structure. This imposes several limitations to the domains, boundary conditions and using orthogonal meshes. Nevertheless, a lot of different mesh schemes with factorized preconditioner of the form (7), satisfying stability property (12) is known (cf., e.g. [24] [27] and the bibliography therein).

The results of this paper can be extended to the parabolic optimal control problems with other state and control constraints, such as, for example, in [14] and [30].

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