Scientific Research Publishing

Applied Mathematics, 2017, 8, 1320-1335
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Projection of the Semi-Axes of the Ellipse of Intersection

P. P. Klein

Computing Center, University of Technology Clausthal, Clausthal-Zellerfeld, Germany
Email: peter.p.klein@t-online.de

How to cite this paper: Klein, P.P. (2017) Projection of the Semi-Axes of the Ellipse of Intersection. *Applied Mathematics*, **8**, 1320-1335.
https://doi.org/10.4236/am.2017.89097

Received: August 22, 2017
Accepted: September 22, 2017
Published: September 25, 2017

Abstract

It is well known that the line of intersection of an ellipsoid and a plane is an ellipse (see for instance [1]). In this note the semi-axes of the ellipse of intersection will be projected from 3d space onto a 2d plane. It is shown that the projected semi-axes agree with results of a method used by Bektas [2] and also with results obtained by Schrantz [3].

Keywords

Ellipsoid and Plane Intersection, Projection of the Semi-Axes of the Ellipse of Intersection

1. Introduction

Let an ellipsoid be given with the three positive semi-axes a_1, a_2, a_3

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}=1 \tag{1}$$

and a plane with the unit normal vector

$$\boldsymbol{n}=\left(n_1,n_2,n_3\right)^{\mathrm{T}},$$

which contains an interior point $\boldsymbol{q}=\left(q_1,q_2,q_3\right)^{\mathrm{T}}$ of the ellipsoid. A plane spanned by vectors $\boldsymbol{r}=\left(r_1,r_2,r_3\right)^{\mathrm{T}}$, $\boldsymbol{s}=\left(s_1,s_2,s_3\right)^{\mathrm{T}}$ and containing the point $\boldsymbol{q}$ is described in parametric form by

$$\boldsymbol{x}=\boldsymbol{q}+t\boldsymbol{r}+u\boldsymbol{s} \quad \text{with } \boldsymbol{x}=\left(x_1,x_2,x_3\right)^{\mathrm{T}}. \tag{2}$$

Inserting the components of $\boldsymbol{x}$ into the equation of the ellipsoid (1) leads to the line of intersection as a quadratic form in the variables t and u. Let the scalar product in $\mathbf{R}^3$ for two vectors $\boldsymbol{v}=\left(v_1,v_2,v_3\right)^{\mathrm{T}}$ and $\boldsymbol{w}=\left(w_1,w_2,w_3\right)^{\mathrm{T}}$ be denoted by

$$\left(\boldsymbol{v},\boldsymbol{w}\right)=v_1w_1+v_2w_2+v_3w_3$$

and the norm of vector $\boldsymbol{v}$ by

$$\left\|\boldsymbol{v}\right\|=\sqrt{\left(v,v\right)}.$$

With the diagonal matrix

$$D_1=\operatorname{diag}\left(\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3}\right)$$

the line of intersection has the form:

$$\begin{aligned}&\left(t,u\right)\begin{pmatrix}\left(D_1\boldsymbol{r},D_1\boldsymbol{r}\right) & \left(D_1\boldsymbol{r},D_1\boldsymbol{s}\right)\\ \left(D_1\boldsymbol{r},D_1\boldsymbol{s}\right) & \left(D_1\boldsymbol{s},D_1\boldsymbol{s}\right)\end{pmatrix}\begin{pmatrix}t\\ u\end{pmatrix}\\ &+2\left(\left(D_1\boldsymbol{q},D_1\boldsymbol{r}\right),\left(D_1\boldsymbol{q},D_1\boldsymbol{s}\right)\right)\begin{pmatrix}t\\ u\end{pmatrix}\\ &=1-\left(D_1\boldsymbol{q},D_1\boldsymbol{q}\right).\end{aligned}\tag{3}$$

As $\boldsymbol{q}$ is an interior point of the ellipsoid the right-hand side of Equation (3) is positive.

Let $\boldsymbol{r}$ and $\boldsymbol{s}$ be unit vectors orthogonal to the unit normal vector $\boldsymbol{n}$ of the plane

$$\begin{aligned}\left(\boldsymbol{r},\boldsymbol{r}\right)&=r_1^2+r_2^2+r_3^2=1,\\ \left(\boldsymbol{n},\boldsymbol{r}\right)&=n_1r_1+n_2r_2+n_3r_3=0,\end{aligned}\tag{4}$$

$$\begin{aligned}\left(\boldsymbol{s},\boldsymbol{s}\right)&=s_1^2+s_2^2+s_3^2=1,\\ \left(\boldsymbol{n},\boldsymbol{s}\right)&=n_1s_1+n_2s_2+n_3s_3=0,\end{aligned}\tag{5}$$

and orthogonal to eachother

$$\left(\boldsymbol{r},\boldsymbol{s}\right)=r_1s_1+r_2s_2+r_3s_3=0.\tag{6}$$

If vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ have the additional property

$$\left(D_1\boldsymbol{r},D_1\boldsymbol{s}\right)=\frac{r_1s_1}{a_1^2}+\frac{r_2s_2}{a_2^2}+\frac{r_3s_3}{a_3^2}=0\tag{7}$$

the 2×2 matrix in (3) has diagonal form. If condition (7) does not hold for vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, it can be fulfilled, as shown in [1], with vectors $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ obtained by a transformation of the form

$$\begin{aligned}\tilde{\boldsymbol{r}}&=\cos\omega\boldsymbol{r}+\sin\omega\boldsymbol{s},\\ \tilde{\boldsymbol{s}}&=-\sin\omega\boldsymbol{r}+\cos\omega\boldsymbol{s}\end{aligned}\tag{8}$$

with an angle ω according to

$$\omega=\frac{1}{2}\arctan\left[\frac{2\left(D_1\boldsymbol{r},D_1\boldsymbol{s}\right)}{\left(D_1\boldsymbol{r},D_1\boldsymbol{r}\right)-\left(D_1\boldsymbol{s},D_1\boldsymbol{s}\right)}\right].\tag{9}$$

Relations (4), (5) and (6) hold for the transformed vectors $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ instead of $\boldsymbol{r}$ and $\boldsymbol{s}$. If plane (2) is written instead of vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ with the transformed vectors $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ the 2×2 matrix in (3) has diagonal form because of condition (7):

$$\begin{aligned}&\left(D_1\tilde{\boldsymbol{r}}, D_1\tilde{\boldsymbol{r}}\right)t^2+\left(D_1\tilde{\boldsymbol{s}}, D_1\tilde{\boldsymbol{s}}\right)u^2+2\left(D_1\boldsymbol{q}, D_1\tilde{\boldsymbol{r}}\right)t+2\left(D_1\boldsymbol{q}, D_1\tilde{\boldsymbol{s}}\right)u\\&=1-\left(D_1\boldsymbol{q}, D_1\boldsymbol{q}\right).\end{aligned}$$

Then the line of intersection reduces to an ellipse in translational form

$$\frac{\left(t-t_0\right)^2}{A^2}+\frac{\left(u-u_0\right)^2}{B^2}=1 \tag{10}$$

with the center $\left(t_0, u_0\right)$

$$t_0=-\frac{\left(D_1\boldsymbol{q}, D_1\tilde{\boldsymbol{r}}\right)}{\left(D_1\tilde{\boldsymbol{r}}, D_1\tilde{\boldsymbol{r}}\right)} \quad \text{and} \quad u_0=-\frac{\left(D_1\boldsymbol{q}, D_1\tilde{\boldsymbol{s}}\right)}{\left(D_1\tilde{\boldsymbol{s}}, D_1\tilde{\boldsymbol{s}}\right)} \tag{11}$$

and the semi-axes

$$A=\sqrt{\frac{1-d}{\left(D_1\tilde{\boldsymbol{r}}, D_1\tilde{\boldsymbol{r}}\right)}} \quad \text{and} \quad B=\sqrt{\frac{1-d}{\left(D_1\tilde{\boldsymbol{s}}, D_1\tilde{\boldsymbol{s}}\right)}}, \tag{12}$$

where

$$d=\left(D_1\boldsymbol{q}, D_1\boldsymbol{q}\right)-\frac{\left(D_1\boldsymbol{q}, D_1\tilde{\boldsymbol{r}}\right)^2}{\left(D_1\tilde{\boldsymbol{r}}, D_1\tilde{\boldsymbol{r}}\right)}-\frac{\left(D_1\boldsymbol{q}, D_1\tilde{\boldsymbol{s}}\right)^2}{\left(D_1\tilde{\boldsymbol{s}}, D_1\tilde{\boldsymbol{s}}\right)}. \tag{13}$$

Because of $1-d \geq 1-\left(D_1\boldsymbol{q}, D_1\boldsymbol{q}\right)>0$ the numerator $1-d$ in (12) is positive. Putting

$$\beta_1=\left(D_1\tilde{\boldsymbol{r}}, D_1\tilde{\boldsymbol{r}}\right) \quad \text{and} \quad \beta_2=\left(D_1\tilde{\boldsymbol{s}}, D_1\tilde{\boldsymbol{s}}\right) \tag{14}$$

the semi-axes A, B given in (12) can be rewritten as

$$A=\sqrt{\frac{1-d}{\beta_1}} \quad \text{and} \quad B=\sqrt{\frac{1-d}{\beta_2}}. \tag{15}$$

In [1] it is shown that β_1 and β_2 according to (14) are solutions of the following quadratic equation

$$\begin{aligned}&\beta^2-\left[n_1^2\left(\frac{1}{a_2^2}+\frac{1}{a_3^2}\right)+n_2^2\left(\frac{1}{a_1^2}+\frac{1}{a_3^2}\right)+n_3^2\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)\right]\beta\\&+\frac{n_1^2}{a_2^2a_3^2}+\frac{n_2^2}{a_1^2a_3^2}+\frac{n_3^2}{a_1^2a_2^2}=0.\end{aligned} \tag{16}$$

Furthermore it is proven in [1] that d according to (13) satisfies

$$d=\frac{\kappa^2}{a_1^2n_1^2+a_2^2n_2^2+a_3^2n_3^2}. \tag{17}$$

2. Projection of the Ellipse of Intersection onto a 2-d Plane

The curve of intersection in 3d space can be described by

$$\boldsymbol{x} = \boldsymbol{m} + (A\cos\theta)\tilde{\boldsymbol{r}} + (B\sin\theta)\tilde{\boldsymbol{s}} \tag{18}$$

with center $\boldsymbol{m} = \boldsymbol{q} + t_0\tilde{\boldsymbol{r}} + u_0\tilde{\boldsymbol{s}}$, where t_0 and u_0 are from (11), semi-axes A and B from (12), $\theta \in [0, 2\pi)$ and vectors $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ obtained after a suitable rotation (8) starting from initial vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ (see for instance [1]).

Without loss of generality the plane of projection of the ellipse (18) shall be the $x_1 - x_2$ plane. The angle between the plane of intersection (2) containing the ellipse (18) and the plane of projection is denoted by Ω. The same angle is to be found between the unit normal $\boldsymbol{n}$ of the plane of intersection (2) and the x_3-direction, normal to the plane of projection. Denoting the unit vector in x_3-direction by $\boldsymbol{e}_3$ the definition of the scalar product (see for instance [4]) yields

$$n_3 = (\boldsymbol{n}, \boldsymbol{e}_3) = \|\boldsymbol{n}\|\|\boldsymbol{e}_3\|\cos\Omega = \cos\Omega \tag{19}$$

where $\cos\Omega > 0$ holds for $0 \le \Omega < \dfrac{\pi}{2}$.

Let us assume that the plane of intersection (2) is not perpendicular to the plane of projection, the $x_1 - x_2$ plane. This means that $0 \le \Omega < \dfrac{\pi}{2}$ is valid and according to (19) $n_3 > 0$ holds.

The ellipse of intersection (18) projected from 3d space onto the $x_1 - x_2$ plane has the following form:

$$\begin{aligned} x_1 &= m_1 + A\cos\theta\tilde{r}_1 + B\sin\theta\tilde{s}_1 \\ x_2 &= m_2 + A\cos\theta\tilde{r}_2 + B\sin\theta\tilde{s}_2. \end{aligned} \tag{20}$$

In general the two dimensional vectors $(\tilde{r}_1, \tilde{r}_2)^{\mathrm{T}}$ and $(\tilde{s}_1, \tilde{s}_2)^{\mathrm{T}}$ are not orthogonal because their orthogonality in 3d space implies

$$\tilde{r}_1\tilde{s}_1 + \tilde{r}_2\tilde{s}_2 = -\tilde{r}_3\tilde{s}_3,$$

which need not be zero. In order to calculate the lenghts of the semi-axes A and B projected from 3d space onto the $x_1 - x_2$ plane the following linear system deduced from (20) with the abbreviations $x_1' = x_1 - m_1$ and $x_2' = x_2 - m_2$ is treated:

$$\begin{pmatrix} A\tilde{r}_1 & B\tilde{s}_1 \\ A\tilde{r}_2 & B\tilde{s}_2 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \tag{21}$$

The determinant of the linear system (21), $AB(\tilde{r}_1\tilde{s}_2 - \tilde{r}_2\tilde{s}_1)$, is different from zero. This can be shown by noting that $\tilde{r}_1\tilde{s}_2 - \tilde{r}_2\tilde{s}_1$ is the third component of the vector $\tilde{\boldsymbol{r}} \times \tilde{\boldsymbol{s}}$. At first this vector is not affected by rotation (8):

$$\begin{aligned} \tilde{\boldsymbol{r}} \times \tilde{\boldsymbol{s}} &= (\cos\omega\boldsymbol{r} + \sin\omega\boldsymbol{s}) \times (-\sin\omega\boldsymbol{r} + \cos\omega\boldsymbol{s}) \\ &= (\cos^2\omega + \sin^2\omega)(\boldsymbol{r} \times \boldsymbol{s}) = \boldsymbol{r} \times \boldsymbol{s}. \end{aligned}$$

This result was obtained by applying the rules for the cross product in $\mathbf{R}^3$. Furthermore one obtains employing the Grassman expansion theorem (see for instance [4]):

$$\boldsymbol{r} \times \boldsymbol{s} = \boldsymbol{r} \times (\boldsymbol{n} \times \boldsymbol{r}) = (\boldsymbol{r}, \boldsymbol{r})\boldsymbol{n} - (\boldsymbol{r}, \boldsymbol{n})\boldsymbol{r} = \boldsymbol{n}$$

because of $(\boldsymbol{r},\boldsymbol{r})=1$ and $(\boldsymbol{r},\boldsymbol{n})=0$. Thus one ends up with

$$\tilde{r}_1\tilde{s}_2-\tilde{r}_2\tilde{s}_1=r_1s_2-r_2s_1=n_3, \tag{22}$$

which is positive because of (19) for angles Ω with $0\le\Omega<\frac{\pi}{2}$.

Solving the linear system (21) leads to

$$\cos\theta=\frac{B\left(x_1'\tilde{s}_2-x_2'\tilde{s}_1\right)}{AB\left(\tilde{r}_1\tilde{s}_2-\tilde{r}_2\tilde{s}_1\right)},$$

$$\sin\theta=\frac{A\left(\tilde{r}_1x_2'-\tilde{r}_2x_1'\right)}{AB\left(\tilde{r}_1\tilde{s}_2-\tilde{r}_2\tilde{s}_1\right)}.$$

Since $\cos^2\theta+\sin^2\theta=1$ together with (22) the following quadratic equation in x_1' and x_2' is obtained:

$$B^2\left(x_1'\tilde{s}_2-x_2'\tilde{s}_1\right)^2+A^2\left(\tilde{r}_1x_2'-\tilde{r}_2x_1'\right)^2=A^2B^2\left(\tilde{r}_1\tilde{s}_2-\tilde{r}_2\tilde{s}_1\right)^2=A^2B^2n_3^2.$$

Expanding the squares on the left side and using the denotations

$$\begin{aligned} l_{11}&=A^2\tilde{r}_2^2+B^2\tilde{s}_2^2,\\ l_{12}&=-\left(A^2\tilde{r}_1\tilde{r}_2+B^2\tilde{s}_1\tilde{s}_2\right),\\ l_{22}&=A^2\tilde{r}_1^2+B^2\tilde{s}_1^2 \end{aligned} \tag{23}$$

arranged as a 2×2 matrix L

$$L=\begin{pmatrix} l_{11} & l_{12}\\ l_{12} & l_{22}\end{pmatrix} \tag{24}$$

leads to

$$\left(x_1',x_2'\right)L\begin{pmatrix}x_1'\\ x_2'\end{pmatrix}=A^2B^2n_3^2. \tag{25}$$

L as a real symmetric matrix can be diagonalized and thus is similar to the diagonal matrix of its eigenvalues $\lambda_1(L)$, $\lambda_2(L)$:

$$L=S^{-1}\operatorname{diag}\left(\lambda_1(L),\lambda_2(L)\right)S$$

with a nonsingular transformation matrix S, being orthogonal, *i.e.* $S^{-1}=S^{\mathrm{T}}$, the inverse of S is equal to the transpose of S. Putting

$$\left(x_1'',x_2''\right)=\left(x_1',x_2'\right)S^{\mathrm{T}},\quad S\begin{pmatrix}x_1'\\ x_2'\end{pmatrix}=\begin{pmatrix}x_1''\\ x_2''\end{pmatrix}$$

the quadratic equation (25) in $\left(x_1',x_2'\right)$ reduces to

$$\left(x_1'',x_2''\right)\operatorname{diag}\left(\lambda_1(L),\lambda_2(L)\right)\begin{pmatrix}x_1''\\ x_2''\end{pmatrix}=A^2B^2n_3^2. \tag{26}$$

The eigenvalues $\lambda_1(L)$, $\lambda_2(L)$ are positive because L is positive definite; this is true since the terms l_{11} and $l_{11}l_{22}-l_{12}^2$ are positive. For l_{11} this is clear; for the second term, the determinant of L, holds because of (22):

$$\begin{aligned}\det L = l_{11}l_{22} - l_{12}^2 &= \left(A^2\tilde{r}_2^2 + B^2\tilde{s}_2^2\right)\left(A^2\tilde{r}_1^2 + B^2\tilde{s}_1^2\right) - \left(A^2\tilde{r}_1\tilde{r}_2 + B^2\tilde{s}_1\tilde{s}_2\right)^2 \\ &= A^2B^2\left(\tilde{r}_1\tilde{s}_2 - \tilde{r}_2\tilde{s}_1\right)^2 = A^2B^2\left(r_1s_2 - r_2s_1\right)^2 = A^2B^2n_3^2.\end{aligned} \tag{27}$$

Dividing (26) by $A^2B^2n_3^2$ yields

$$\frac{\lambda_1(L)}{A^2B^2n_3^2}\left(x_1''\right)^2 + \frac{\lambda_2(L)}{A^2B^2n_3^2}\left(x_2''\right)^2 = 1.$$

This is an ellipse projected from 3d space (18) onto the $x_1 - x_2$ plane with the semi-axes

$$A_L = \frac{ABn_3}{\sqrt{\lambda_1(L)}}, \quad B_L = \frac{ABn_3}{\sqrt{\lambda_2(L)}}. \tag{28}$$

With (19) one obtains from (28)

$$A_L = \frac{AB\cos\Omega}{\sqrt{\lambda_1(L)}}, \quad B_L = \frac{AB\cos\Omega}{\sqrt{\lambda_2(L)}}. \tag{29}$$

3. Calculation of Semi-Axes According to a Method Used by Bektas

Let the ellipsoid (1) be given and a plane in the form

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4 = 0. \tag{30}$$

The unit normal vector of the plane is:

$$\boldsymbol{n} = \frac{1}{\sqrt{A_1^2 + A_2^2 + A_3^2}}\left(A_1, A_2, A_3\right). \tag{31}$$

The distance between the plane and the origin is given by

$$\kappa = -\frac{A_4}{\sqrt{A_1^2 + A_2^2 + A_3^2}}. \tag{32}$$

The plane written in Hessian normal form then reads:

$$n_1x_1 + n_2x_2 + n_3x_3 - \kappa = 0.$$

Without loss of generality $A_3 \neq 0$ shall be assumed. Then $n_3 \neq 0$ holds:

$$x_3 = \frac{1}{n_3}\left(\kappa - n_1x_1 - n_2x_2\right).$$

Forming x_3^2 and substituting into equation (1) gives:

$$m_{11}x_1^2 + 2m_{12}x_1x_2 + m_{22}x_2^2 + 2m_{13}x_1 + 2m_{23}x_2 + m_{33} = 0 \tag{33}$$

with

$$\begin{aligned} m_{11} &= \frac{1}{a_1^2} + \frac{n_1^2}{a_3^2n_3^2}, \quad m_{12} = \frac{n_1n_2}{a_3^2n_3^2}, \\ m_{22} &= \frac{1}{a_2^2} + \frac{n_2^2}{a_3^2n_3^2}, \quad m_{13} = -\frac{n_1\kappa}{a_3^2n_3^2}, \\ m_{23} &= -\frac{n_2\kappa}{a_3^2n_3^2}, \quad m_{33} = \frac{\kappa^2}{a_3^2n_3^2} - 1. \end{aligned} \tag{34}$$

In the sequel the determinant of the following matrix will be needed:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$$

$$\begin{aligned} \det M = m_{11}m_{22} - m_{12}^2 &= \left(\frac{1}{a_1^2} + \frac{n_1^2}{a_3^2 n_3^2}\right)\left(\frac{1}{a_2^2} + \frac{n_2^2}{a_3^2 n_3^2}\right) - \frac{n_1^2 n_2^2}{a_3^4 n_3^4} \\ &= \frac{n_3^2}{a_1^2 a_2^2 n_3^2} + \frac{n_1^2}{a_2^2 a_3^2 n_3^2} + \frac{n_2^2}{a_1^2 a_3^2 n_3^2} = \frac{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2}{a_1^2 a_2^2 a_3^2 n_3^2}. \end{aligned} \tag{35}$$

In order to get rid of the linear terms x_1 and x_2 in (33) the following translation can be performed: $x_1 = x_1' + h$, $x_2 = x_2' + k$ with parameters h and k to be determined later. After substitution into (33) one obtains:

$$\begin{aligned} & m_{11}x_1'^2 + 2m_{12}x_1'x_2' + m_{22}x_2'^2 + 2\left(m_{11}h + m_{12}k + m_{13}\right)x_1' \\ & + 2\left(m_{12}h + m_{22}k + m_{23}\right)x_2' + m_{11}h^2 + 2m_{12}hk + m_{22}k^2 \\ & + 2m_{13}h + 2m_{23}k + m_{33} = 0. \end{aligned} \tag{36}$$

The terms x_1' and x_2' in (36) vanish if h and k are determined by the linear system:

$$\begin{aligned} m_{11}h + m_{12}k &= -m_{13}, \\ m_{12}h + m_{22}k &= -m_{23}. \end{aligned} \tag{37}$$

The linear system (37) has M as matrix of coefficients, the determinant of which is given in (35). It is nonzero because of the assumption $n_3 \neq 0$. Solving the linear system (37) yields:

$$\begin{aligned} h &= \frac{-m_{13}m_{22} + m_{23}m_{12}}{m_{11}m_{22} - m_{12}^2}, \\ k &= \frac{-m_{11}m_{23} + m_{12}m_{13}}{m_{11}m_{22} - m_{12}^2}. \end{aligned} \tag{38}$$

Substituting the terms (34) into (38) gives the result:

$$\begin{aligned} h &= \frac{a_1^2 n_1 \kappa}{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2}, \\ k &= \frac{a_2^2 n_2 \kappa}{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2}. \end{aligned} \tag{39}$$

With the terms h and k from (39) the constant term in (36) turns out to be, together with (17):

$$\begin{aligned} & m_{11}h^2 + 2m_{12}hk + m_{22}k^2 + 2m_{13}h + 2m_{23}k + m_{33} \\ & = \frac{\kappa^2}{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2} - 1 = -\left(1 - d\right). \end{aligned}$$

Thus the quadratic equation (36) reduces to:

$$\left(x_1', x_2'\right) M \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = 1 - d. \tag{40}$$

M as a real symmetric matrix can be diagonalized and thus is similar to the diagonal matrix of its eigenvalues $\lambda_1(M)$, $\lambda_2(M)$:

$$M = T^{-1}\mathrm{diag}\left(\lambda_1(M), \lambda_2(M)\right)T$$

with a nonsingular transformation matrix T, being orthogonal, *i.e.* $T^{-1} = T^{\mathrm{T}}$, the inverse of T is equal to the transpose of T. Putting

$$\left(x_1'', x_2''\right) = \left(x_1', x_2'\right)T^{\mathrm{T}}, \quad T\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}$$

the quadratic equation (40) in $\left(x_1', x_2'\right)$ reduces to

$$\left(x_1'', x_2''\right)\mathrm{diag}\left(\lambda_1(M), \lambda_2(M)\right)\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = 1 - d. \tag{41}$$

The eigenvalues $\lambda_1(M)$, $\lambda_2(M)$ are positive because M is positive definite; this is true since the terms m_{11} and $m_{11}m_{22} - m_{12}^2$ are positive. For m_{11} this is clear; the second term, the determinant of M, is given in (35). If a point of the plane (30) exists which is an interior point of the ellipsoid (1), then $1-d$ is positive (see Section 1). Dividing (41) by $1-d$ yields

$$\frac{\lambda_1(M)}{1-d}\left(x_1''\right)^2 + \frac{\lambda_2(M)}{1-d}\left(x_2''\right)^2 = 1.$$

This is an ellipse in the $x_1 - x_2$ plane with the semi-axes

$$A_M = \sqrt{\frac{1-d}{\lambda_1(M)}}, \quad B_M = \sqrt{\frac{1-d}{\lambda_2(M)}}. \tag{42}$$

4. Calculation of Projected Semi-Axes According to Schrantz

In [3] the ellipse

$$x_1 = A\cos t, \quad x_2 = B\sin t, \quad t \in [0, 2\pi) \tag{43}$$

with the semi-axes A and B is projected from plane E onto plane E'. As in Section 2 the angle between the two planes is denoted by Ω, with $0 \le \Omega \le \frac{\pi}{2}$.

Let α, with $0 \le \alpha \le \frac{\pi}{2}$, be the angle between the major axis of the original ellipse (43) and the straight line of intersection of the two planes E and E' $\left(E \cap E'\right)$ and let ψ be a phase-shift with $0 \le \psi \le \frac{\pi}{2}$ and $\psi = \tau - \sigma$ where the angles τ and σ are determined by

$$\begin{aligned}
\cos\sigma &= \frac{A\cos\alpha}{\sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha}}, \\
\sin\sigma &= \frac{B\sin\alpha}{\sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha}}, \\
\cos\tau &= \frac{B\cos\alpha}{\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha}}, \\
\sin\tau &= \frac{A\sin\alpha}{\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha}}.
\end{aligned} \tag{44}$$

The projected ellipse in the plane E' is given by

$$\bar{x}_1 = \bar{A}\cos(\bar{t}+\psi),\ \bar{x}_2 = \bar{B}\sin\bar{t},\ \bar{t}\in[0,2\pi) \tag{45}$$

with

$$\bar{A} = \sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha},$$
$$\bar{B} = \cos\Omega\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha}. \tag{46}$$

Eliminating parameter $\bar{t}$ from (45) yields a quadratic equation in $\bar{x}_1$ and $\bar{x}_2$

$$\left(\frac{\bar{x}_1}{\bar{A}}\right)^2 + 2\sin\psi\left(\frac{\bar{x}_1}{\bar{A}}\right)\left(\frac{\bar{x}_2}{\bar{B}}\right) + \left(\frac{\bar{x}_2}{\bar{B}}\right)^2 = \cos^2\psi$$

or written with the elements

$$g_{11} = \frac{1}{\bar{A}^2},\quad g_{12} = \frac{\sin\psi}{\bar{A}\bar{B}},\quad g_{22} = \frac{1}{\bar{B}^2} \tag{47}$$

forming matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$$

one obtains

$$(\bar{x}_1,\bar{x}_2)\,G\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix} = \cos^2\psi. \tag{48}$$

G as a real symmetric matrix can be diagonalized and thus is similar to the diagonal matrix of its eigenvalues $\lambda_1(G)$, $\lambda_2(G)$:

$$G = R^{-1}\operatorname{diag}\left(\lambda_1(G),\lambda_2(G)\right)R$$

with a nonsingular transformation matrix R, being orthogonal, *i.e.* $R^{-1} = R^{\mathrm{T}}$, the inverse of R is equal to the transpose of R. Putting

$$\left(\bar{\bar{x}}_1,\bar{\bar{x}}_2\right) = (\bar{x}_1,\bar{x}_2)R^{\mathrm{T}},\quad R\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix} = \begin{pmatrix}\bar{\bar{x}}_1\\ \bar{\bar{x}}_2\end{pmatrix}$$

the quadratic equation (48) in $(\bar{x}_1,\bar{x}_2)$ reduces to

$$\left(\bar{\bar{x}}_1,\bar{\bar{x}}_2\right)\operatorname{diag}\left(\lambda_1(G),\lambda_2(G)\right)\begin{pmatrix}\bar{\bar{x}}_1\\ \bar{\bar{x}}_2\end{pmatrix} = \cos^2\psi. \tag{49}$$

The eigenvalues $\lambda_1(G)$, $\lambda_2(G)$ are positive, if G is positive definite; this is the case if the terms g_{11} and $g_{11}g_{22} - g_{12}^2$ are positive. For g_{11} this is true; the second term, the determinant of G, given by

$$\det G = g_{11}g_{22} - g_{12}^2 = \frac{1}{\bar{A}^2\bar{B}^2} - \frac{\sin^2\psi}{\bar{A}^2\bar{B}^2} = \frac{\cos^2\psi}{\bar{A}^2\bar{B}^2} \tag{50}$$

is positive for $0 \le \psi < \frac{\pi}{2}$. Dividing (49) by $\cos^2\psi$ for $0 \le \psi < \frac{\pi}{2}$ yields

$$\frac{\lambda_1(G)}{\cos^2\psi}\left(\bar{\bar{x}}_1\right)^2+\frac{\lambda_2(G)}{\cos^2\psi}\left(\bar{\bar{x}}_2\right)^2=1.$$

This is an ellipse in the $\bar{x}_1-\bar{x}_2$ plane with the semi-axes

$$A_G=\frac{\cos\psi}{\sqrt{\lambda_1(G)}},\quad B_G=\frac{\cos\psi}{\sqrt{\lambda_2(G)}}. \tag{51}$$

5. Some Auxiliary Means

Let H stand for the following 2×2 matrix:

$$H=\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \tag{52}$$

and be a place holder for the matrices M and G used above. The semi-axes A_L, B_L projected onto the x_1-x_2 plane, given in (28), are compared with the semi-axes A_H, B_H. It will be shown that the two polynomials

$$\begin{aligned} Q_L(z)&=z^2-(A_L+B_L)z+A_LB_L, \\ Q_H(z)&=z^2-(A_H+B_H)z+A_HB_H, \end{aligned} \tag{53}$$

have the same coefficients and thus have the same zeros:

$$\begin{aligned} Q_L(z)&=(z-A_L)(z-B_L), \\ Q_H(z)&=(z-A_H)(z-B_H). \end{aligned} \tag{54}$$

In the first step $A_LB_L=A_HB_H$ will be proven. In the second step

$$A_L^2+B_L^2=A_H^2+B_H^2 \tag{55}$$

will be shown. This is sufficient, since by adding $2A_LB_L=2A_HB_H$ to both sides of (55) one obtains:

$$(A_L+B_L)^2=A_L^2+2A_LB_L+B_L^2=A_H^2+2A_HB_H+B_H^2=(A_H+B_H)^2$$

which yields $A_L+B_L=A_H+B_H$ since the semi-axes are positive.
$\lambda_1(L)$, $\lambda_2(L)$ are the zeros of the characteristic polynomial of L. This can be expressed in two ways:

$$P_L(\lambda)=(l_{11}-\lambda)(l_{22}-\lambda)-l_{12}^2=\lambda^2-(l_{11}+l_{22})\lambda+l_{11}l_{22}-l_{12}^2,$$

$$P_L(\lambda)=(\lambda-\lambda_1(L))(\lambda-\lambda_2(L))=\lambda^2-(\lambda_1(L)+\lambda_2(L))\lambda+\lambda_1(L)\lambda_2(L).$$

Comparing the coefficients one obtains

$$\begin{aligned} \lambda_1(L)+\lambda_2(L)&=l_{11}+l_{22}, \\ \lambda_1(L)\lambda_2(L)&=l_{11}l_{22}-l_{12}^2. \end{aligned} \tag{56}$$

Similarly the results for matrix H instead of L are

$$\begin{aligned} \lambda_1(H)+\lambda_2(H)&=h_{11}+h_{22}, \\ \lambda_1(H)\lambda_2(H)&=h_{11}h_{22}-h_{12}^2. \end{aligned} \tag{57}$$

6. Comparison of the Semi-Axes A_L, B_L with A_M, B_M

In the first step $A_L B_L = A_M B_M$ will be proven. According to (28) and (42) holds:

$$A_L B_L = \frac{A^2 B^2 n_3^2}{\sqrt{\lambda_1(L)\lambda_2(L)}}, \tag{58}$$

$$A_M B_M = \frac{1-d}{\sqrt{\lambda_1(M)\lambda_2(M)}}. \tag{59}$$

In the case of matrix L combining (56) and (27) yields:

$$\lambda_1(L)\lambda_2(L) = l_{11}l_{22} - l_{12}^2 = A^2 B^2 n_3^2. \tag{60}$$

In the case of matrix M combining (57), where M is substituted for H, and (35) leads to:

$$\lambda_1(M)\lambda_2(M) = m_{11}m_{22} - m_{12}^2 = \frac{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2}{a_1^2 a_2^2 a_3^2 n_3^2}. \tag{61}$$

Because β_1 and β_2 are solutions of (16)

$$\beta_1\beta_2 = \frac{n_1^2}{a_2^2 a_3^2} + \frac{n_2^2}{a_1^2 a_3^2} + \frac{n_3^2}{a_1^2 a_2^2} = \frac{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2}{a_1^2 a_2^2 a_3^2} \tag{62}$$

holds and because of (60), (15), (62) and (61)

$$\lambda_1(L)\lambda_2(L) = \frac{1-d}{\beta_1}\frac{1-d}{\beta_2} n_3^2 = \frac{(1-d)^2 a_1^2 a_2^2 a_3^2 n_3^2}{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2} = \frac{(1-d)^2}{\lambda_1(M)\lambda_2(M)}. \tag{63}$$

Thus with (58), (60), (63) and (59) one concludes

$$\begin{aligned} A_L B_L &= \frac{A^2 B^2 n_3^2}{\sqrt{\lambda_1(L)\lambda_2(L)}} = \frac{\lambda_1(L)\lambda_2(L)}{\sqrt{\lambda_1(L)\lambda_2(L)}} = \sqrt{\lambda_1(L)\lambda_2(L)} \\ &= \frac{1-d}{\sqrt{\lambda_1(M)\lambda_2(M)}} = A_M B_M. \end{aligned}$$

In the second step because of (28) and (60) holds

$$\begin{aligned} A_L^2 + B_L^2 &= A^2 B^2 n_3^2 \left(\frac{1}{\lambda_1(L)} + \frac{1}{\lambda_2(L)} \right) \\ &= \frac{A^2 B^2 n_3^2}{\lambda_1(L)\lambda_2(L)} \left(\lambda_2(L) + \lambda_1(L) \right) = \lambda_1(L) + \lambda_2(L). \end{aligned} \tag{64}$$

Because of (42), (61) and (62) holds

$$\begin{aligned} A_M^2 + B_M^2 &= \frac{1-d}{\lambda_1(M)} + \frac{1-d}{\lambda_2(M)} = \frac{1-d}{\lambda_1(M)\lambda_2(M)} \left(\lambda_2(M) + \lambda_1(M) \right) \\ &= \frac{(1-d) a_1^2 a_2^2 a_3^2 n_3^2}{a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2} \left(\lambda_1(M) + \lambda_2(M) \right) \\ &= \frac{(1-d) n_3^2}{\beta_1\beta_2} \left(\lambda_1(M) + \lambda_2(M) \right). \end{aligned} \tag{65}$$

Together with

$$\lambda_1(M)+\lambda_2(M)=m_{11}+m_{22}=\frac{1}{n_3^2}\left(\frac{n_3^2}{a_1^2}+\frac{n_3^2}{a_2^2}+\frac{n_1^2+n_2^2}{a_3^2}\right) \tag{66}$$

(65) yields

$$A_M^2+B_M^2=\frac{(1-d)}{\beta_1\beta_2}\left(\frac{n_3^2}{a_1^2}+\frac{n_3^2}{a_2^2}+\frac{n_1^2+n_2^2}{a_3^2}\right). \tag{67}$$

In continuation of (64), because $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ are fulfilling (4) and (5), the following relations hold:

$$\begin{aligned}\lambda_1(L)+\lambda_2(L)&=l_{11}+l_{22}=A^2\left(\tilde{r}_1^2+\tilde{r}_2^2\right)+B^2\left(\tilde{s}_1^2+\tilde{s}_2^2\right)\\&=A^2\left(1-\tilde{r}_3^2\right)+B^2\left(1-\tilde{s}_3^2\right)=\frac{1-d}{\beta_1}\left(1-\tilde{r}_3^2\right)+\frac{1-d}{\beta_2}\left(1-\tilde{s}_3^2\right)\\&=\frac{1-d}{\beta_1\beta_2}\left(\beta_2\left(1-\tilde{r}_3^2\right)+\beta_1\left(1-\tilde{s}_3^2\right)\right)=\frac{1-d}{\beta_1\beta_2}\left(\beta_1+\beta_2-\beta_2\tilde{r}_3^2-\beta_1\tilde{s}_3^2\right)\end{aligned} \tag{68}$$

with

$$\beta_1+\beta_2=n_1^2\left(\frac{1}{a_2^2}+\frac{1}{a_3^2}\right)+n_2^2\left(\frac{1}{a_1^2}+\frac{1}{a_3^2}\right)+n_3^2\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right) \tag{69}$$

because β_1 and β_2 are solutions of (16). Combining (64), (68), (69) and (67) one obtains:

$$A_L^2+B_L^2-\left(A_M^2+B_M^2\right)=\frac{1-d}{\beta_1\beta_2}\left(\frac{n_1^2}{a_2^2}+\frac{n_2^2}{a_1^2}-\beta_2\tilde{r}_3^2-\beta_1\tilde{s}_3^2\right). \tag{70}$$

To simplify the term in round brackets of (70) the following relations are used:

$$n_1=\tilde{r}_2\tilde{s}_3-\tilde{r}_3\tilde{s}_2,\quad n_2=\tilde{r}_3\tilde{s}_1-\tilde{r}_1\tilde{s}_3,$$

because of $\tilde{\boldsymbol{r}}\times\tilde{\boldsymbol{s}}=\boldsymbol{r}\times\boldsymbol{s}=\boldsymbol{n}$ (see Section 2), and

$$\beta_2=\left(D_1\tilde{\boldsymbol{s}},D_1\tilde{\boldsymbol{s}}\right),\quad \beta_1=\left(D_1\tilde{\boldsymbol{r}},D_1\tilde{\boldsymbol{r}}\right)$$

according to (14). The term in round brackets of (70) thus becomes:

$$\begin{aligned}&\frac{1}{a_2^2}\left(\tilde{r}_2\tilde{s}_3-\tilde{r}_3\tilde{s}_2\right)^2+\frac{1}{a_1^2}\left(\tilde{r}_3\tilde{s}_1-\tilde{r}_1\tilde{s}_3\right)^2-\left(\frac{\tilde{s}_1^2}{a_1^2}+\frac{\tilde{s}_2^2}{a_2^2}+\frac{\tilde{s}_3^2}{a_3^2}\right)\tilde{r}_3^2-\left(\frac{\tilde{r}_1^2}{a_1^2}+\frac{\tilde{r}_2^2}{a_2^2}+\frac{\tilde{r}_3^2}{a_3^2}\right)\tilde{s}_3^2\\&=-2\tilde{r}_3\tilde{s}_3\left(\frac{\tilde{r}_1\tilde{s}_1}{a_1^2}+\frac{\tilde{r}_2\tilde{s}_2}{a_2^2}+\frac{\tilde{r}_3\tilde{s}_3}{a_3^2}\right)=-2\tilde{r}_3\tilde{s}_3\left(D_1\tilde{\boldsymbol{r}},D_1\tilde{\boldsymbol{s}}\right)=0,\end{aligned}$$

because $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ have been chosen in such a way that condition (7) is fulfilled.

7. Comparison of the Semi-Axes A_L, B_L with A_G, B_G

In the first step $A_LB_L=A_GB_G$ will be proven. According to (29) and (51) holds:

$$A_L B_L = \frac{A^2 B^2 \cos^2 \Omega}{\sqrt{\lambda_1(L)\lambda_2(L)}}, \tag{71}$$

$$A_G B_G = \frac{\cos^2 \psi}{\sqrt{\lambda_1(G)\lambda_2(G)}}. \tag{72}$$

In the case of matrix L combining (56), (27) and (19) yields:

$$\lambda_1(L)\lambda_2(L) = l_{11}l_{22} - l_{12}^2 = A^2 B^2 \cos^2 \Omega. \tag{73}$$

In the case of matrix G combining (57), where G is substituted for H, and (50) leads to:

$$\lambda_1(G)\lambda_2(G) = g_{11}g_{22} - g_{12}^2 = \frac{\cos^2 \psi}{\bar{A}^2 \bar{B}^2}. \tag{74}$$

Substitution of (73) into (71) and (74) into (72) yield

$$A_L B_L - A_G B_G = AB\cos\Omega - \bar{A}\bar{B}\cos\psi. \tag{75}$$

According to the definition of $\psi = \tau - \sigma$ given in the beginning of Section 4 together with (44) and (46) one obtains:

$$\cos\psi = \cos(\tau - \sigma) = \frac{AB\cos\Omega}{\bar{A}\bar{B}}.$$

Substituting this into (75) one ends up with $A_L B_L - A_G B_G = 0$.
In the second step because of (64), (56) and (23) holds

$$\begin{aligned} A_L^2 + B_L^2 &= \lambda_1(L) + \lambda_2(L) = l_{11} + l_{22} = A^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + B^2\left(\tilde{s}_1^2 + \tilde{s}_2^2\right) \\ &= A^2\left(1 - \tilde{r}_3^2\right) + B^2\left(1 - \tilde{s}_3^2\right) = A^2 + B^2 - \left(A^2\tilde{r}_3^2 + B^2\tilde{s}_3^2\right). \end{aligned} \tag{76}$$

Because of (51), (74), (57), where matrix G is substituted for matrix H, and (47) holds

$$\begin{aligned} A_G^2 + B_G^2 &= \frac{\cos^2\psi}{\lambda_1(G)} + \frac{\cos^2\psi}{\lambda_2(G)} = \frac{\cos^2\psi}{\lambda_1(G)\lambda_2(G)}\left(\lambda_2(G) + \lambda_1(G)\right) \\ &= \bar{A}^2\bar{B}^2\left(\lambda_1(G) + \lambda_2(G)\right) = \bar{A}^2\bar{B}^2\left(g_{11} + g_{22}\right) \\ &= \bar{A}^2\bar{B}^2\left(\frac{1}{\bar{A}^2} + \frac{1}{\bar{B}^2}\right) = \bar{B}^2 + \bar{A}^2; \end{aligned} \tag{77}$$

(77) is continued by substituting $\bar{B}$ and $\bar{A}$ from (46)

$$\begin{aligned} &\cos^2\Omega\left(A^2\sin^2\alpha + B^2\cos^2\alpha\right) + A^2\cos^2\alpha + B^2\sin^2\alpha \\ &= A^2\left(\cos^2\alpha + \cos^2\Omega\sin^2\alpha\right) + B^2\left(\sin^2\alpha + \cos^2\Omega\cos^2\alpha\right) \\ &= A^2\left(\cos^2\alpha + \left(1 - \sin^2\Omega\right)\sin^2\alpha\right) + B^2\left(\sin^2\alpha + \left(1 - \sin^2\Omega\right)\cos^2\alpha\right) \\ &= A^2\left(\cos^2\alpha + \sin^2\alpha - \sin^2\Omega\sin^2\alpha\right) + B^2\left(\sin^2\alpha + \cos^2\alpha - \sin^2\Omega\cos^2\alpha\right) \\ &= A^2\left(1 - \sin^2\Omega\sin^2\alpha\right) + B^2\left(1 - \sin^2\Omega\cos^2\alpha\right) \\ &= A^2 + B^2 - \sin^2\Omega\left(A^2\sin^2\alpha + B^2\cos^2\alpha\right) \end{aligned} \tag{78}$$

Comparing (76) and (78), in order to show equality $A_L^2 + B_L^2 = A_G^2 + B_G^2$, it has to be proven:

$$A^2 \tilde{r}_3^2 + B^2 \tilde{s}_3^2 = \sin^2 \Omega \left(A^2 \sin^2 \alpha + B^2 \cos^2 \alpha \right). \tag{79}$$

As already described in the beginning of Section 4 the ellipse (43) is projected from the original plane E onto the plane E'. Both planes are forming an angle Ω with $0 \le \Omega \le \frac{\pi}{2}$. Without loss of generality the intersection of E and E', $E \cap E'$, shall be the $\bar{x}_1$-axis of the coordinate system in plane E'. The original plane E thus contains the following three points: $(-1,0,0)$, $(1,0,0)$, $(0,\cos\Omega,\sin\Omega)$ and can therefore be described by the following equation:

$$-\sin\Omega \bar{x}_2 + \cos\Omega \bar{x}_3 = 0. \tag{80}$$

The unit normal vector $\boldsymbol{n}$ of plane (80) given by (31) is

$$\boldsymbol{n} = (0, -\sin\Omega, \cos\Omega). \tag{81}$$

In order to describe a unit vector $\boldsymbol{r}$ in the plane E the equations (4) must hold:

$$\begin{aligned} (\boldsymbol{r},\boldsymbol{r}) &= r_1^2 + r_2^2 + r_3^2 = 1, \\ (\boldsymbol{n},\boldsymbol{r}) &= -\sin\Omega r_2 + \cos\Omega r_3 = 0. \end{aligned} \tag{82}$$

The second equation of (82) yields $r_3 = r_2 \tan\Omega$. Substituting this into the first equation of (82) results in:

$$r_1^2 + r_2^2 \left(1 + \tan^2 \Omega\right) = 1$$

or

$$r_1^2 + \frac{r_2^2}{\cos^2 \Omega} = 1. \tag{83}$$

If the unit vector $\boldsymbol{r}$ is forming the angle α with the $\bar{x}_1$-axis and $\boldsymbol{e}_1$ is designating a unit vector in $\bar{x}_1$-direction according to the definition of the scalar product (see for instance [4]) holds

$$r_1 = (\boldsymbol{r}, \boldsymbol{e}_1) = \|\boldsymbol{r}\| \|\boldsymbol{e}_1\| \cos\alpha = \cos\alpha.$$

From (83) one obtains

$$r_2^2 = \left(1 - \cos^2\alpha\right)\cos^2\Omega = \sin^2\alpha \cos^2\Omega,$$

yielding $r_2 = \pm\sin\alpha\cos\Omega$ and furthermore with the first equation of (82) $r_3 = \pm\sin\alpha\sin\Omega$. From

$$\boldsymbol{r} = \left(\cos\alpha, \pm\sin\alpha\cos\Omega, \pm\sin\alpha\sin\Omega\right)$$

and $\boldsymbol{s} = \boldsymbol{n} \times \boldsymbol{r}$ one obtains

$$\boldsymbol{s} = \left(\mp\sin\alpha, \cos\alpha\cos\Omega, \cos\alpha\sin\Omega\right).$$

By transformation (8) one obtains

$$\tilde{r}_3 = \cos\omega r_3 + \sin\omega s_3 = \sin(\omega \pm \alpha)\sin\Omega,$$
$$\tilde{s}_3 = -\sin\omega r_3 + \cos\omega s_3 = \cos(\omega \pm \alpha)\sin\Omega.$$

Thus equation (79) turns into

$$\begin{aligned} &\left(A^2\sin^2(\omega\pm\alpha)+B^2\cos^2(\omega\pm\alpha)\right)\sin^2\Omega \\ &=\sin^2\Omega\left(A^2\sin^2\alpha+B^2\cos^2\alpha\right). \end{aligned} \tag{84}$$

Equation (84) is fulfilled if $\omega \pm \alpha = \alpha$ holds. The $+$-case leads to $\omega = 0$, which means that (84) is fulfilled if transformation (8) is the identity, *i.e.* $\tilde{\boldsymbol{r}} = \boldsymbol{r}$, $\tilde{\boldsymbol{s}} = \boldsymbol{s}$; the $-$-case leads to $\omega = 2\alpha$, meaning that if α, the angle between the major axis of the ellipse (43) and the $\bar{x}_1$-axis, is chosen to be $\frac{\omega}{2}$ then (84) is true.

8. Numerical Example

The following numerical example is taken from [2]. Let the semi-axes of the ellipsoid (1) be

$$a_1 = 5, \quad a_2 = 4, \quad a_3 = 3$$

and let the plane be given by

$$x_1 + 2x_2 + 3x_3 + 4 = 0.$$

The following calculations have been performed with Mathematica. According to (31) the unit normal vector $\boldsymbol{n}$ of the plane is

$$\boldsymbol{n} = \frac{1}{\sqrt{1^2+2^2+3^2}}(1,2,3).$$

Furthermore in (32) the distance κ of the plane to the origin is given

$$\kappa = -\frac{4}{\sqrt{1^2+2^2+3^2}}.$$

According to (17) d can be calculated.
Starting with an arbitrary unit vector $\boldsymbol{r}$ orthogonal to the unit normal vector $\boldsymbol{n}$, for instance

$$\boldsymbol{r} = \frac{1}{\sqrt{1^2+2^2}}(2,-1,0)^{\mathrm{T}},$$

calculating $\boldsymbol{s}$ to be orthogonal to both according to $\boldsymbol{s} = \boldsymbol{n} \times \boldsymbol{r}$ and, as $(D_1\boldsymbol{r}, D_1\boldsymbol{s}) \neq 0$, perform a rotation with angle ω given in (9), yielding new vectors $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$ according to (8), which are plugged into $(D_1\tilde{\boldsymbol{r}}, D_1\tilde{\boldsymbol{r}})$ and $(D_1\tilde{\boldsymbol{s}}, D_1\tilde{\boldsymbol{s}})$.
The semi-axes A and B in 3d space according to (12) can be calculated to be

$$A = 4.59157, \quad B = 3.39705.$$

Furthermore having calculated the eigenvalues $\lambda_1(L)$ and $\lambda_2(L)$ the semi-axes A_L and B_L projected onto the $x_1 - x_2$ plane according to (28) are

$$A_L = 4.56667, \quad B_L = 2.73855.$$

The same results are obtained calculating A_M and B_M according to (42) by the method used by Bektas.

9. Conclusion

The intention of this paper was, to show that the semi-axes of the ellipse of intersection projected from 3d space onto a 2d plane are the same as those calculated by a method used by Bektas. Furthermore they are also equal to the semi-axes of the projected ellipse obtained by Schrantz.

References

[1] Klein, P.P. (2012) On the Ellipsoid and Plane Intersection Equation. *Applied Mathematics*, **3**, 1634-1640. https://doi.org/10.4236/am.2012.311226

[2] Bektas, S. (2016) On the Intersection of an Ellipsoid and a Plane. *International Journal of Research in Engineering and Applied Sciences*, **6**, 273-283.

[3] Schrantz, G.R. (2004) Projections of Ellipses and Circles. Hamline University. (To be found in the internet)

[4] Bronshtein, I.N., Semendyayev, K.A., *et al.* (2007) Handbook of Mathematics. 5th Edition, Springer-Verlag Berlin Heidelberg.