Exact Solutions to the Boussinesq-Burgers Equations

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Abstract

A nonlinear transformation from the solution of a linear equation to the solution of the Boussinesq-Burgers equations is derived by using the simplified homogeneous balance method. Based on the nonlinear transformation and various given solutions of the linear equation, various exact solutions, including solitary wave solutions, rational solutions, the solutions containing hyperbolic functions and the solutions containing trigonometric functions, of the Boussinesq-Burgers equations are obtained.

Keywords

1. Introduction

In the present paper, we will investigate the well-known Boussinesq-Burgers equations in the form

$$u_t + 2uu_x - \alpha v_x = 0, \tag{1}$$

$$v_t + 2(uv)_x - \alpha u_{xxx} = 0, \tag{2}$$

where α = constant. Equations ((1) & (2)) emerge in the investigation of fluid flow, and describe the proliferation of shallow water waves. $u = u(x,t)$ represents horizontal velocity; $v = v(x,t)$ represents the height of the water surface above horizontal at bottom. Equations ((1) & (2)) have been investigated by many authors with different methods (see [1]-[6] and references therein).

In this paper, we will propose a somewhat different method to find various exact solutions of Equations ((1) & (2)). First of all, a nonlinear transformation from the solution of a linear equation to the solution of Equations ((1) & (2)) is

derived by using the simplified homogeneous balance method [7] [8] [9]; then based on the nonlinear transformation, various exact solutions of Equations ((1) & (2)) can be obtained by the given various solutions of the linear equation.

2. Derivation of Nonlinear Transformation

Considering the homogeneous balance between uu_x and v_x in Equation (1), and between $(uv)_x$ and u_{xxx} in Equation (2):

$$2m+1=n+1, m+n+1=m+3 \Rightarrow m=1, n=2,$$

according to the simplified homogeneous balance method [7] [8] [9], we can suppose that the solution of Equations ((1) & (2)) is of the form

$$u(x,t) = A(\ln\phi)_x, \tag{3}$$

$$v(x,t) = B(\ln\phi)_{xx}, \tag{4}$$

where the constants A and B, as well as the function $\phi = \phi(x,t)$ are to be determined later. Substituting (3), (4) into the left hand sides of Equations ((1) & (2)), yields

$$u_t + 2uu_x - \alpha v_x = A\frac{\partial}{\partial x}\left[\frac{\phi_t - \alpha\frac{B}{A}\phi_{xx}}{\phi} + \left(A + \alpha\frac{B}{A}\right)\left(\frac{\phi_x}{\phi}\right)^2\right], \tag{5}$$

$$v_t + 2(uv)_x - \alpha u_{xxx} = B\frac{\partial^2}{\partial_x^2}\left[\frac{\phi_t - \alpha\frac{A}{B}\phi_{xx}}{\phi} + \left(A + \alpha\frac{A}{B}\right)\left(\frac{\phi_x}{\phi}\right)^2\right]. \tag{6}$$

In expressions (5), (6), setting the coefficients of $\left(\dfrac{\phi_x}{\phi}\right)^2$ to zero, yields

$$A + \alpha\frac{B}{A} = 0, A + \alpha\frac{A}{B} = 0 \Rightarrow A = \pm\alpha, B = -\alpha. \tag{7}$$

Using the results (7), the expressions (3), (4) becomes

$$u(x,t) = \pm\alpha(\ln\phi)_x, \tag{8}$$

$$v(x,t) = -\alpha(\ln\phi)_{xx}, \tag{9}$$

and the expressions (5), (6) can be simplified as

$$u_t + 2uu_x - \alpha v_x = \pm\alpha\frac{\partial}{\partial x}\left[\frac{\phi_t \pm \alpha\phi_{xx}}{\phi}\right] = 0, \tag{10}$$

$$v_t + 2(uv)_x - \alpha u_{xxx} = -\alpha\frac{\partial^2}{\partial_x^2}\left[\frac{\phi_t \pm \alpha\phi_{xx}}{\phi}\right] = 0, \tag{11}$$

provided that the function $\phi = \phi(x,t)$ satisfies the linear equation

$$\phi_t \pm \alpha\phi_{xx} = 0. \tag{12}$$

From (8)-(12), we come to the conclusion that if $\phi = \phi(x,t)$ is a solution of

the Equation (12), substituting it into the expressions (8), (9), we have the solution of Equations ((1) & (2)). Thus the expressions (8), (9) and Equation (12) together comprise a nonlinear transformation that from the solution of Equation (12) to the solution of Equations ((1) & (2)). According to the nonlinear transformation composed of expressions (8), (9) and Equation (12), in order to obtain the solution of Equations ((1) & (2)), we have to give the solution of Equation (12). Many solutions of Equation (12) are easily given by

$$\phi_1 = 1 + e^{kx \mp \alpha k^2 t},$$

$$\phi_2 = 1 \mp \alpha xt + \frac{1}{6}x^3,$$

$$\phi_3 = 1 + e^{\mp \alpha t}\left(A\cosh x + B\sinh x\right),$$

$$\phi_4 = 1 + e^{\pm \alpha t}\left(A\cos x + B\sin x\right),$$

A and *B* are constants, $\cdots$, and so on.

3. Various Exact Solutions of Equations ((1) & (2))

Substituting the solution $\phi_1 = 1 + e^{kx \mp \alpha k^2 t}$, of Equtions (12) into (8), (9), we have the solitary wave solutions of Equations ((1) & (2)):

$$u(x,t) = \pm \frac{1}{2}\alpha k\left\{1 + \tanh\left[\frac{1}{2}\left(kx \mp \alpha k^2 t\right)\right]\right\}, \tag{13}$$

$$v(x,t) = -\frac{1}{4}\alpha k^2 \text{sech}^2\left[\frac{1}{2}\left(kx \mp \alpha k^2 t\right)\right]. \tag{14}$$

Substituting the solution $\phi_2 = 1 \mp \alpha xt + \frac{1}{6}x^3$, of Equation (12) into (8), (9), we have rational solutions of Equations ((1) & (2)):

$$u(x,t) = \frac{\pm\dfrac{\alpha}{2}x^2 - \alpha t}{1 \mp \alpha xt + \dfrac{1}{6}x^3}, \tag{15}$$

$$v(x,t) = -\frac{\alpha x}{1 \mp \alpha xt + \dfrac{1}{6}x^3} + \alpha\frac{\left(\dfrac{1}{2}x^2 \mp \alpha t\right)^2}{\left(1 \mp \alpha xt + \dfrac{1}{6}x^3\right)^2}. \tag{16}$$

Substituting the solution $\phi_3 = 1 + e^{\mp \alpha t}\left(A\cosh x + B\sinh x\right)$, of Equation (12) into (8), (9), we have the exact solutions containing hyperbolic functions of Equations ((1) & (2)):

$$u(x,t) = \pm\alpha\frac{e^{\mp\alpha t}\left(A\sinh x + B\cosh x\right)}{1 + e^{\mp\alpha t}\left(A\cosh x + B\sinh x\right)}, \tag{17}$$

$$v(x,t) = \frac{-\alpha e^{\mp\alpha t}\left(A\cosh x + B\sinh x\right)}{1 + e^{\mp\alpha t}\left(A\cosh x + B\sinh x\right)} + \frac{\alpha e^{\mp 2\alpha t}\left(A\sinh x + B\cosh x\right)^2}{\left[1 + e^{\mp\alpha t}\left(A\cosh x + B\sinh x\right)\right]^2}. \tag{18}$$

Substituting the solution $\phi_4 = 1 + e^{\pm \alpha t}\left(A\cos x + B\sin x\right)$, of Equation (12)

into (8), (9), we have the exact solutions containing trigonometric functions of Equations ((1) & (2)):

$$u(x,t) = \pm\alpha e^{\pm\alpha t}\frac{(-A\sin x + B\cos x)}{1 + e^{\pm\alpha t}(A\cos x + B\sin x)}, \tag{19}$$

$$v(x,t) = \frac{\alpha e^{\pm\alpha t}(A\cos x + B\sin x)}{1 + e^{\pm\alpha t}(A\cos x + B\sin x)} + \frac{\alpha e^{\pm 2\alpha t}(-A\sin x + B\cos x)^2}{\left[1 + e^{\pm\alpha t}(A\cos x + B\sin x)\right]^2} \tag{20}$$

$\cdots$, and so on.

It is well known that the linear Equation (12) can admit an infinite many solutions, based on the nonlinear transformation composed of (8), (9) and (12), we can obtain more solutions of Equations ((1) & (2)), provided that more solutions of Equation (12) are given.

4. Conclusion

In this paper, the nonlinear transformation composed of (8), (9) and (12) for the Boussinesq-Burgers equations has been derived by using the simplified homogeneous balance method. The important role of the nonlinear transformation is that the problem of solving nonlinear Boussinesq-Burgers equations becomes that of solving a linear equation, and the latter is much easily to solve for the researchers.

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