Hermite Solution of Bagley-Torvik Equation of Fractional Order

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Abstract

In this paper, a new methodology of fractional derivatives based upon Hermite polynomial is projected. The fractional derivatives are demonstrated according to Caputo sense. Hermite collocation technique is introduced to express the definite results of Bagley-Torvik Equations. The appropriateness and straightforwardness of numerical plan is presented by graphs and error tables.

Keywords

MAPLE 13, Bagley-Torvik Equations, Hermite Polynomials, Fractional Calculus

1. Introduction

Numerical analysis is the study of set of rules that use numerical estimation for the problems of mathematical analysis as distinguished from discrete mathematics. Fractional differential equations are operational and most effective tool to describe different physical phenomena such as rheology, diffusion processes, damping laws, and so on. Many technics have been delegated to solve differential equation of fractional order. Different structures are used to resolve the issues of nonlinear physical models of fractional orders like Finite element method [1], Finite difference method [2], differential transformation method [3] [4], Adomian's decomposition method [5] [6] [7], variational iteration method [8] [9] [10], Homotopy perturbation technique [11], Zubair decomposition method (ZDM) [12], (G'/G)-expansion method [13], (U'/U)-expansion method [14], U-expansion method [15], Fractional sub numerical announcement method [16] [17], Legendre wavelets technique [18], Chebyshev wavelets framework [19] [20] [21], Haar wavelets schema [22], Legendre Method [23], Chebyshev strategy [24], Jacobi polynomial scheme [25] and collocation scheme [26] [27] [28] [29]. All

the mentioned approaches have certain limitations like excessive computational work, less efficiency to tackle nonlinearity and divergent solution due to which many issues arise. All these disputes can be fixed with the help of orthogonal polynomials, which is a vital thought in close estimation and structures. These orthogonal polynomials are the reason of powerful strategies of spectral methods [30] [31] [32]. Starting late, Khader [33] displayed a capable numerical procedure for enlightening the fractional order physical problems using the Chebyshev polynomials. In the [34] two Chebyshev spectral frameworks for measuring multi-term fractional problems are displayed. The author (Tamour Zubair) devolve a new wavelets algorithm to construct the numerical solution of nonlinear Bagley-Torvik equation of fractional order which will have less computational works, straight forward and better accuracy as compare to the existing technique. It is to be emphasized that proposed algorithm is tremendously simple but highly effective Moreover, this new pattern is proficient for reducing the computational work to a tangible level while still retaining a very high level of accuracy.

2. Basic Definitions

Fractional Calculus [35]-[40]

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1. A real function $f(t), t > 0$ is said to be in the space $c_\mu, \mu \in R$ if there exists a real number $p(>\mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m iff $f^m \in C_i, m \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \ \alpha > 0, t > 0,$$

$$J^0 f(t) = f(t).$$

Properties of the operator J^α can be found in literature, we mention only the following: For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:
1) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$.
2) $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$.
3) $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain drawbacks when trying to model real-world processes with fractional differential equations. Therefore, we shall introduce a improved fractional differential operator D^α proposed by M.

Definition 3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^m(t) \mathrm{d}t,$$

for $m-1 < \alpha \leq 1m, m \in N, t > 0, f \in C_{-1}^m$. For the Caputo's derivative we have

$D^{\alpha}C = 0$, C is a constant,

$$D^{\alpha}x^k = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil; \\ \dfrac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}x^{k-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0,1,2,\cdots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

3. Bagley-Torvik Equations

Bagley-Torvik equation assumes an extremely vital part to study the performance of different material by application of fractional calculus [40] [41]. It has increased its significance in many fields of industrial and applied sciences. Precisely, the equation with 1/2 order derivative or 3/2 order derivative can be model the frequency dependent damping materials. The summed up form of Bagley-Torvik equation is given

$$\sigma \frac{\mathrm{d}^{\alpha}u(t)}{\mathrm{d}t^{\alpha}} + \omega \frac{\mathrm{d}^{\beta}u(t)}{\mathrm{d}t^{\beta}} + \mu(y(t))^n = f(t), \quad 0 < t \leq T. \tag{1}$$

with initial condition

$$\frac{\mathrm{d}^p}{\mathrm{d}t^p}u(0) = l_p, \quad p = 0,1.$$

with boundary condition at $t = t_0$, for $0 < t_0 \leq T$, is given by

$$\frac{\mathrm{d}^p}{\mathrm{d}t^p}u(t_0) = m_p, \quad p = 0,1.$$

where n is the nonlinear operator of the equation, $u(t)$ is unknown function. σ, ω and μ are the constant coefficients, T is the constant representing the span of input in close interval [0,T], and l_k, m_k are contents. When we have

$$n = 1, \sigma = M, \mu = k, \omega = 2S\sqrt{\epsilon\rho}$$

where M is mass of the rigid plate, k is stiffness of the spring, S is the area of plate immersed in Newtonian fluid, ϵ is the velocity, and ρ is the fluid density then equation (1) represents the motion of large thin plate in a Newtonian fluid [39]. Similarly, linearly damped fractional oscillator with the damping term has the fraction derivative $\beta = \dfrac{3}{2}$.

Further, we will discuss mathematical modeling of BT equation with feed-forward artificial neural network. The solution u of the fractional differential equation along with its v arbitrary order derivative $\dfrac{\mathrm{d}^v u}{\mathrm{d}t^v}$ can be approximated by the following continuous mapping as a neural network methodology [41] [42] [43] [44]:

$$\tilde{u}(t) = \sum_{i=1}^{h} \gamma_i \vartheta(\aleph_i t + \chi_i),$$

$$\frac{\mathrm{d}^v \tilde{u}}{\mathrm{d}t^v} = \sum_{i=1}^{h} \gamma_i \frac{\mathrm{d}^v}{\mathrm{d}t^v}\vartheta(\aleph_i t + \chi_i),$$

where $\gamma_i, \aleph_i$ and χ_i are bounded real valued adaptive parameters, h is the number of neurons and f is the active function taken as exponential function.

Fractional differential equation neural networks (FDN-NNs) can be approximate as

$$\tilde{u}(t) = \sum_{i=1}^{h} \gamma_i e^{(\aleph_i t + \chi_i)},$$

$$\frac{d^v \tilde{u}}{dt^v} = \sum_{i=1}^{h} \gamma_i \aleph_i^v e^{(\aleph_i t + \chi_i)},$$

for $v = 2$, we get

$$\frac{d^2 \tilde{u}}{dt^2} = \sum_{i=1}^{h} \gamma_i \aleph_i^2 e^{(\aleph_i t + \chi_i)},$$

Using Definition 4, for $v = \dfrac{3}{2}$, we get

$$\frac{d^{\frac{3}{2}} \tilde{u}}{dt^{\frac{3}{2}}} = \sum_{i=1}^{h} \gamma_i e^{(\chi_i)} t^{-\frac{3}{2}} E_{1,-\frac{1}{2}} (\aleph_i t).$$
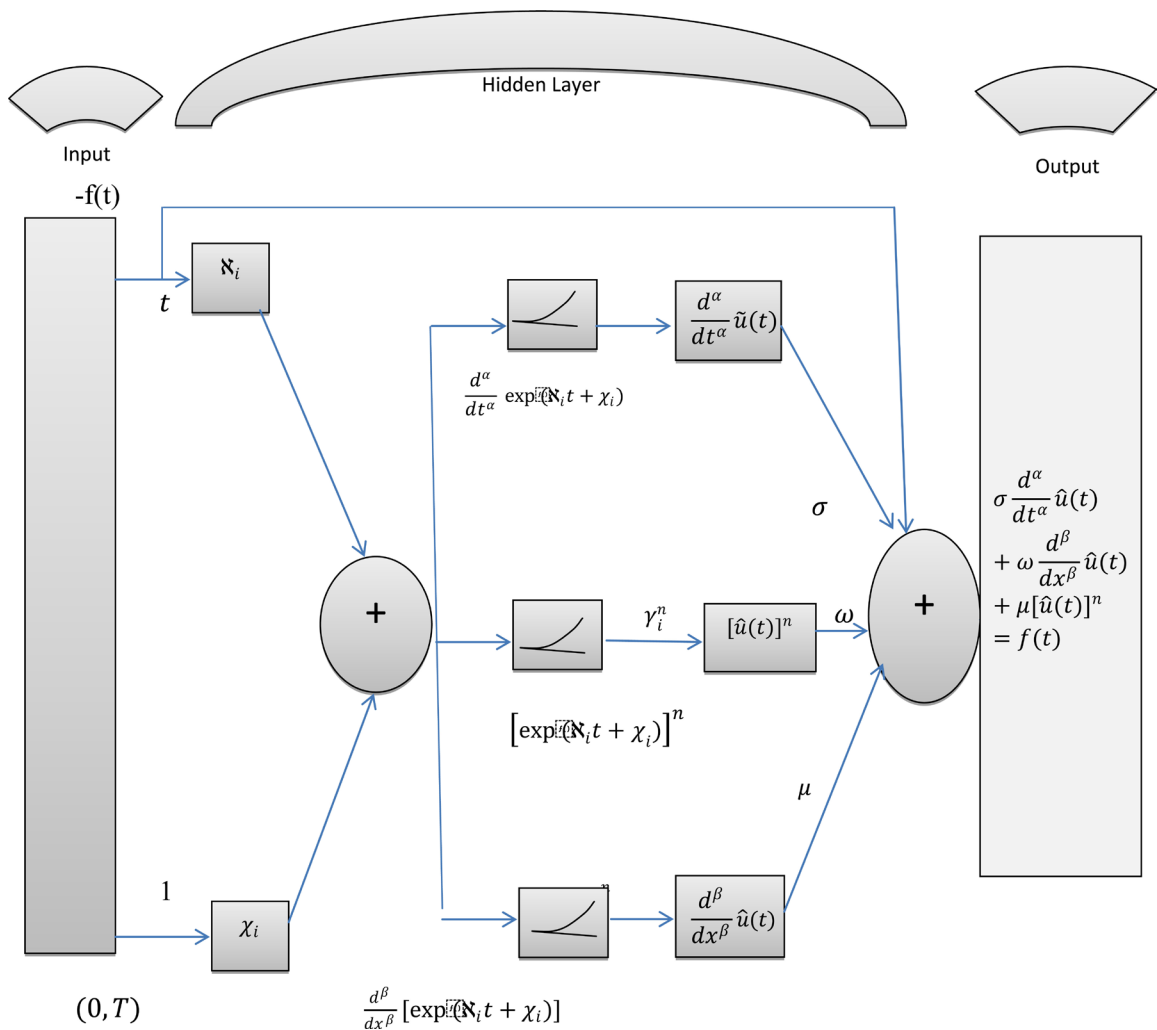

Figure 1. FDE-NN architecture of Bagley-Torvik equation.

The mathematical model can be the linear combinations of the networks represented above. The FDE-NN architecture formulated for Bagley-Torvik equation can be seen in **Figure 1**. It is clear that the solution u can be approximated with $\tilde{u}$ subject to finding appropriate unknown weights.

4. Hermite Polynomials [45]

It is classical orthogonal polynomials play very important role in probability. It has wide applications in numerical analysis as finite element methods as shape functions for beams. They are also applicable in physical quantum theory. Hermite polynomials are categorized into two kinds

The Probabilists Hermite polynomials are the solutions of

$$\left(e^{\varnothing}y'\right)' + \lambda\left(e^{\varnothing}y\right) = 0, \varnothing < 0, \lambda > 0.$$

where $\varnothing = \dfrac{x^2}{2}$ and λ is a constant, with the boundary conditions that y should be polynomially bounded at infinity. The above equation can be written in the form of eigen value problem

$$L[y] = y'' + xy' = -\lambda y,$$

solutions are the Eigen functions of the differential operator L. This equation is called Hermite equation, although the term is also used for the closely related equation

$$y'' - 2xy' = 2\lambda y.$$

whose solutions are the Physicists Hermites Polynomials, which is the second kind of Hermite polynomials.

The Hermite polynomials is given by

$$H_n(x) = (-1)^n e^{\varnothing_1} \frac{d^n}{dx^n} e^{\varnothing_2}, \varnothing_1 > 0, \varnothing_2 < 0.$$

where $\varnothing_1 = \begin{cases} x^2, & \text{for Physicists Hermites Polynomials,} \\ \dfrac{x^2}{2}, & \text{for Probabilists Hermites Polynomials.} \end{cases}$

and also $\varnothing_1 = -\varnothing_2$.

Probabilists Hermites Polynomials	Physicists Hermites Polynomials
$He_0 = 1,$	$H_0 = 1,$
$He_1 = x,$	$H_1 = 2x,$
$He_2 = x^2 - 1,$	$H_2 = 4x^2 - 2,$
$He_3 = x^3 - 3x,$	$H_3 = 8x^3 - 12x,$
$He_4 = x^4 - 6x^2 + 3,$	$H_4 = x^4 - 6x^2 + 3,$
$\vdots$	$\vdots$

$H_n(x)$ and $He_n(x)$ the two branches of Hermite polynomial of degree n, which are orthogonal with respect to weigh function.

$$w(x) = \begin{cases} e^{\frac{\varnothing_1}{2}}, & \text{for Physicists Hermites Polynomials,} \\ e^{2\varnothing_1}, & \text{for Probabilists Hermites Polynomials.} \end{cases}$$

Here we have $\varnothing_1 < 0$.

Further we have orthogonality $R(x)$ is given by

$$R(x) = \begin{cases} \int_{-\infty}^{\infty} He_m(x) He_n(x) w(x) dx = \sqrt{2\pi} n! \delta_{nm}, & \text{for Probabilists,} \\ \int_{-\infty}^{\infty} H_m(x) H_n(x) w(x) dx = \sqrt{\pi} 2^n n! \delta_{nm}, & \text{for Physicists.} \end{cases}$$

A function $u(x) \in L^2_{w(x)}(\infty, -\infty)$ can be express in term of Hermite polynomials

$$u(x) = \sum_{i=0}^{\infty} a_i H_i(x).$$

where a_i coefficients is given by

$$a_i = \frac{1}{\varrho_i} \int_{-\infty}^{\infty} u(x) H_i(x) w(x) dx.$$

where $\varrho_i = \sqrt{\pi} 2^n n!$.

5. Fractional Form of Hermite Polynomials [35]-[40]

The explicit formula of Hermites polynomials is

$$P_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ \frac{(-1)^m}{m!(n-m)!} X' \right\}. \tag{1*}$$

where X' is given by

$$X' = \begin{cases} (2x)^{n-2m}, & \text{for Physicists Hermites Polynomials,} \\ \dfrac{x^{n-2m}}{2^m}, & \text{for Probabilists Hermites Polynomials.} \end{cases}$$

Further we have

$$P_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \{\delta_m X'\}. \tag{2}$$

where δ_m is given by

$$\delta_m = \frac{(-1)^m}{m!(n-m)!}.$$

A function $u(x) \in L^2_{w(x)}(\infty, -\infty)$ can be express in term of Hermite polynomials

$$\tilde{u}(x) = \sum_{n=0}^{M} a_k P_n(x). \tag{3}$$

where $P(x)$ are Hermites polynomials. Using (1*)-(3) and definition of fractional derivative, we get the following

$$D^\alpha \tilde{u}(x) \cong \sum_{n=\alpha}^{M} \left(n! \sum_{m=0}^{\xi} a_n \delta_m \chi_{nm}^\alpha \aleph'(X')^{-\alpha} \right). \tag{4}$$

where $\aleph', \chi_{nm}^{\alpha}$ and ξ is given by $\aleph' = 2^{\alpha}, \chi_{nm}^{\alpha} = \dfrac{\Gamma(n-2m+1)}{\Gamma(n-2m-\alpha+1)}$ and

$$\xi = \frac{n-\alpha}{2}.$$

Note that only for $\aleph'$, we have following

$$\aleph' = \begin{cases} \alpha > 0, & \text{for Physicists Hermites Polynomials,} \\ \alpha < 0, & \text{for Probabilists Hermites Polynomials.} \end{cases}$$

a) Methodology

Consider the multi order fractional differential equation (1) as

$$D^{\alpha}u(x) = \varphi\left(x; u(x), D^{\beta_1}u(x), D^{\beta_2}u(x), \cdots, D^{\beta_n}u(x)\right), \tag{5}$$

$$u^{(n)}(0) = u_n, \quad n = 0, 1, 2, \cdots, m-1,$$

where $u(x)$ is the unknown function, to be determined. The proposed technique for solving Equation (5) proceeds in the following three steps:

Step 1: According to the proposed algorithm we assume the following trial solution

$$\tilde{u}(x) = \sum_{k=0}^{M} a_k P_k(x) = \boldsymbol{U}^{\mathrm{T}} \boldsymbol{P}(x), \tag{6}$$

where $\boldsymbol{U} = [a_0, a_1, a_2, \cdots]^{\mathrm{T}}$ and $\boldsymbol{P}(x) = [P_0(x), P_1(x), P_2(x), \cdots]^{\mathrm{T}}$.
where $P_k(x)$ are Hermite polynomials of degree k defined in Equation (6) and a_k are unknown parameters, to be determined.

Step 2: Substituting Equation (6) into Equation (5), we get

$$\boldsymbol{U}^{\mathrm{T}} D^{\alpha} \boldsymbol{P}(x) = \varphi\left(x; \boldsymbol{U}^{\mathrm{T}} \boldsymbol{P}(x), \boldsymbol{U}^{\mathrm{T}} D^{\beta_1} \boldsymbol{P}(x), \boldsymbol{U}^{\mathrm{T}} D^{\beta_2} \boldsymbol{P}(x), \cdots, \boldsymbol{U}^{\mathrm{T}} D^{\beta_n} \boldsymbol{P}(x)\right), \beta_1$$

$$\boldsymbol{U}^{\mathrm{T}} \boldsymbol{P}(x)(0) = u_n,$$

Using (4) we have

$$\sum_{n=\lceil \alpha \rceil}^{h}\left(n! \sum_{m=0}^{\xi} a_n \delta_m \chi_{nm}^{\alpha} \aleph'(X')^{-\alpha}\right)$$
$$= \varphi\left(x; \sum_{k=0}^{M} a_k P_k(x), \sum_{n=\lceil \beta_1 \rceil}^{M}\left(n! \sum_{m=0}^{\xi} a_n \delta_m \chi_{nm}^{\beta_1} \aleph'(X')^{-\beta_1}\right),\right.$$
$$\sum_{n=\lceil \beta_2 \rceil}^{M}\left(n! \sum_{m=0}^{\xi} a_n \delta_m \chi_{nm}^{\beta_2} \aleph'(X')^{-\beta_2}\right), \cdots,$$
$$\left.\sum_{n=\lceil \beta_n \rceil}^{M}\left(n! \sum_{m=0}^{\xi} a_n \delta_m \chi_{nm}^{\beta_n} \aleph'(X')^{-\beta_n}\right)\right), \tag{7}$$

$$\boldsymbol{U}^{\mathrm{T}} \boldsymbol{P}(x)(0) = u_n.$$

Step 3: Further we Assume suitable collocation point for Equation (7). Therefore, we obtained system has $M+1$ equations and $M+1$ unknowns. Solving this system gives the unknown coefficients using Conjugate Gradient Method. Putting these constant into trial solution, we can obtained the approximate/exact solutions of linear/nonlinear fractional differential Equation (5).

b) Approximation by Hermite Polynomials [45]

Let us define $\Omega = \{x, -\infty < x < \infty\}$ and $\gamma_N = span\{P_0(x), P_1(x), \cdots, P_N(x)\}$. The $L_{w(x)}^2(\Omega)$-orthogonal projection $\pi_N : L^2(\Omega) \to \gamma_N$ be the mapping and we

have

$$\left\langle \pi_N(z) - z \right\rangle = 0, \ \forall \theta \in \gamma_N.$$

Due to the orthogonality property, we can write it as

$$\pi_N(z) = \sum_{k=0}^{N-1} a_k P_k(x),$$

where $a_i (i = 0, 1, \cdots, N-1)$ are the constants in the following form

$$a_i = \frac{1}{\varrho_i} \left\langle u(x), P_k \right\rangle L^2_{w(x)}.$$

6. Numerical Simulation

In this section, we apply new algorithm to construct approximate/exact solutions fractional differential equation. Numerical results are very encouraging.

Case 1 In Equation (1), we take $\alpha = 2$, $\beta = \frac{3}{2}$, $\sigma = \omega = \mu = n = 1$,

$f(t) = 2 + 4\sqrt{\dfrac{t}{\pi}} + t^2$, $l_p = 0$, $t_0 = 1$, $m_p = 1, 2$. The close form solution is t^2.

Consider the trial solutions for $M = 2$ as

$$\tilde{u}(t) = \sum_{n=0}^{M} a_k P_n(t).$$

Using the trail solution into Equation (1) and proceed it according to Step 1 and Step 2, then we collocate it further to generate the system of equations. Solve the system of equations along with initial conditions, we get the values of constants

Hermite's	a_0	a_1	a_2
Physicists	$\dfrac{1}{2}$	0	$\dfrac{1}{4}$
Probabilists	1	0	1

Finally, we get the approximate solution

$$u(t) = t^2.$$

which is exact solution.

Case 2 In Equation (1), we take $\alpha = 2$, $\beta = \frac{3}{2}$, $\sigma = \omega = \mu = n = 1$,

$f(t) = 2 + 4\sqrt{\dfrac{t}{\pi}} + t^2$, $l_0 = 0$, $t_0 = 5$, $m_0 = 25$. The close form solution is t^2.

Consider the trial solutions for $M = 2$ as

$$\tilde{u}(t) = \sum_{n=0}^{M} a_k P_n(t).$$

Using the trail solution into Equation (1) and proceed it according to Step 1 and Step 2, then we collocate it further to generate the system of equations. Solve the system of equations along with initial conditions, we get the values of constants

Hermite's	a_0	a_1	a_2
Physicists	$\frac{1}{2}$	0	$\frac{1}{4}$
Probabilists	1	0	1

Finally, we get the approximate solution

$$u(t) = t^2.$$

which is exact solution.

Case 3 In Equation (1), we take $\alpha = 2$, $\beta = \frac{3}{2}$, $\sigma = \omega = \mu = n = 1$, $f(t) = t + 1$, $l_p = 1$. The close form solution is $1 + t$.

This equation can be simplify by using

$$U(t) = u(t) - 1 - t.$$

Consider the trial solutions for $M = 2$ as

$$\tilde{u}(t) = \sum_{n=0}^{M} a_k P_n(t).$$

Using the trail solution into Equation (1) and proceed it according to Step 1 and Step 2, then we collocate it further to generate the system of equations. Solve the system of equations along with initial conditions, we get the values of constants

Hermite's	a_0	a_1	a_2
Physicists	1	$\frac{1}{2}$	0
Probabilists	1	1	0

Finally, we get the approximate solution

$$u(t) = 1 + t.$$

which is exact solution.

Case 4 In Equation (1), we take $\alpha = 2$, $\sigma = n = \mu = 1$, $\omega = \frac{1}{2}$,

$$f(t) = 3 + t^2 \left(\frac{t^{-\beta}}{\Gamma(3 - \beta)} + 1 \right), \quad p = 0, \quad l_0 = 1, \quad t_0 = 1, \quad m_0 = 2.$$ The close form solution is $1 + t^2$.

Using the trail solution into Equation (1) and proceed it according to Step 1 and Step 2, then we collocate it further to generate the system of equations. Solve the system of equations along with initial conditions, we get the values of constants

Hermite's	a_0	a_1	a_2
Physicists	$\frac{3}{2}$	0	$\frac{1}{4}$
Probabilists	2	0	1

Finally, we get the approximate solution

$$u(t) = 1 + t^2.$$

which is exact solution.

Case 5. In Equation (1), we take $\alpha = 2$, $\sigma = n = \mu = 1$, $\omega = \dfrac{1}{2}$,

$$f(t) = 4t^2(5t-3) + \omega t^{4-\beta}\left(\frac{120}{\Gamma(6-\beta)}t - \frac{24}{\Gamma(5-\beta)}\right) + \mu t^4(t-1), \quad p = 0, \quad l_0 = 0,$$

$t_0 = 1$, $m_0 = 0$. The close form solution is $t^4(t-1)$.

The numerical solution is represented in **Table 1** in case of $M = 2,5$ and $\beta = 0.3$, while the error for various values of $\beta = 0$ and $\beta = 0.5$ are represented in **Table 2**. There is a graphical comparison between exact and approximate solution represented in **Figure 2**.

Table 1. Numerical comparison between exact and approximate solution for deferent values of M.

x	$M = 2$	$M = 5$
0	0.00000E+00	0.00000E+00
0.1	9.00000E−05	6.42250E−45
0.2	1.28000E−03	9.13422E−44
0.3	5.67000E−03	4.04617E−43
0.4	1.53600E−02	1.09611E−42
0.5	3.12500E−02	2.23003E−42
0.6	5.18400E−02	3.69936E−42
0.7	7.20300E−02	5.14014E−42
0.8	8.19200E−02	5.84590E−42
0.9	6.56100E−02	4.68200E−42
1.0	0.00000E+00	0.00000E+00

Error = |Exact Solution − Approximate solution|.

Table 2. Numerical comparison between exact approximate solutions for different values of β.

x	$\beta = 0$	$\beta = 0.5$
0	0.00000E+00	4.00000E−100
0.1	9.08224E−32	7.57685E−45
0.2	1.80074E−31	1.07760E−43
0.3	2.65365E−31	4.77341E−43
0.4	3.42665E−31	1.29312E−42
0.5	4.05488E−31	2.63085E−42
0.6	4.44066E−31	4.36426E−42
0.7	4.44538E−31	6.06400E−42
0.8	3.88121E−31	6.89662E−42
0.9	2.50300E−31	5.52352E−42
1.0	8.00000E−100	2.00000E−99

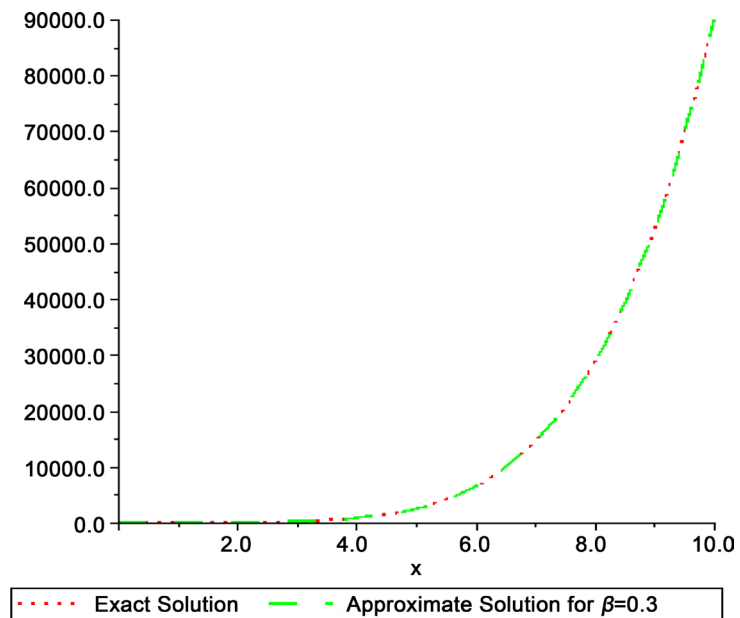Error = |Exact Solution − Approximate solution|.

Figure 2. Graphysical comparision between exact and approximted solution.

7. Conclusions

All the facts and findings of the paper are summarized as follow:

- This paper provides novel study of Bagley-Torvik equations of fractional order in different situations by using newly suggested Hermite Polynomial scheme.
- Implementation of this methodology is moderately relaxed and with the help of this suggested algorithm, complicated problems can be tackled.
- It is to be highlighted that the suggested comparison gives attentive respond regarding some particular issues for values of *M*, which demonstrates viability of the proposed framework. Likewise, the reliability of the application provided this technique a more comprehensive suitability.

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