# Asymptotic Theory for a General Second-Order Differential Equation 

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## Abstract

An asymptotic theory developed for a second-order differential equation. We obtain the form of solutions for some class of the coefficients for large $x$.

## Keywords

Asymptotic Form of Solutions, Second-Order

## 1. Introduction

In this paper, we examine the asymptotic form of two linearly independant solutions of the general second-order differential equation.

$$
\begin{equation*}
\left(p y^{\prime}\right)^{\prime}+q y^{\prime}+r y=0 \tag{1}
\end{equation*}
$$

as $x \rightarrow \infty$, where $x$ is the independant variable and the prime denotes $\frac{\mathrm{d}}{\mathrm{d} x}$.
The coefficients $p, q$ and $r$ are nowhere zero in some interval $[a, \infty)$. We shall consider the situation where $p$ and $r$ are small compared to $q$ see (15) to identify the following case:

$$
\begin{equation*}
\frac{q^{\prime}}{q}=0\left(\frac{r}{q}\right),(x \rightarrow \infty) \tag{2}
\end{equation*}
$$

and under (2) we shall obtain the forms of the asymptotic solutions for (1) as $x \rightarrow \infty$ which is given in Theorem 1.

If $p=1$, then (1) reduces to the differential equation considered by Walker [1]. We do not investigate the case where $q^{2}=o(p r)$, the analysis for this case is already known for the Sturm-Liouville equation

$$
\left(p y^{\prime}\right)^{\prime}+r y=0
$$

see Eastham [2] and Atkinson [3].

We shall use the asymptotic Theorem of Eastham ([3], Section 2), [4] to obtain our main result of (1) in Section 4. The general feature of our method are given in Sections (2) and (3), with some examples in Section (5).

## 2. The General Method

We write (1) in a standard way [5] as a first-order system:

$$
\begin{equation*}
Y^{\prime}=A Y \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
Y=\binom{y}{p y^{\prime}} \tag{4}
\end{equation*}
$$

and the matrix is given by

$$
A=\left(\begin{array}{cc}
0 & p^{-1}  \tag{5}\\
-r & -q p^{-1}
\end{array}\right)
$$

As in [6] we express the matrix $A$ in the diagonal form:

$$
\begin{equation*}
T^{-1} A T=\Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}\right) \tag{6}
\end{equation*}
$$

and we therefore require the eigenvalues $\lambda_{j}$ and the eigenvectors $v_{j}$ of $A$, $j=1,2$.

The characteristic equation of is given by:

$$
\begin{equation*}
p \lambda^{2}+q \lambda+r=0 \tag{7}
\end{equation*}
$$

An eigenvector $v_{j}$ corresponding to $\lambda_{j}$ is

$$
v_{j}=\left(\begin{array}{ll}
1 & p \lambda_{j} \tag{8}
\end{array}\right)^{*}
$$

where the superscript * denote the transpose.
Now by (7)

$$
\begin{equation*}
\lambda_{j}=-\frac{q}{2 p} \pm \frac{\left(q^{2}-4 p r\right)^{1 / 2}}{2 p} \quad(j=1,2) \tag{9}
\end{equation*}
$$

Now we define the matrix $T$ in (6) by

$$
T=\left[\begin{array}{cc}
1 & 1  \tag{10}\\
p \lambda_{1} & p \lambda_{2}
\end{array}\right]
$$

Hence by (6), the transformation

$$
\begin{equation*}
Y=T Z \tag{11}
\end{equation*}
$$

takes (3) into

$$
\begin{equation*}
Z^{\prime}=\left(\Lambda-T^{-1} T^{\prime}\right) Z \tag{12}
\end{equation*}
$$

Now if we write

$$
\begin{equation*}
T^{-1} T^{\prime}=\left(t_{j k}\right) \tag{13}
\end{equation*}
$$

then by (7) and (10)

$$
\begin{align*}
& t_{1 j}=\left(\lambda_{1}-\lambda_{2}\right)^{-1}\left[\left(p^{\prime} \lambda_{j}^{2}+q^{\prime} \lambda_{j}+r^{\prime}\right)\left(2 p \lambda_{j}+q\right)^{-1}-\frac{p^{\prime}}{p} \lambda_{j}\right]  \tag{14}\\
& t_{2 j}=-t_{1 j} \quad(j=1,2) .
\end{align*}
$$

Now we need to work (14) in terms of $r, p$ and $q$ in order to determine (12) and then make progress for (1).

## 3. The Matrices $\Lambda$ and $T^{-1} T^{\prime}$

At this stage we require the following conditions in the coefficients $r, p$ and $q$ as $x \rightarrow \infty$.

Condition I. $r, p$ and $q$ are nowhere zero in some interval $[a, \infty)$, and

$$
\begin{equation*}
r p=0\left(q^{2}\right), \quad(x \rightarrow \infty) \tag{15}
\end{equation*}
$$

we write

$$
\begin{equation*}
\delta=\frac{r p}{q^{2}} \rightarrow 0(x \rightarrow \infty) \tag{16}
\end{equation*}
$$

## Condition II.

$$
\begin{equation*}
\delta \frac{r^{\prime}}{r}, \delta \frac{p^{\prime}}{p}, \delta \frac{q^{\prime}}{q} \text { are all } L(a, \infty) \tag{17}
\end{equation*}
$$

Now if we let

$$
\begin{equation*}
D=\frac{\left(q^{2}-4 p r\right)^{1 / 2}}{2 p} \tag{18}
\end{equation*}
$$

then (9) gives

$$
\begin{equation*}
\lambda_{j}=-\frac{q}{2 p} \pm D \quad(j=1,2) \tag{19}
\end{equation*}
$$

where by(18) and (16)

$$
\begin{equation*}
D=\frac{q}{2 p}(1-4 \delta)^{1 / 2} \sim \frac{q}{2 p}(x \rightarrow \infty) \tag{20}
\end{equation*}
$$

Now by (19) and (20)

$$
\begin{equation*}
\lambda_{1}=-\frac{r}{q}\left[1+\delta+\mathrm{O}\left(\delta^{2}\right)\right] \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{2}=-\frac{q}{p}\left[1-\delta+\mathrm{O}\left(\delta^{2}\right)\right] \tag{22}
\end{equation*}
$$

Now using (14), (21) and (22) we obtain

$$
\begin{gather*}
t_{11}=t_{21}=\mathrm{O}(\Delta)  \tag{23}\\
t_{12}=-t_{22}=-\frac{q^{\prime}}{q}+\mathrm{O}(\Delta) \tag{24}
\end{gather*}
$$

where

$$
\begin{equation*}
\Delta=\left(\left|\frac{r^{\prime}}{r} \delta\right|+\left|\frac{p^{\prime}}{p} \delta\right|+\left|\frac{q^{\prime}}{q} \delta\right|\right) \tag{25}
\end{equation*}
$$

Hence by (17),

$$
\begin{equation*}
\Delta \in L(a, \infty) \tag{26}
\end{equation*}
$$

Therefore, by (23), (24) and (26), we can write (12) as:

$$
\begin{equation*}
Z^{\prime}=(\Lambda+R+S) Z \tag{27}
\end{equation*}
$$

where

$$
R=\left[\begin{array}{cc}
0 & \frac{q^{\prime}}{q}  \tag{28}\\
0 & -\frac{q^{\prime}}{q}
\end{array}\right],
$$

and $S$ is $L(a, \infty)$ by (26).

## 4. The Asymptotic Form of Solutions

Theorem 1. Let the coefficients $r$ and $p$ in (1) be $C^{1}[a, \infty)$ while q to be $C^{2}[a, \infty)$.

Let (15) and (17) hold.
Let

$$
\begin{gather*}
\frac{q^{\prime}}{q}=0\left(\frac{r}{q}\right) \quad(x \rightarrow \infty)  \tag{29}\\
\left(\frac{q^{\prime} p}{q^{2}}\right)^{\prime}, \frac{r^{2} p}{q^{3}} \text { are } L(a, \infty) \tag{30}
\end{gather*}
$$

Let

$$
\begin{equation*}
\operatorname{Re}\left[\frac{q}{p}-2 \frac{r}{q}+\frac{q^{\prime}}{q}\right] \text { be of one sign in }[a, \infty) \tag{31}
\end{equation*}
$$

Then (1) has solutions $y_{1}$ and $y_{2}$ such that

$$
\begin{gather*}
y_{1} \sim \exp \left(-\int_{a}^{x} \frac{r}{q} \mathrm{~d} t\right),  \tag{32}\\
y_{1}^{\prime}=\mathrm{o}\left[q p^{-1} \exp \left(-\int_{a}^{x} \frac{r}{q} \mathrm{~d} t\right)\right] \tag{33}
\end{gather*}
$$

while

$$
\begin{align*}
& y_{2} \sim q^{-1} \exp \left(\int_{a}^{x}\left[-\frac{q}{p}+\frac{r}{q}\right] \mathrm{d} t\right),  \tag{34}\\
& y_{2}^{\prime} \sim p^{-1} \exp \left(\int_{a}^{x}\left[-\frac{q}{p}+\frac{r}{q}\right] \mathrm{d} t\right) . \tag{35}
\end{align*}
$$

Proof. As in [6], we apply the Eastham theorem ([3], section 2) to the system (27) provided only that $\Lambda$ and $R$, satisfy the required conditions.

We shall use (15), (17), (29), and (31).
We first require that

$$
\begin{equation*}
\frac{q^{\prime}}{q}=o\left(\lambda_{1}-\lambda_{2}\right), \tag{36}
\end{equation*}
$$

this being [2] for our system,

$$
\begin{equation*}
\lambda_{1}-\lambda_{2}=\frac{q}{p}(1-4 \delta)^{1 / 2} \tag{37}
\end{equation*}
$$

Thus (36) holds by (15) and (29).
Second, we need

$$
\begin{equation*}
\left[\left(\lambda_{1}-\lambda_{2}\right)^{-1} \frac{q^{\prime}}{q}\right]^{\prime} \in L(a, \infty) \tag{38}
\end{equation*}
$$

this being [2] for our system. By (38), this requirement is implied by (17) and (30).

Finally we show that the eigenvalues $\mu_{k}$ of $\Lambda+R$ satisfy the dichotomy condition [2].

As in [6] and [7], the dichotomy condition holds if

$$
\begin{equation*}
\operatorname{Re}\left(\mu_{1}-\mu_{2}\right)=f+g \tag{39}
\end{equation*}
$$

where $f$ has one sign in $[a, \infty)$ and $g$ is $L(a, \infty)$ [2].
Now by (6) and (28):

$$
\begin{equation*}
\mu_{1}(x)=\lambda_{1}(x), \mu_{2}(x)=\lambda_{2}(x)-\frac{q^{\prime}}{q} \tag{40}
\end{equation*}
$$

then by (21), (22) and (40)

$$
\begin{equation*}
\operatorname{Re}\left(\mu_{1}-\mu_{2}\right)=\operatorname{Re}\left(\frac{q}{p}-2 \frac{r}{p}+\frac{q^{\prime}}{q}\right)+\mathrm{O}\left(\frac{r^{2} p}{q^{3}}\right) \tag{41}
\end{equation*}
$$

Thus, by (31) and (30), (39) holds. Since (27) satisfies all the conditions for the asymptotic result [3, section 2], it follows that as $x \rightarrow \infty$, (27) has two linearly independant solutions.

$$
\begin{equation*}
Z_{k}(x)=\left[e_{k}+\mathrm{o}(1)\right] \exp \left(\int_{a}^{x} \mu_{k}(t) \mathrm{d} t\right) \tag{42}
\end{equation*}
$$

with $e_{k}$ the coordinate vector with $k$-th coponment unity and other coponments zero.

Finally, on transforming back to $y$ via (10), (11), (4) and making use of (40), (21), (22) and (30), we obtain (33), also (32) after adjusing $y_{1}$ by a constant multiple, and similary for $y_{2}$ and $y_{2}^{\prime} \cdot \square$

## 5. Examples

Example 1. We consider the cofficients in (1) given by

$$
r(x)=c_{1} x^{\alpha_{1}}, q(x)=c_{2} x^{\alpha_{2}}, p(x)=c_{3} x^{\alpha_{3}} .
$$

$\alpha_{i}$ and $c_{i} \quad(1 \leq i \leq 3)$ are real constants with $c_{i} \neq 0$. Then (15) and (17) of

Theorem 4.1 hold under the conditions

$$
\begin{equation*}
2 \alpha_{2}-\alpha_{1}-\alpha_{3}>0 \tag{43}
\end{equation*}
$$

Also (29) true if

$$
\begin{equation*}
\alpha_{1}-\alpha_{2}+1>0 \tag{44}
\end{equation*}
$$

Now in (30) $\left(\frac{q^{\prime} p}{q^{2}}\right)^{\prime}$ is $L(a, \infty)$ if

$$
\begin{equation*}
\alpha_{2}-\alpha_{3}+1>0 \tag{45}
\end{equation*}
$$

wich is true by (43) and (44).
Also, in (30), $\frac{r^{2} p}{q^{3}}$ is $L(a, \infty)$ if

$$
\begin{equation*}
3 \alpha_{2}-2 \alpha_{1}-\alpha_{3}>1 \tag{46}
\end{equation*}
$$

So all conditions of theorem 4.1 are true under (43), (44) and (46). For example if we take $\alpha_{1}=\alpha_{2}$.

Then all condition are true if

$$
\begin{equation*}
\alpha_{2}-\alpha_{3}>1 \tag{47}
\end{equation*}
$$

Example 2. Let $r(x)=c_{1} x^{\alpha_{1}} \exp \left(x^{a}\right), \quad p(x)=c_{2} x^{\alpha_{2}} \exp \left(-4 x^{b}\right)$, $q(x)=c_{3} x^{\alpha_{3}} \exp \left(-x^{b}\right)$
where $b \geq a>0, \alpha_{i}$ and $c_{i} \quad(1 \leq i \leq 3)$ are real constants with $c_{i} \neq 0$.
Again it is easy to check that all conditions of Theorem 4.1 are satisfied.

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