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The Number of Maximal Independent Sets in Quasi-Tree Graphs and Quasi-Forest Graphs

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Abstract

A maximal independent set is an independent set that is not a proper subset of any other independent set. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). In this paper, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. In addition, we further look into the problem of determining the third largest number of maximal independent sets among all quasi-trees and quasi-forests. Extremal graphs achieving these values are also given.

Keywords

Maximal Independent Set, Quasi-Tree Graph, Quasi-Forest Graph, Extremal Graph

1. Introduction and Preliminary

Let $G = (V, E)$ be a simple undirected graph. An *independent set* is a subset S of V such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [1]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [2]. Jin and Li [3] investigated the second largest number of $mi(G)$ among all graphs of order n; Jou and Lin [4] further explored the same problem for trees and forests; Jin and Yan [5] solved the third largest number of

$mi(G)$ among all trees of order *n*. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by Liu and Lu in [6]. Recently, the problem of determining the largest and the second largest numbers of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order *n* was solved by Lin [7] [8].

In this paper, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. In addition, we further look into the problem of determining the third largest number of maximal independent sets among all quasi-trees and quasi-forests. Extremal graphs achieving these values are also given.

For a graph $G = (V, E)$, the *neighborhood* $N_G(x)$ of a vertex *x* is the set of vertices adjacent to *x* in *G* and the *closed neighborhood* $N_G[x]$ is $\{x\} \cup N_G(x)$. The *degree* of *x* is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \varnothing$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. nG is the short notation for the union of *n* copies of disjoint graphs isomorphic to *G*. Denote by C_n a *cycle* with *n* vertices and P_n a *path* with *n* vertices.

Throughout this paper, for simplicity, let $r = \sqrt{2}$.

Lemma 1.1 ([9]) *For any vertex x in a graph G,* $mi(G) \le mi(G - x) + mi(G - N_G[x])$.

Lemma 1.2 ([10]) *If G is the union of two disjoint graphs* G_1 *and* G_2*, then* $mi(G) = mi(G_1)mi(G_2)$.

2. Survey on the Large Numbers of Maximal Independent Sets

In this section, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.1 and 2.2, respectively.

Theorem 2.1 ([10] [11]) *If T is a tree with* $n \ge 1$ *vertices, then* $mi(T) \le t_1(n)$*, where*

$$t_1(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even,} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_1(n)$ if and only if $T \in T_1(n)$, where

$$T_1(n) = \begin{cases} B\left(2, \frac{n-2}{2}\right) \text{ or } B\left(4, \frac{n-4}{2}\right), & \text{if } n \text{ is even,} \\ B\left(1, \frac{n-1}{2}\right), & \text{if } n \text{ is odd,} \end{cases}$$

where $B(i, j)$ is the set of batons, which are the graphs obtained from the basic path P of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the end-points of P in all possible ways (see Figure 1).

Theorem 2.2 ([10] [11]) *If F is a forest with $n \geq 1$ vertices, then $mi(F) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even,} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_1(n)$ if and only if $F \in F_1(n)$, where

$$F_1(n) = \begin{cases} \frac{n}{2} P_2, & \text{if } n \text{ is even,} \\ B\left(1; \frac{n-1-2s}{2}\right) \cup sP_2 \text{ for some } s \text{ with } 0 \leq s \leq \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.3 and 2.4, respectively.

Theorem 2.3 ([4]) *If T is a tree with $n \geq 4$ vertices having $T \notin T_1(n)$, then $mi(T) \leq t_2(n)$, where*

$$t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \geq 4 \text{ is even,} \\ 3, & \text{if } n = 5, \\ 3r^{n-5} + 1, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_2(n)$ if and only if $T = T_2'(8), T_2''(8), P_{10}$ or $T \in T_2(n)$, where $T_2(n)$ and $T_2'(8)$, $T_2''(8)$ are shown in Figure 2 and Figure 3, respectively.

Theorem 2.4 ([4]) *If F is a forest with $n \geq 4$ vertices having $F \notin F_1(n)$, then $mi(F) \leq f_2(n)$, where*

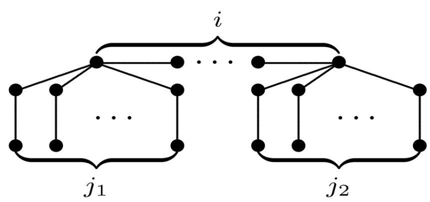

Figure 1. The baton $B(i, j)$ with $j = j_1 + j_2$.

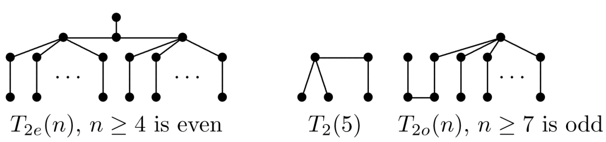

Figure 2. The trees $T_2(n)$.

Figure 3. The trees $T_2'(8)$ and $T_2''(8)$.

$$f_2(n)=\begin{cases}3r^{n-4}, & \text{if } n\ge 4 \text{ is even},\\ 3, & \text{if } n=5,\\ 7r^{n-7}, & \text{if } n\ge 7 \text{ is odd}.\end{cases}$$

Furthermore, $mi(F)=f_2(n)$ if and only if $F\in F_2(n)$, where

$$F_2(n)=\begin{cases}P_4\cup\frac{n-4}{2}P_2, & \text{if } n\ge 4 \text{ is even},\\ T_2(5) \text{ or } P_4\cup P_1, & \text{if } n=5,\\ P_7\cup\frac{n-7}{2}P_2, & \text{if } n\ge 7 \text{ is odd}.\end{cases}$$

The results of the third largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.5 and 2.6, respectively.

Theorem 2.5 ([5]) *If T is a tree with* $n\ge 7$ *vertices having* $T\notin T_i(n)$, $i=1,2$, *then* $mi(T)\le t_3(n)$, *where*

$$t_3(n)=\begin{cases}3r^{n-5}, & \text{if } n\ge 7 \text{ is odd},\\ 7, & \text{if } n=8,\\ 15, & \text{if } n=10,\\ 7r^{n-8}+2, & \text{if } n\ge 12 \text{ is even}.\end{cases}$$

Furthermore, $mi(T)=t_3(n)$ if and only if $T=T_3(8),T_3'(10),T_3''(10)$ or $T\in T_3(n)$, where $T_3(8),T_3'(10),T_3''(10),T_3(n)$ are shown in **Figure 4** and **Figure 5**, respectively.

Theorem 2.6 ([12]) *If F is a forest with* $n\ge 8$ *vertices having* $F\notin F_i(n)$, $i=1,2$, *then* $mi(F)\le f_3(n)$, *where*

$$f_3(n)=\begin{cases}5r^{n-6}, & \text{if } n \text{ is even},\\ 13r^{n-9}, & \text{if } n \text{ is odd}.\end{cases}$$

Furthermore, $mi(F)=f_3(n)$ if and only if $F\in F_3(n)$, where

$$F_3(n)=\begin{cases}T_1(6)\cup\frac{n-6}{2}P_2, & \text{if } n \text{ is even},\\ T_2(9)\cup\frac{n-9}{2}P_2, & \text{if } n \text{ is odd}.\end{cases}$$

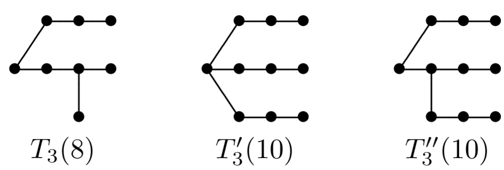

Figure 4. The trees $T_3(8),T_3'(10)$ and $T_3''(10)$.

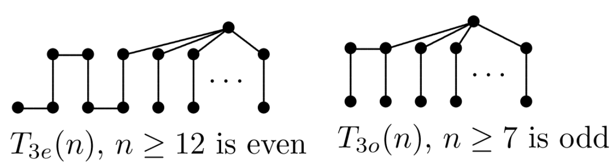

Figure 5. The trees $T_3(n)$.

The results of the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.7 and 2.8, respectively.

Theorem 2.7 ([7]) *If Q is a quasi-tree graph with* $n \ge 5$ *vertices, then* $mi(Q) \le q_1(n)$, *where*

$$q_1(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even,} \\ r^{n-1}+1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q_1(n)$ if and only if $Q = C_5$ or $Q \in Q_1(n)$, where $Q_1(n)$ is shown in **Figure 6**.

Theorem 2.8 ([7]) *If Q is a quasi-forest graph with* $n \ge 2$ *vertices, then* $mi(Q) \le \overline{q}_1(n)$, *where*

$$\overline{q}_1(n) = \begin{cases} r^{n}, & \text{if } n \text{ is even,} \\ 3r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = \overline{q}_1(n)$ if and only if $Q \in \overline{Q}_1(n)$, where

$$\overline{Q}_1(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even,} \\ C_3 \cup \frac{n-3}{2}P_2, & \text{if } n \text{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.9 and 2.10, respectively.

Theorem 2.9 ([8]) *If Q is a quasi-tree graph with* $n \ge 6$ *vertices having* $Q \notin Q_1(n)$, *then* $mi(Q) \le q_2(n)$, *where*

$$q_2(n) = \begin{cases} 5r^{n-6}+1, & \text{if } n \text{ is even,} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q_2(n)$ if and only if $Q \in Q_2(n)$, where

$$Q_2(n) = \begin{cases} Q_{2e}^{(1)}(n), Q_{2e}^{(2)}(n), Q_{2e}^{(3)}(n), Q_{2e}^{(4)}(n), & \text{if } n \text{ is even,} \\ B\left(1, \frac{n-1}{2}\right), Q_{2o}^{(1)}(7), Q_{2o}^{(2)}(7), Q_{2o}^{(3)}(7), Q_{2o}^{(4)}(7), & \text{if } n \text{ is odd,} \end{cases}$$

where $Q_2(n)$ is shown in **Figure 7** and **Figure 8**.

Theorem 2.10 ([8]) *If Q is a quasi-forest graph with* $n \ge 4$ *vertices having* $Q \notin \overline{Q}_1(n)$, *then* $mi(Q) \le \overline{q}_2(n)$, *where*

$$\overline{q}_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even,} \\ 5r^{n-5}, & \text{if } n \text{ is odd.} \end{cases}$$

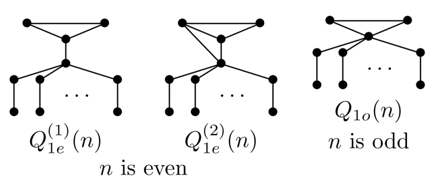

Figure 6. The graph $Q_1(n)$.

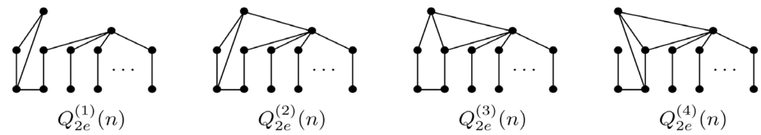

Figure 7. The graphs $Q_{2e}^{(i)}(n)$, $1 \le i \le 4$.

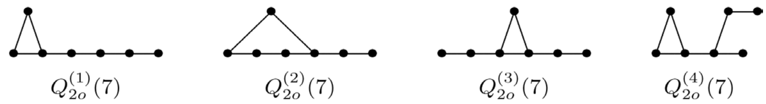

Figure 8. The graphs $Q_{2o}^{(i)}(7)$, $1 \le i \le 4$.

Furthermore, $mi(Q) = \overline{q}_2(n)$ if and only if $Q \in \overline{Q}_2(n)$, where

$$\overline{Q}_2(n) = \begin{cases} P_4 \cup \dfrac{n-4}{2} P_2, Q_1(n-2s) \cup sP_2, & \\ Q_2(6) \cup \dfrac{n-6}{2} P_2, C_3 \cup B\left(1, \dfrac{n-4-2s}{2}\right) \cup sP_2, & \text{if } n \text{ is even,} \\ Q_1(5) \cup \dfrac{n-5}{2} P_2, W \cup \dfrac{n-5}{2} P_2, C_5 \cup \dfrac{n-5}{2} P_2, & \text{if } n \text{ is odd,} \end{cases}$$

where W is a bow, that is, two triangles C_3 having one common vertex.

A graph is said to be *unicyclic* if it contains exactly one cycle. The result of the second largest number of maximal independent sets among all connected unicyclic graphs are described in Theorems 2.11.

Theorem 2.11 ([13]) *If U is a connected unicyclic graph of order* $n \ge 6$ *with* $U \ne C_5$ *and* $Q \notin Q_1(n)$*, then* $mi(G) \le u_2(n)$*, where*

$$u_2(n) = \begin{cases} 5r^{n-6} + 1, & \text{if } n \text{ is even,} \\ 3r^{n-5} + 2, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(G) = u_2(n)$ if and only if $U \in U_2(n)$, where

$$U_2(n) = \begin{cases} Q_{2e}^{(1)}(n), & \text{if } n \text{ is even,} \\ U_{2o}^{(1)}(n), U_{2o}^{(2)}(n), U_{2o}^{(3)}(n), U_{2o}^{(4)}(n), U_{2o}^{(5)}(n), U_{2o}^{(6)}(n), & \text{if } n \text{ is odd,} \end{cases}$$

where $U_{2o}^{(i)}(n)$ is shown in Figure 9.

3. Main Results

In this section, we determine the third largest values of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order $n \ge 7$, respectively. Moreover, the extremal graphs achieving these values are also determined.

Theorem 3.1 *If Q is a quasi-tree graph of odd order* $n \ge 7$ *having* $Q \notin Q_1(n), Q_2(n)$*, then* $mi(Q) \le 3r^{n-5} + 2$*. Furthermore, the equality holds if and only if* $Q = U_{2o}^{(i)}$*,* $1 \le i \le 6$*, where* $U_{2o}^{(i)}(n)$ *is shown in* Figure 9.

Proof. It is straightforward to check that $mi\left(U_{2o}^{(i)}(n)\right) = 3r^{n-5} + 2$, $1 \le i \le 6$. Let Q be a quasi-tree graph of odd order $n \ge 7$ having $Q \notin Q_1(n), Q_2(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \ge 3r^{n-5} + 2$. If Q is a tree, by Theorems 2.1, 2.3 and $Q \notin Q_2(n)$, we have that

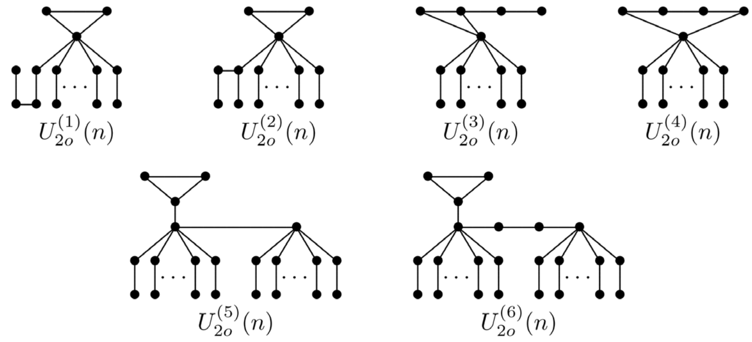

Figure 9. The graphs $U_{2o}^{(i)}(n)$, $1 \le i \le 6$.

$3r^{n-5} + 2 \le mi(Q) \le t_2(n) = 3r^{n-5} + 1$. This is a contradiction.

Suppose that Q contains at least two cycles and x is the vertex such that $Q - x$ is a tree. Then $\deg_Q(x) \ge 3$. By Lemma 1.1, Theorems 2.1 and 2.2, $3r^{n-5} + 2 \le mi(Q) \le mi(Q-x) + mi(Q - N_Q[x]) \le r^{(n-1)-2} + 1 + r^{(n-4)-1} = 3r^{n-5} + 1$, which is a contradiction. We obtain that Q is a connected unicyclic graph, thus the result follows from Theorem 2.11.

Theorem 3.2 *If Q is a quasi-tree graph of even order* $n \ge 8$ *having* $Q \notin Q_1(n), Q_2(n)$, *then* $mi(Q) \le 5r^{n-6}$. *Furthermore, the equality holds if and only if* $Q = \mathbb{Q}'(8)$, $\mathbb{Q}''(8)$, $\mathbb{Q}'''(10)$, $Q_{3e}^{(i)}(n)$, $1 \le i \le 12$, *where* $\mathbb{Q}'(8)$, $\mathbb{Q}''(8)$, $\mathbb{Q}'''(10)$ *and* $Q_{3e}^{(i)}(n)$ *are shown in* **Figure 10**.

Proof. It is straightforward to check that $mi(\mathbb{Q}'(8)) = mi(\mathbb{Q}''(8)) = 10$, $mi(\mathbb{Q}'''(10)) = 20$ and $mi(Q_{3e}^{(i)}(n)) = 5r^{n-6}$, $1 \le i \le 12$. Let Q be a quasi-tree graph of even order $n \ge 8$ having $Q \notin Q_1(n), Q_2(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \ge 5r^{n-6}$. If Q is a tree, by Theorem 2.1, we have that $5r^{n-6} \le mi(Q) \le t_1(n) = r^{n-2} + 1$. This is a contradiction, so Q contains at least one cycle. Let x be the vertex such that $Q - x$ is a tree. Then x is on some cycle of Q, it follows that $\deg_Q(x) \ge 2$. In addition, by Lemma 1.1, Theorems 2.2 and 2.5, $mi(Q-x) \ge 5r^{n-6} - r^{(n-3)-1} = 3r^{n-6} = t_3(n-1)$. We consider the following three cases.

Case 1. $Q - x \in T_1(n-1)$. If $\deg_Q(x) \ge 6$ then $Q - N_Q[x]$ is a forest with at most $n-7$ vertices, by Lemma 1.1, Theorems 2.1 and 2.2, $5r^{n-6} \le mi(Q) \le mi(Q-x) + mi(Q - N_Q[x]) \le r^{(n-1)-1} + r^{(n-7)-1} = 9r^{n-8}$. This is a contradiction. So we assume that $2 \le \deg_Q(x) \le 5$.

- $\deg x = 2$. There are 6 possibilities for graph Q. See **Figure 11**. Note that $Q_1^* = Q_1(n)$. By simple calculation, we have that $mi(Q_i^*) \le r^{n-2} + 1$ for $2 \le i \le 6$, a contradiction to $mi(Q) \ge 5r^{n-6}$.
- $\deg x = 3$. Suppose that there exists an isolated vertex y in $Q - N_Q[x]$ and $Q - N_Q[x] - y \notin F_1(n-5)$, then $mi(Q) \le mi(Q-x) + mi(Q - N_Q[x]) < r^{(n-1)-1} + r^{(n-4)-1-1} = 5r^{n-6}$. Hence there are 4 possibilities for graph Q. See **Figure 12**.

Note that $Q_8^* = Q_{2e}^{(2)}(n)$, $Q_9^* = Q_{3e}^{(7)}(n)$ and $Q_{10}^* = Q_{1e}^{(2)}(n)$. By simple calculation, we have $mi(Q_7^*) = r^{n-2} + 1$, a contradiction to $mi(Q) \ge 5r^{n-6}$.

- $4 \le \deg x \le 5$. Since $Q - N_Q[x]$ is a forest of odd order $n-5$ or even or-

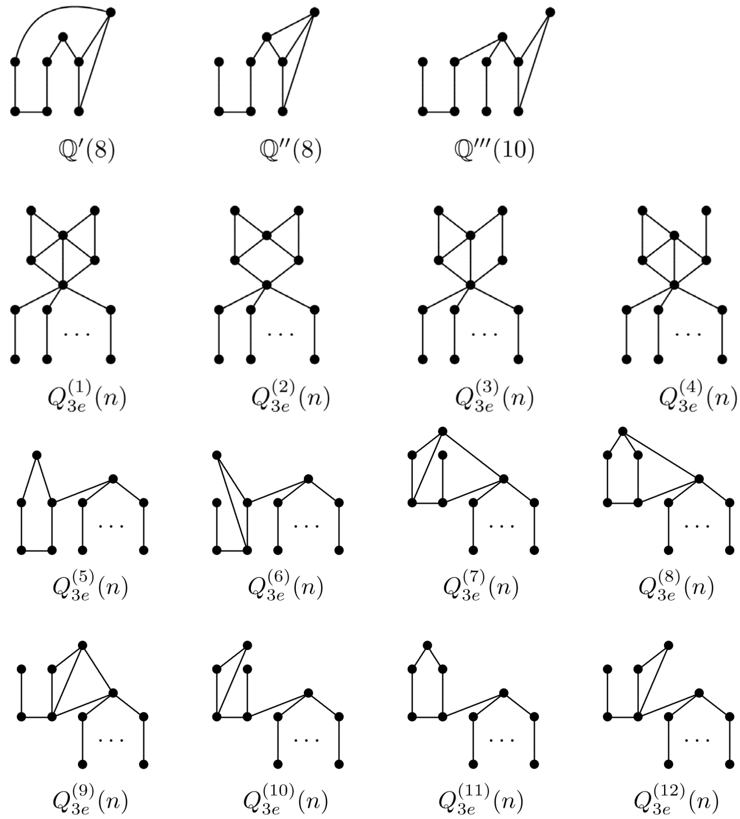

Figure 10. The graphs $\mathbb{Q}'(8)$, $\mathbb{Q}''(8)$, $\mathbb{Q}'''(10)$ and $Q_{3e}^{(i)}(n)$, $1 \le i \le 12$.

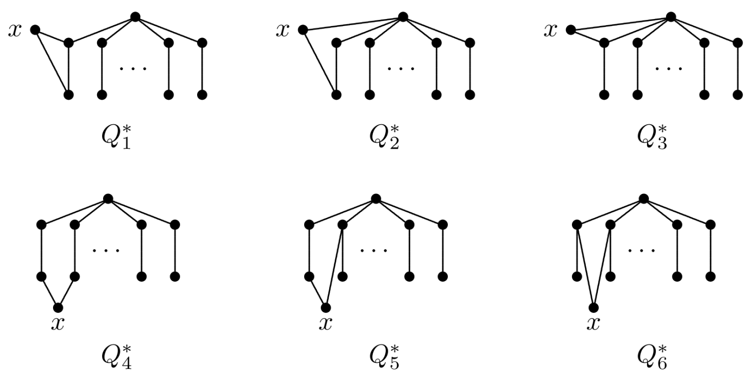

Figure 11. The graphs Q_i^*, $1 \le i \le 6$.

der $n-6$, by Lemma 1.1, Theorems 2.1 and 2.2, we have $5r^{n-6} \le mi(Q) \le mi(Q-x) + mi(Q-N_Q[x]) \le r^{n-2} + r^{n-6} = 5r^{n-6}$. The equalities holding imply that $Q-x = T_1(n-1)$ and $Q-N_Q[x] = F_1(n-5)$ or $F_1(n-6)$. Hence we obtain that $Q = Q_{3e}^{(i)}(n)$, $1 \le i \le 4$.

Case 2. $Q-x \in T_2(n-1)$. If $\deg_Q(x) \ge 4$ then $Q-N_Q[x]$ is a forest with at most $n-5$ vertices, by Lemma 1.1, Theorems 2.2 and 2.3, we have that $5r^{n-6} \le mi(Q) \le mi(Q-x) + mi(Q-N_Q[x]) \le 3r^{(n-1)-5} + 1 + r^{(n-5)-1} = 4r^{n-6} + 1$. This is a contradiction. So we assume that $2 \le \deg_Q(x) \le 3$.

- $\deg x = 2$. Suppose that $Q-N_Q[x] \notin F_1(n-3)$, by Lemma 1.1, Theorems 2.3 and 2.4, we have that

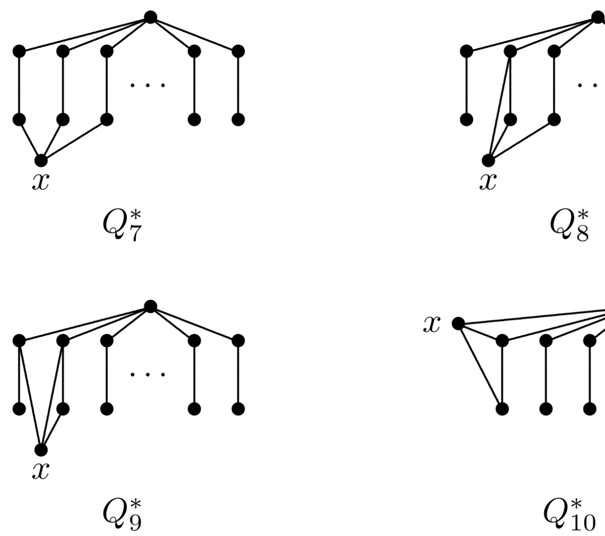

Figure 12. The graphs Q_i^*, $7 \le i \le 10$.

$5r^{n-6} \le mi(Q) \le mi(Q-x) + mi(Q-N_Q[x]) \le 3r^{(n-1)-5} + 1 + 7r^{(n-3)-7} = 19r^{n-10} + 1$. The equalities holding imply that $n=10$, that is, $Q-x=T_2(9)$ and $Q-N_Q[x]=F_2(7)$. Hence we obtain that $Q=\mathbb{Q}'''(10)$. Now we assume that $Q-N_Q[x]\in F_1(n-3)$. There are 7 possibilities for graph *Q*. See Figure 13.

Note that $Q_{11}^*=Q_{2e}^{(1)}(n)$, $Q_{12}^*=Q_{3e}^{(5)}(n)$ and $Q_{13}^*=Q_{3e}^{(6)}(n)$. By simple calculation, we have $mi(Q_i^*)\le r^{n-2}+2$ for $14\le i\le 17$, a contradiction to $mi(Q)\ge 5r^{n-6}$ when $n\ge 10$. In addition, $r^{8-2}+2=mi(Q_{17}^*)=5r^{8-6}$ when $n=8$, it follows that $Q=Q_{3e}^{(6)}(8)$.

- $\deg x=3$. Suppose that $Q-N_Q[x]\notin F_1(n-4)$, by Lemma 1.1, Theorems 2.3 and 2.4, we have that

$5r^{n-6} \le mi(Q) \le mi(Q-x) + mi(Q-N_Q[x]) \le 3r^{(n-1)-5} + 1 + 3r^{(n-4)-4} = 9r^{n-8} + 1$. The equalities holding imply that $n=8$, that is, $Q-x=T_2(7)$ and $Q-N_Q[x]=F_2(4)$. Hence we obtain that $Q=\mathbb{Q}'(8),\mathbb{Q}''(8)$. Now we assume that $Q-N_Q[x]\in F_1(n-4)$. Since $Q-x\in T_2(n-1)$ and $Q-N_Q[x]\in F_1(n-4)$, it follows that $Q=Q_{2e}^{(i)}(n)$, $2\le i\le 4$, a contradiction to $Q\notin Q_2(n)$.

Case 3. $Q-x\in T_3(n-1)$. Since $Q-N_Q[x]$ is a forest with at most $n-3$ vertices, by Lemma 1.1, Theorems 2.2 and 2.5, we have $5r^{n-6} \le mi(Q) \le mi(Q-x) + mi(Q-N_Q[x]) \le 3r^{(n-1)-5} + r^{(n-3)-1} = 5r^{n-6}$. The equalities holding imply that $Q-x\in T_3(n-1)$ and $Q-N_Q[x]\in F_1(n-3)$ or $F_1(n-4)$. For the case that $Q-N_Q[x]\in F_1(n-4)$, we obtain that $Q=Q_{3e}^{(i)}$, $7\le i\le 9$. For the other case that $Q-N_Q[x]\in F_1(n-3)$ There are 7 possibilities for graph *Q*. See Figure 14.

Note that $Q_{18}^*=Q_{3e}^{(10)}(n)$, $Q_{19}^*=Q_{3e}^{(11)}(n)$ and $Q_{20}^*=Q_{3e}^{(12)}(n)$. By simple calculation, we have $mi(Q_i^*)\le r^{n-2}+1$ for $21\le i\le 24$, a contradiction to $mi(Q)\ge 5r^{n-6}$.

In the following, we will investigate the same problem for quasi-forest graphs.

Theorem 3.3 *If Q is a quasi-forest graph of odd order* $n\ge 7$ *having*

$Q \notin \overline{Q}_1(n), \overline{Q}_2(n)$, then $mi(Q) \le 9r^{n-7}$. Furthermore, the equality holds if and only if $Q = \overline{Q}_{3o}^{(i)}(n)$, $1 \le i \le 4$, where $\overline{Q}_{3o}^{(i)}(n)$ is shown in Figure 15.

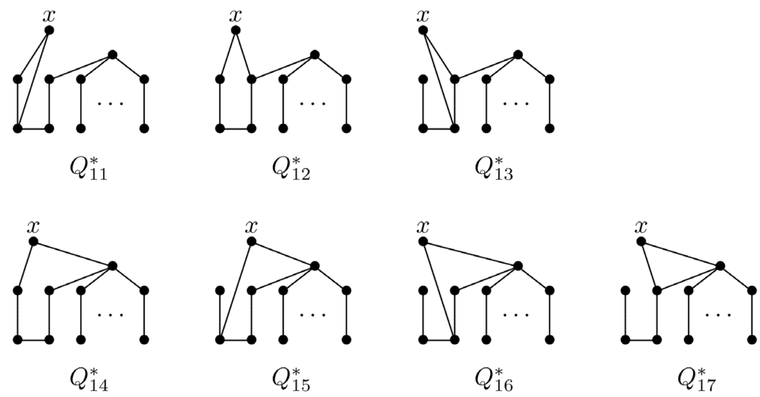

Figure 13. The graphs Q_i^*, $11 \le i \le 17$.

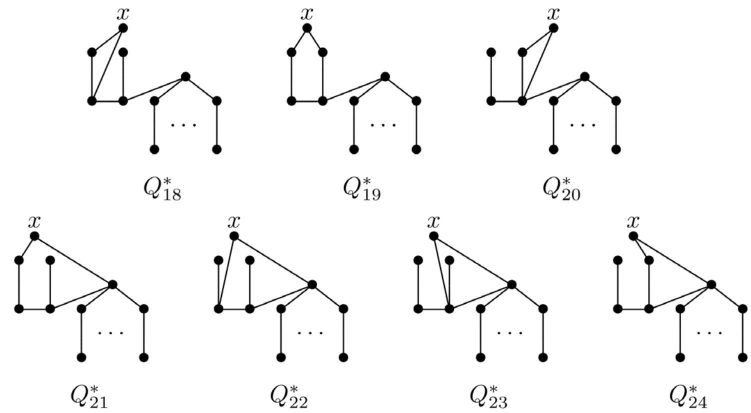

Figure 14. The graphs Q_i^*, $18 \le i \le 24$.

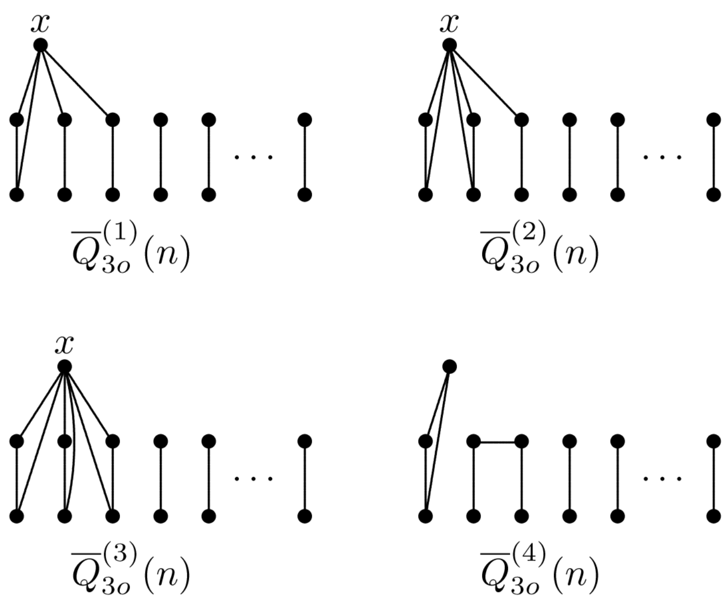

Figure 15. The graphs $\overline{Q}_{3o}^{(i)}(n)$, $1 \le i \le 4$.

Proof. It is straightforward to check that $mi\left(\bar{Q}_{3o}^{(i)}(n)\right)=9r^{n-7}$, $1\le i\le 4$. Let Q be a quasi-forest graph of odd order $n\ge 7$ having $Q\notin \bar{Q}_1(n),\bar{Q}_2(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q)\ge 9r^{n-7}$. If Q is a forest, by Theorem 2.2, we have that $9r^{n-7}\le mi(Q)\le f_1(n)=r^{n-1}$. This is a contradiction, so Q contains at least one cycle. Let x be a vertex such that $Q-x$ is a forest. Then x is on some cycle of Q, it follows that $\deg_Q(x)\ge 2$ and $Q-N_Q[x]$ is a forest with at most $n-3$ vertices. By Lemma 1.1, Theorem2.2 and 2.6, we obtain that $mi(Q-x)\ge mi(Q)-mi(Q-N_Q[x])\ge 9r^{n-7}-r^{n-3}=5r^{n-7}=f_3(n-1)$. We consider the following three ases.

Case 1. $Q-x\in F_1(n-1)$. If $\deg_Q(x)\ge 7$ then $Q-N_Q[x]$ is a forest with at most $n-8$ vertices, by Lemma 1.1 and Theorem 2.2, we have that
$9r^{n-7}\le mi(Q)\le mi(Q-x)+mi(Q-N_Q[x])\le r^{n-1}+r^{(n-8)-1}=17r^{n-9}$. This is a contradiction. So we assume that $2\le \deg_Q(x)\le 6$. There are 9 possibilities for graph Q. See Figure 16.

Note that $\bar{Q}_1^*\in\bar{Q}_1(n)$, $\bar{Q}_2^*\in\bar{Q}_2(n)$, $\bar{Q}_3^*\in\bar{Q}_3(n)$, $\bar{Q}_4^*=\bar{Q}_{3o}^{(1)}(n)$, $\bar{Q}_5^*=\bar{Q}_{3o}^{(2)}(n)$, $\bar{Q}_7^*=\bar{Q}_{3o}^{(3)}(n)$. By simple calculation, we have $mi\left(\bar{Q}_i^*\right)\le 17r^{n-9}$, $i=6,8,9$, a contradiction to $mi(Q)\ge 9r^{n-7}$.

Case 2. $Q-x=F_2(n-1)$. If $\deg_Q(x)\ge 3$ then $Q-N_Q[x]$ is a forest with at most $n-4$ vertices, by Lemma 1.1, Theorems 2.2 and 2.4, we have that
$9r^{n-7}\le mi(Q)\le mi(Q-x)+mi(Q-N_Q[x])\le 3r^{(n-1)-4}+r^{(n-4)-1}=4r^{n-5}$. This is a contradiction. So we assume that $\deg_Q(x)=2$. There are 5 possibilities for graph Q. See Figure 17.

Note that $\bar{Q}_{10}^*=\bar{Q}_2(n)$, $\bar{Q}_{12}^*=\bar{Q}_2(n)$, $\bar{Q}_{14}^*=\bar{Q}_{3o}^{(4)}(n)$. By simple calculation, we have $mi\left(\bar{Q}_i^*\right)\le 3r^{n-5}+1$, $i=11,13$, a contradiction to $mi(Q)\ge 9r^{n-7}$.

Case 3. $Q-x\in F_3(n-1)$. Since $Q-N_Q[x]$ is a forest with at most $n-3$ vertices, by Lemma 1.1, Theorems 2.2 and 2.6, we have that
$9r^{n-7}\le mi(Q)\le mi(Q-x)+mi(Q-N_Q[x])\le 5r^{(n-1)-6}+r^{n-3}=9r^{n-7}$. The equali-

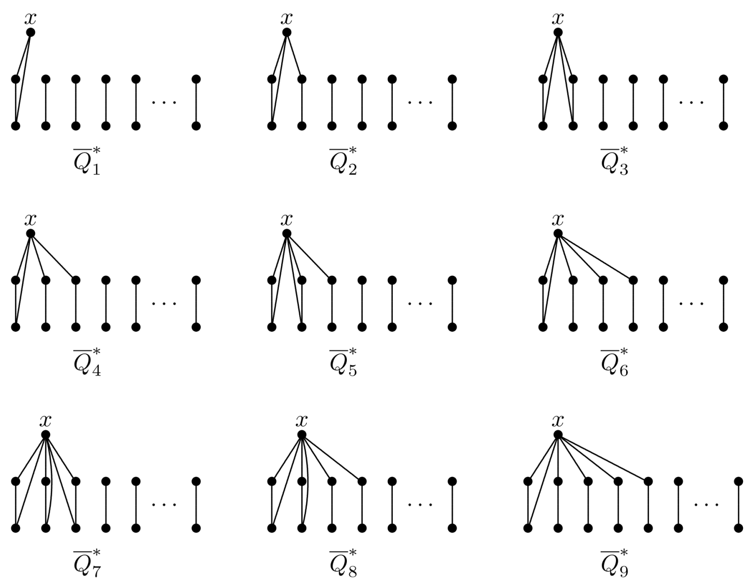

Figure 16. The graphs $\bar{Q}_i^*$, $1\le i\le 9$.

ties holding imply that $Q-x\in F_3(n-1)$ and $Q-N_Q[x]\in F_1(n-3)$. There are 3 possibilities for graph Q. See Figure 18.

Note that $\bar{Q}_{17}^{*}=\bar{Q}_{3o}^{(1)}(n)$. By simple calculation, we have $mi(\bar{Q}_i^{*})=8r^{n-7}$, $15\le i\le 16$, a contradiction to $mi(Q)\ge 9r^{n-7}$.

Theorem 3.4 *If Q is a quasi-forest graph of even order $n\ge 8$ having $Q\notin \bar{Q}_1(n),\bar{Q}_2(n)$, then $mi(Q)\le 11r^{n-8}$. Furthermore, the equality holds if and only if $Q=Q_2(8)\cup\frac{n-8}{2}P_2$.*

Proof. It is straightforward to check that $mi\left(Q_2(8)\cup\frac{n-8}{2}P_2\right)=11r^{n-8}$. Let Q be a quasi-forest graph of even order $n\ge 8$ having $Q\notin \bar{Q}_1(n),\bar{Q}_2(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q)\ge 11r^{n-8}$. If Q is a forest, by Theorems 2.2, 2.4, 2.6, 2.8 and 2.10, we have that

$11r^{n-8}\le mi(Q)\le f_3(n)=5r^{n-6}$. This is a contradiction, so Q contains a coponent $\hat{Q}$ with at least one cycle.

Let $|\hat{Q}|=s$. Suppose that $Q-\hat{Q}\ne\frac{n-s}{2}P_2$. Since $\hat{Q}$ is not a tree and $Q\notin \bar{Q}_1(n),\bar{Q}_2(n)$, by Lemma 1.2, Theorems 2.2, 2.4 and 2.7, we have that

$$
\begin{aligned}
mi(Q) &= mi(\hat{Q})\cdot mi(Q-\hat{Q})\\
&\le \begin{cases} 3r^{s-4}\cdot 3r^{(n-s)-4}, & \text{if } s\ge 4 \text{ is even},\\ 3\cdot 7r^{(n-3)-7}, & \text{if } s=3,\\ (r^{s-1}+1)\cdot r^{(n-s)-1}, & \text{if } s\ge 5 \text{ is odd},\end{cases}\\
&\le \begin{cases} 9r^{n-8}, & \text{if } s\ge 4 \text{ is even},\\ 21r^{n-10}, & \text{if } s=3,\\ 5r^{n-6}, & \text{if } s\ge 5 \text{ is odd},\end{cases}\\
&< 11r^{n-8},
\end{aligned}
$$

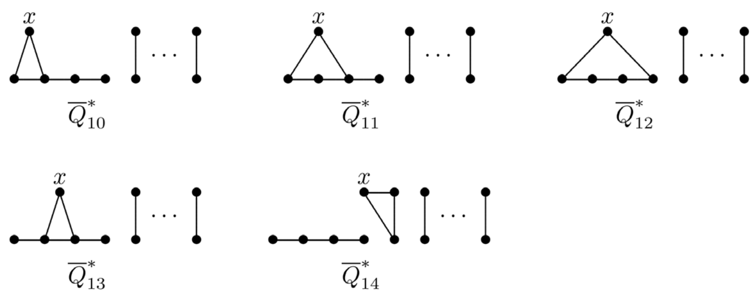

Figure 17. The graphs $\bar{Q}_i^{*}$, $10\le i\le 14$.

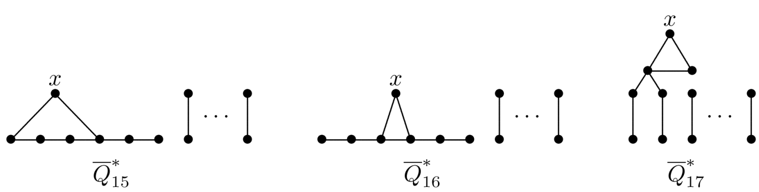

Figure 18. The graphs $\bar{Q}_i^{*}$, $15\le i\le 17$.

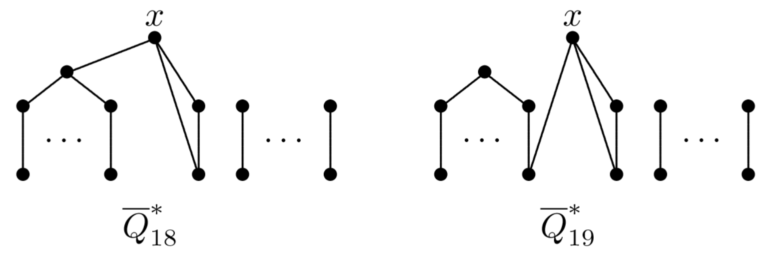

Figure 19. The graphs $\overline{Q}_i^*$, $18 \le i \le 19$.

which is a contradiction. Hence we obtain that s is even and $Q - \hat{Q} = \frac{n-s}{2} P_2$. Let x be the vertex in $\hat{Q}$ such that $\hat{Q} - x$ is a forest and $w(\hat{Q} - x)$ be the number of components of $\hat{Q} - x$. We consider the following two cases.

Case 1. $w(\hat{Q} - x) = 1$. Then $\hat{Q}$ is a quasi-tree graph. Since $Q \notin \overline{Q}_1(n), \overline{Q}_2(n)$ it follows that $s \ge 8$. By Lemma 1.2 and Theorem 2.9, it follows that $mi(Q) = (5r^{s-6} + 1) \cdot r^{n-s} = 5r^{n-6} + r^{n-s} \le 11r^{n-8}$. The equality holding imply that $s = 8$. In conclusion, $Q = Q_2(8) \cup \frac{n-8}{2} P_2$.

Case 2. $w(\hat{Q} - x) \ge 2$. Then $\deg x \ge 3$. In addition, suppose that $Q - N_Q[x]$ has a isolated vertex or $\deg_Q(x) \ge 4$, by Lemma 1.1 and Theorem 2.2, we have that $11r^{n-8} \le mi(Q) \le mi(Q - x) + mi(Q - N_Q[x]) \le r^{(n-1)-1} + r^{(n-5)-1} = 5r^{n-6}$. This is a contradiction, hence, we have that $\deg_Q(x) = 3$ and $Q - N_Q[x]$ has no isolated vertex. For the case that $Q - x \notin F_1(n-1)$, by Lemma 1.1, Theorems 2.2 and 2.4, we have that $11r^{n-8} \le mi(Q) \le mi(Q - x) + mi(Q - N_Q[x]) \le 7r^{(n-1)-7} + r^{n-4} = 11r^{n-8}$. The equalities holding imply that $Q - x \in F_2(n-1)$ and $Q - N_Q[x] \in F_1(n-4)$. Since $w(\hat{Q} - x) \ge 2$, there no such graph Q. For the other case that $Q - x \in F_1(n-1)$, there are 2 possibilities for graph Q. See Figure 19.

Note that $\overline{Q}_{18}^* = \overline{Q}_2(n)$ and $\overline{Q}_{19}^* = Q_{2e}^{(1)}(8) \cup \frac{n-8}{2} P_2$ when $s = 8$. On the other hand, $mi(\overline{Q}_{19}^*) \le 21r^{n-10}$ when $s \ge 10$, a contradiction to $mi(Q) \ge 11r^{n-8}$.

References

[1] Moon, J.W. and Moser, L. (1965) On Cliques in Graphs. *Israel Journal of Mathematics*, **3**, 23-28. https://doi.org/10.1007/BF02760024

[2] Jou, M.J. and Chang, G.J. (1995) Survey on Conunting Maximal Independent Sets. In: Tangmance, S. and Schulz, E., Eds., *Proceedings of the Second Asian Mathematical Conference*, World Scientific, Singapore, 265-275.

[3] Jin, Z. and Li, X. (2008) Graphs with the Second Largest Number of Maximal Independent Sets. *Discrete Mathematics*, **308**, 5864-5870. https://doi.org/10.1016/j.disc.2007.10.032

[4] Jou, M.J. and Lin, J.J. (2009) Trees with the Second Largest Number of Maximal Independent Sets. *Discrete Mathematics*, **309**, 4469-4474. https://doi.org/10.1016/j.disc.2009.02.007

[5] Jin, Z. and Yan, H.F. (2009) Trees with the Second and Third Largest Number of Maximal Independent Sets. *Ars Combinatoria*, **93**, 341-351.

[6] Liu, H. and Lu, M. (2008) On the Spectral Radius of Quasi-Tree Graphs. *Linear Algebra and Its Applications*, **428**, 2708-2714. https://doi.org/10.1016/j.laa.2007.12.017

[7] Lin, J.J. (2010) Quasi-Tree Graphs with the Largest Number of Maximal Independent Sets. *Ars Combinatoria*, **97**, 27-32.

[8] Lin, J.J. (2013) Quasi-Tree Graphs with the Second Largest Number of Maximal Independent Sets. *Ars Combinatoria*, **108**, 257-267.

[9] Sagan, B.E. and Vatter, V.R. (2006) Maximal and Maximum Independent Sets in Graphs with at Most r Cycles. *Journal of Graph Theory*, **53**, 283-314. https://doi.org/10.1002/jgt.20186

[10] Jou, M.J. (1991) The Number of Maximal Independent Sets in Graphs. Master Thesis, Department of Mathematics, National Central University, Taiwan.

[11] Jou, M.J. and Chang, G.J. (1997) Maximal Independent Sets in Graphs with at Most One Cycle. *Discrete Applied Mathematics*, **79**, 67-73. https://doi.org/10.1016/S0166-218X(97)00033-4

[12] Jou, M.J. and Lin, J.J. (2010) Forests with the Third Largest Number of Maximal Independent Sets. *Ling Tung Journal*, **27**, 203-212.

[13] Lin, J.J. and Jou, M.J. The Numbers of Maximal Independent Sets in Connected Unicyclic Graphs. Utilitas Math., to Appear.