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Cake Filtration Equation Using $\tan^n\theta$ Reduction Method

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Abstract

This study was completed by an extensive mathematical analysis. New equation to sludge filtration processes has been proposed for use in routine laboratory. The equation has been suggested to replace Ademiluyi's cake filtration equation in view of the limitations of the latter. The new equation can be used for sludges whose compressibility factor is more than one but Ademiluyi's cake filtration equation can only be used for sludges whose compressibility coefficient is less than one. The new sludge filtration equation was derived using $\tan^n\theta$ reduction method. The generalized equation thus obtained resembles Ademiluyi's equation in the mode of parameter combination except the presence of summation notation in the new equation.

Keywords

$\tan^n\theta$ Method, Reduction Formula, Sludge Filtration Equation, Piezometric Positions and Compressibility Factor

1. Introduction

Sludge is produced from the treatment of waste water in on-site (e.g. septic tank) and off-site (e.g. activated sludge) systems. This is inherently so because a primary aim of waste water treatment is removing solids from the waste water. Sludge is also produced from the treatment of storm water, although it is likely to be less organic in nature compared to waste water sludge [1].

Bucket latrine and vault latrines store faecal sludge, which needs to be collected and treated. These two types of latrine are not discussed in this research work because no treatment is involved at the latrines.

The characteristics of sludge vary widely from relatively fresh faecal materials generated in bucket latrines to sludge which has undergone bacterial decomposition for over a year in a double pit latrine. The treatment required is therefore dependent on the characteristics of the sludge. Sludge maybe contaminated with heavy metals and other pollutants, especially when industrial waste are disposed into the sewer. Prevention of contamination of the sludge by industrial wastes is preferable. A conversion process to produce oil from sludge has been developed, which can be suitable for heavily contaminated sludge. The costs of treatment of sludge are generally of the same order as the costs of removing the sludge from the waste water. Also sludge volume is usually less than 1% of the total plant influent, sludge handling costs are 21% - 50% of total plant operating and maintenance costs. Sludge contains solids, oil, fat, protein, phosphates, carbohydrate, nitrogen, water, etc with a specific gravity of 1.02, 1.06 (for organic fraction) and 2.5 for inorganic fraction [2]. [3] assumed that the pressure across the cake has no impact on specific cake resistance. He stressed that the void fraction for most cakes can be significantly affected by pressure, because the cake is often compressible. He concluded that because the pressure drop changes with time the void fraction, can also be function of time, at least in principle. The sludge generated by waste treatment must undergo dewatering before disposal to the environment. This can be achieved by filtration. Filtration is the process of separating a heterogeneous mixture of fluid and particles of solid by means of a filter medium which permits the passage of the filtrate but retains the particles during the process.

Sewage sludge treatment describes the processes used to manage and dispose of sewage sludge produced during sewage treatment. Sludge is mostly water with lesser amount of solid materials removed from liquid sewage.

The various types of sludge treatment are; stabilization, thickening, dewatering, drying and incineration. The latter is most costly, because fuel is needed and air pollution control requires extensive treatment of the combustion gases. It can be used when the sludge is heavily contaminated with heavy metals or other undesirable pollutants. The organic carbon in the sludge, once stabilized is desirable as a soil conditioner, because it provides improved soil structure for plant. Also, faecal sludge contains essential nutrients (nitrogen and phosphorus) and is potentially beneficial as fertilizers for plants. Biosolids produced from dried or treated sludge act as a fertilizer for crop harvesting. The treated sludge can also be used as top dressing on golf course fairways and a soil substitute in final landfill cover. Also recovers struvite in the form of crystalline pellets from sludge dewatering stream, the resulting crystalline product is sold to the agriculture turf and ornamental plants sectors as fertilizer under the registered trade name “Crystal Green” [4].

In the course of waste water treatment process, some amount of sludge is usually generated. The bulky and compressible nature of this waste water solid coupled with its high water content of about 97.5% [5] make it necessary to undergo dewatering. Sludge dewatering is an expensive venture especially if the design Engineer recommends mechanical dewatering. Among the mechanical filter in use, vacuum filter has been found to give filter cake which facilitates easier handling and transportation [6].

Sludge dewatering is an important process finding application in many manufacturing industries and in plants designed for water and waste treatments. Three main concepts have been suggested to evaluate the filterability of sludge. These include specific resistance [7], filtration coefficient [8] and sludge dewaterability number [9]. The concept of specific resistance is the oldest among the three. Carman pioneered the introduction of the concept of specific resistance for measuring sludge filterability by suggesting the equation:

$$t = \frac{rcvf^2}{2PA^2} + \frac{\mu RV_F}{PA} \tag{1}$$

Equation (1) has been modified several times [10] [11] [12]. Various reasons so far advanced for such modifications have been enumerated and discussed [13].

One of the ways of achieving sludge dewatering process is through vacuum filtration which is a mechanical process. Many researchers, [14] [15] etc have proposed different models aimed at improving the performance of the vacuum filtration process but the exact mathematical formulation governing sludge filtration process is not yet known. Even the commonly used equation as proposed by Carman has been described to be in error [16]. Carman's equation was formulated from a combination of Poiseuille's and Darcy's laws. It is therefore applicable to rigid rather than compressible cakes. Also the specific resistance parameter in Carman's equation is usually treated as a constant parameter. This parameter is expected to be a variable since it is a function of cake porosity. The cake porosity decreases from the sludge cake surface to the cake layer closet to the filter septum, there should be a corresponding decrease of specific resistance from the sludge-cake interface to the cake layer closet to the filter septum [17]; [18]. The concept of average specific resistance proposed by Ruth has been descried to be inapplicable [19]. Also the relationship between the filtrate volume and area of filtration as proposed by Carman has been experimentally found to be invalid [20].

The parabolic relationship between filtrate volume and time of filtration does not hold throughout the filtration cycle [21]. The variable head and compressibility factor believed to influence sludge dewatering have not been accounted for in Carman's equation [22]. In the light of problems mentioned above, an equation to describe sludge filtration process was derived [23]. The equation described resistance as a local variable. This equation is given as

$$t = \alpha\beta^{s\gamma-(s-2)}\left(\frac{\beta}{s+1}\left(\varphi^{S+1} - \varphi_o^{S+1}\right) + \beta + YH_0\right)e_n\frac{1+\varphi}{1+\varphi} \tag{2}$$

Equation (2) has some limitations. Ademiluyi's theory is not applicable during the early period in cake filtration before the vacuum has assumed a constant value and before enough cake has formed to become the dominant filter resistance. Also his theory is only applicable for sludge whose Terzaghi's compressibility coefficient has been found to be less than 1 (one) (Ademiluyi, 1985). That is, the equation can only been used for sludges whose Terzaghi's compressibility coefficient is very much less than 1. The research in sludge filtration should continue until an acceptable equation which governs sludge filtration phenomenon is derived [11]. Little or no studies have attempted to integrate the aforementioned problems to derive an acceptable equation which governs sludge filtration phenomenon. A large body of research on sludge filtration among the concept of specific resistance, filtration coefficient, sludge dewaterability number, the variable head and compressibility factor, the parabolic relationship between filtrate volume and time of filtration, the average resistance and also the relationship between the filtrate volume and area of filtration models with no agreement regarding which one is better. [24] stated that in other to obtain a valid sludge filtration equation, compressibility attribute of the sludge cake in question should be properly accounted for. The objective of this study is to develop a new filtration equation which will solve the problem of compressibility attribute of sludge called compressibility factor 's' in the cake filtration theory using $\tan^n \theta$ reduction method. An equation of this nature would be useful in the treatment of sludge since it contains the most important filtration parameters

2. Mathematical Formulation to Describe the New Proposed Filtration Equation

The basic equation Carman and Tiller's equations describing sludge dewatering is given as

$$\frac{I\mathrm{d}v}{A\mathrm{d}t} = \frac{P}{\mu\left(Rw + R_m\right)} \tag{3}$$

where p is the vacuum pressure (kN/m^2), v is the volume of filtration (m^3), t is the time taken to obtain filtrate (s), A is the area of filtration (m^2), R_m is the medium or septum resistance (m^{-2}), w is the mass of dry solid deposited per unit area (kg/m^2), R is the average specific resistance (m/kg) and μ is the viscosity of filtrate (poise).

Equation (3) is frequently the starting point for development of filtration equations. Accounting for the hydrostatic pressure and compressibility coefficient and by appropriate substitution Ademiluyi transformed Equation (3) to

$$\frac{-\mathrm{d}H}{\mathrm{d}t} = \frac{\beta + \gamma H}{\alpha\left(H_0 - H\right)\left(H\right)^s} \tag{4}$$

And gave the solution of Equation (4) as

$$t = 2\beta^s \alpha \gamma^{-s-2} \int_{\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}}^{\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2}} \left[\beta \tan^{2s+3}\theta - \gamma H_0 \tan^{2s+1}\theta\right] \mathrm{d}\theta \tag{5}$$

where, H is the driving head at any time (t) in (m), β is the vacuum pressure (kN/m^2), H_0 is the initial head (m), γ is the specific weight of filtrate (N/m^3), exponent, s is the compressibility characteristic called the coefficient of compressibility (cm^2/g).

Solution of Equation (5) by $\tan^n\theta$ Reduction Method

Equation (5) can be integrated by reducing the expression to a simple integratable form by the application of $\tan^n\theta$ reduction method. In order to derive the new sludge filtration equation, the $\tan^n\theta$ method of reduction was employed.

Thus;

$$\int \tan^n \theta \mathrm{d}\theta = \frac{1}{n-1}\tan^{n-1}\theta - \int \tan^{n-2}\theta \mathrm{d}\theta \tag{6}$$

$$\int \tan^n \theta \mathrm{d}\theta = \frac{1}{n-1}\tan^{n-1}\theta - \left[\frac{1}{n-3}\tan^{n-3}\theta - \int \tan^{n-4}\theta \mathrm{d}\theta\right] \tag{7}$$

$$\int \tan^n \theta \mathrm{d}\theta = \frac{1}{n-1}\tan^{n-1}\theta - \frac{1}{n-3}\tan^{n-3}\theta + \int \tan^{n-4}\theta \mathrm{d}\theta \tag{8}$$

$$\int \tan^n \theta \mathrm{d}\theta = \frac{1}{n-1}\tan^{n-1}\theta - \frac{1}{n-3}\tan^{n-3}\theta - \frac{1}{n-5}\tan^{n-5}\theta + \int \tan^{n-6}\theta \mathrm{d}\theta \tag{9}$$

Therefore,

$$\int \tan^n \theta \mathrm{d}\theta = \sum_{i=0}^{\left[\frac{n}{2}\right]-1} \frac{(-1)^i}{n-(2i+1)}\tan^{n-(2i+1)}\theta + (-1)^{n/2}\ln|\cos\theta| \tag{10}$$

where $\left[\frac{n}{2}\right]$ is the greatest integer value of $\frac{n}{2}$, n is an odd number and $n \geq 3$, $n \in \mathbb{Z}^+$

Application of Equation (10), in solving Equation (5)

Thus,

$$t = 2\alpha\beta^s\gamma^{-s-2}\int_{\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}}^{\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2}} \left[\beta\tan^{2s+3}\theta - \gamma H_o\tan^{2s+1}\theta\right]\mathrm{d}\theta$$

Let

$$\theta_1 = \tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2} \tag{11}$$

$$\theta_0 = \tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2} \tag{12}$$

$$\frac{t}{2\alpha\beta^s\gamma^{-s-2}} = \int_{\theta_0}^{\theta_1}\left[\beta\tan^{2s+3}\theta - \gamma H_0\tan^{2s+1}\theta\right]\mathrm{d}\theta \tag{13}$$

Where s is a positive integer

$$\frac{t}{2\alpha\beta^s\gamma^{-s-2}} = \int_{\theta_0}^{\theta_1}\tan^{2s+1}\theta\left(\beta\tan^2\theta - \gamma H_0\right)\mathrm{d}\theta \tag{14}$$

$$\frac{t}{2\alpha\beta^s\gamma^{-s-2}} = \int_{\theta_0}^{\theta_1}\tan^{2s+1}\theta\left[\beta\left(\sec^2\theta - 1\right) - \gamma H_0\right]\mathrm{d}\theta \tag{15}$$

$$\frac{t}{2\alpha\beta^s\gamma^{-s-2}} = \int_{\theta_0}^{\theta_1}\left[\beta\tan^{2s+1}\theta\sec^2\theta - \left(\beta + \gamma H_0\right)\tan^{2s+1}\theta\right]\mathrm{d}\theta \tag{16}$$

$$\frac{t}{2\alpha\beta^s\gamma^{-s-2}} = \int_{\theta_0}^{\theta_1}\beta\tan^{2s+1}\theta\sec^2\theta\mathrm{d}\theta - \left(\beta + \gamma H_0\right)\int_{\theta_0}^{\theta_1}\tan^{2s+1}\theta\mathrm{d}\theta \tag{17}$$

$$\frac{t}{2\alpha\beta^s\gamma^{-s-2}} = \frac{\beta}{2s+2}\tan^{2s+2}\theta\Big|_{\theta_0}^{\theta_1} - \left(\beta + \gamma H_0\right)\int_{\theta_0}^{\theta_1}\tan^{2s+1}\theta\mathrm{d}\theta \tag{18}$$

$$\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}\frac{\beta}{2s+2}\left(\tan^{2s+2}\theta_1-\tan^{2s+2}\theta_0\right)-\left(\beta+\gamma H_0\right)\int_{\theta_0}^{\theta_1}\tan^{2s+1}\theta\mathrm{d}\theta \tag{19}$$

Let n in Equation (10) = $2s+1$ (20)

Substituting Equation (20), (19) in Equation (10) gives

$$\begin{aligned}&\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}\\&=\frac{\beta}{2s+2}\left(\tan^{2s+2}\theta_1-\tan^{2s+2}\theta_0\right)\\&-\left(\beta+\gamma H_0\right)\left[\sum_{i=0}^{\left[\frac{2s+1}{2}\right]-1}\frac{(-1)^{i}}{(2s+1)-(2i+1)}\tan^{2s+1-(2i+1)}\theta+(-1)^{2s+1/2}\ln\left|\cos\theta\right|\right]\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}&=\frac{\beta}{2s+2}\left(\tan^{2s+2}\theta_1-\tan^{2s+2}\theta_0\right)\\&-\left(\beta+\gamma H_0\right)\left[\sum_{i=0}^{[s+1/2]-1}\frac{(-1)^{i}}{2(s-i)}\tan^{2(s-i)}\theta\Big|_{\theta_0}^{\theta_1}+(-1)^{[s+1/2]}\ln\left|\cos\theta\right|\Big|_{\theta_0}^{\theta_1}\right]\end{aligned} \tag{22}$$

$$\begin{aligned}&\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}\\&=\frac{\beta}{2s+2}\left(\tan^{2s+2}\theta_1-\tan^{2s+2}\theta_0\right)\\&-\left(\beta+\gamma H_0\right)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^{i}}{2(s-i)}\left(\tan^{2(s-i)}\theta_1-\tan^{2(s-i)}\theta_0\right)+(-1)^{[s+1/2]}\ln\frac{\cos\theta_1}{\cos\theta_0}\right]\end{aligned} \tag{23}$$

Since $\theta_1=\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2}$

$$\theta_0=\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}$$

$$\left(\tan\theta_1\right)^{2s+2}=\left(\tan\left(\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{\frac{1}{2}}\right)\right)^{2s+2}=\left(\left(\frac{\gamma H}{\beta}\right)^{1/2}\right)^{2s+2}=\left(\frac{\gamma H}{\beta}\right)^{s+1} \tag{24}$$

Also

$$\left(\tan\theta_0\right)^{2s+2}=\tan\left(\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{\frac{1}{2}}\right)^{2s+2}=\left(\left(\frac{\gamma H_0}{\beta}\right)^{1/2}\right)^{2s+2}=\left(\frac{\gamma H_0}{\beta}\right)^{s+1} \tag{25}$$

Similarly,

$$\left(\tan\theta_1\right)^{2(s-i)}=\left(\tan\left(\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{\frac{1}{2}}\right)\right)^{2(s-i)}=\left(\left(\frac{\gamma H}{\beta}\right)^{1/2}\right)^{2(s-i)}=\left(\frac{\gamma H}{\beta}\right)^{s-i} \tag{26}$$

$$\left(\tan\theta_0\right)^{2(s-i)}=\left(\tan\left(\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}\right)\right)^{2(s-i)}=\left(\left(\frac{\gamma H_0}{\beta}\right)^{1/2}\right)^{2(s-i)}=\left(\frac{\gamma H_0}{\beta}\right)^{s-i} \tag{27}$$

Substituting Equations ((24)-(27)) in Equation (23) gives

$$\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}=\frac{\beta}{2s+2}\left[\left(\frac{\gamma H}{\beta}\right)^{s+1}-\left(\frac{\gamma H_0}{\beta}\right)^{s+1}\right]$$

$$-\left(\beta+\gamma H_0\right)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^{i}}{2(s-i)}\left(\left(\frac{\gamma H}{\beta}\right)^{s-i}-\left(\frac{\gamma H_0}{\beta}\right)^{s-i}\right)\right. \tag{28}$$

$$\left.+(-1)^{[s+1/2]}\ln\frac{\cos\left(\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2}\right)}{\cos\left(\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}\right)}\right]$$

Let

$$\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2}=\phi_1 \tag{29}$$

$$\cos\left(\tan^{-1}\left(\frac{\gamma H}{\beta}\right)^{1/2}\right)=\cos\phi_1 \tag{30}$$

$$\tan\phi_1=\left(\frac{\gamma H}{\beta}\right)^{1/2} \tag{31}$$

$$\tan^2\phi_1=\frac{\gamma H}{\beta} \tag{32}$$

Let

$$\frac{\gamma H}{\beta}=x \tag{33}$$

$$\tan^2\phi_1=x \tag{34}$$

$$\frac{\sin^2\phi_1}{\cos^2\phi_1}=x \tag{35}$$

$$\frac{1-\cos^2\phi_1}{\cos^2\phi_1}=x \tag{36}$$

$$\frac{1}{\cos^2\phi_1}-1=x \tag{37}$$

$$\frac{1}{\cos^2\phi_1}=1+x \tag{38}$$

$$\cos^2\phi_1=\frac{1}{1+x} \tag{39}$$

$$\cos\phi_1=\frac{1}{\sqrt{1+x}} \tag{40}$$

Also

$$\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}=\phi_0 \tag{41}$$

$$\cos\left(\tan^{-1}\left(\frac{\gamma H_0}{\beta}\right)^{1/2}\right)=\cos\phi_0 \tag{42}$$

$$\tan\phi_0=\left(\frac{\gamma H_0}{\beta}\right)^{1/2} \tag{43}$$

$$\tan^2\phi_0=\left(\frac{\gamma H_0}{\beta}\right) \tag{44}$$

Let

$$\frac{\gamma H_0}{\beta}=x_0 \tag{45}$$

$$\tan^2\phi_0=x_0 \tag{46}$$

$$\frac{\sin^2\phi_0}{\cos^2\phi_0}=x_0 \tag{47}$$

$$\frac{1-\cos^2\phi_0}{\cos^2\phi_0}=x_0 \tag{48}$$

$$\frac{1}{\cos^2\phi_0}-1=x_0 \tag{49}$$

$$\frac{1}{\cos^2\phi_0}=1+x_0 \tag{50}$$

$$\cos^2\phi_0=\frac{1}{1+x_0} \tag{51}$$

$$\cos\phi_0=\frac{1}{\sqrt{1+x_0}} \tag{52}$$

Substituting Equations ((40) and (52)) in Equation (28) gives

$$\begin{aligned}\frac{t}{2\alpha\beta^s\gamma^{-s-2}}=&\frac{\beta}{2s+2}\left[\left(\frac{\gamma H}{\beta}\right)^{s+1}-\left(\frac{\gamma H_0}{\beta}\right)^{s+1}\right]\\&-(\beta+\gamma H_0)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(\left(\frac{\gamma H}{\beta}\right)^{s-i}-\left(\frac{\gamma H_0}{\beta}\right)^{s-i}\right)\right.\\&\left.+(-1)^{[s+1/2]}\ln\frac{(1+x)^{1/2}}{(1+x_0)^{1/2}}\right]\end{aligned} \tag{53}$$

Substituting for $\frac{\gamma H}{\beta}=x$ and $\frac{\gamma H_0}{\beta}=x_0$ in Equation (53) gives

$$\begin{aligned}&\frac{t}{2\alpha\beta^s\gamma^{-s-2}}\\&=\frac{\beta}{2s+2}\left[x^{s+1}-x_0^{s+1}\right]-(\beta+\gamma H_0)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(x^{s-i}-x_0^{s-i}\right)\right.\\&\left.+\frac{1}{2}(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right]\end{aligned} \tag{54}$$

$$\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}=\frac{\beta}{2s+2}\left[x^{s+1}-x_0^{s+1}\right]-\beta\left(1+\frac{\gamma H_0}{\beta}\right)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(x^{s-i}-x_0^{s-i}\right)\right]+\frac{1}{2}\beta\left(1+\frac{\gamma H_0}{\beta}\right)\left[(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right] \quad (55)$$

Substituting for $\frac{\gamma H_0}{\beta}=x_0$ in Equation (55) gives

$$\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}=\frac{\beta}{2s+2}\left[x^{s+1}-x_0^{s+1}\right]-\beta(1+x_0)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(x^{s-i}-x_0^{s-i}\right)\right]+\frac{1}{2}\beta(1+x_0)\left[(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right] \quad (56)$$

$$\frac{t}{2\alpha\beta^{s}\gamma^{-s-2}}=\frac{1}{2}\left(\frac{\beta}{s+1}x^{s+1}-\frac{\beta}{s+1}x_0^{s+1}\right)-\beta(1+x_0)\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(x^{s-i}-x_0^{s-i}\right)\right]+\beta(1+x_0)\left[\frac{1}{2}(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right] \quad (57)$$

$$t=\frac{2\alpha\beta^{s}\gamma^{-s-2}}{2}\left(\frac{\beta}{s+1}x^{s+1}-\frac{\beta}{s+1}x_0^{s+1}\right)-2\alpha\beta^{s}\gamma^{-s-2}\left[\beta(1+x_0)\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(x^{s-i}-x_0^{s-i}\right)\right]+\frac{2\alpha\beta^{s}\gamma^{-s-2}}{2}\left[\beta(1+x_0)(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right] \quad (58)$$

$$t=\alpha\beta^{s}\gamma^{-s-2}\left(\frac{\beta}{s+1}x^{s+1}-\frac{\beta}{s+1}x_0^{s+1}\right)-2\alpha\beta^{s}\gamma^{-s-2}\left[\beta(1+x_0)\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{2(s-i)}\left(x^{s-i}-x_0^{s-i}\right)\right]+\alpha\beta^{s}\gamma^{-s-2}\left[\beta(1+x_0)(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right] \quad (59)$$

$$t=\alpha\beta^{s}\gamma^{-s-2}\left(\frac{\beta}{s+1}x^{s+1}-\frac{\beta}{s+1}x_0^{s+1}\right)-\alpha\beta^{s}\gamma^{-s-2}\left[\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}x^{s-i}\beta(1+x_0)-\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}x_0^{s-i}\beta(1+x_0)\right]+\alpha\beta^{s}\gamma^{-s-2}\left[\beta(1+x_0)(-1)^{[s+1/2]}\ln\left(\frac{1+x_0}{1+x}\right)\right] \quad (60)$$

$$t=\alpha\beta^{s}\gamma^{-s-2}\left(\frac{\beta}{s+1}x^{s+1}-\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}\beta(1+x_0)x^{s-i}-\beta(1+x_0)(-1)^{[s+1/2]}\ln(1+x)\right)+\alpha\beta^{s}\gamma^{-s-2}\left(-\frac{\beta}{s+1}x_0^{s+1}+\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}\beta(1+x_0)x_0^{s-i}+\beta(1+x_0)(-1)^{[s+1/2]}\ln(1+x_0)\right) \quad (61)$$

Substituting for

$$\alpha\beta^{s}\gamma^{-s-2}=k_1 \tag{62}$$

and

$$\beta\left(1+x_0\right)=k_2 \tag{63}$$

in Equation (61)

$$\begin{aligned}t=&k_1\left(\frac{\beta}{s+1}x^{s+1}-\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}x^{s-i}k_2-k_2(-1)^{[s+1/2]}\ln(1+x)\right)\\&+k_1\left(-\frac{\beta}{s+1}x_0^{s+1}+\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}x_0^{s-i}k_2+k_2(-1)^{[s+1/2]}\ln(1+x_0)\right)\end{aligned} \tag{64}$$

Equation (64) can be written in the form

$$t=k_1f(x)+k_1f(x_0) \tag{65}$$

where

$$f(x)=\frac{\beta}{s+1}x^{s+1}-\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}k_2x^{s-i}-k_2(-1)^{[s+1/2]}\ln(1+x) \tag{66}$$

$$f(x_0)=-\frac{\beta}{s+1}x_0^{s+1}+\sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^i}{(s-i)}k_2x_0^{s-i}+k_2(-1)^{[s+1/2]}\ln(1+x_0) \tag{67}$$

Equation (61) is the new sludge filtration equation obtained. It can also be written in the form in Equations ((64) and (65)).

3. Application of the New Equation in Sludge Filtration Theories

In applying the new Equation (61) in sludge filtration theories, it is expected that measured time, $t(s)$ are measured after the formation of the cake so that the septum resistance become negligible as it is required by the theory. The implication is that time t, was measured after the formation of the cake. Many parameters are involved in the new Equation (61) therefore data were stored in the computer using soft ware allowing later use of the soft ware to evaluate the data using the new Equation (61). The change in driving head H, can be measured directly or by measuring the filtrate volume to the drop in head. All parameters in Equation (61) were gotten from Ademiluyi, J.O, 1985 experimental results except alpha α, which was assumed and R^* which was determined.

Validation of the New Model

Equation (61) derived in this work shows that the plot of t versus $f(x)$ should yield a straight line of slope k_1 and intercept $k_1 f(x_0)$. The experimental apparatus, procedure and the use of the new Equation (61) in routine laboratory sludge dewatering has been described. This is the only way by which the new sludge filtration theory can be taken to describe sludge dewatering process. Another alternative could have been done by comparing predicted values of specific resistance parameter with actual values from experiments but the specific resistance parameter cannot be measured directly from experiments. It is in fact the ultimate goal of sludge filtration laboratory experiment.

The result of the experiments, [23] (Ademiluyi, 1984 experimental data) used to validate the new theory are displaced in figures, these graphs are shown in Figures 1-12. The similarity in the shape of this graphs shows that sludge dewaterability process goes on in the two sludges tested in the same way. Each graph

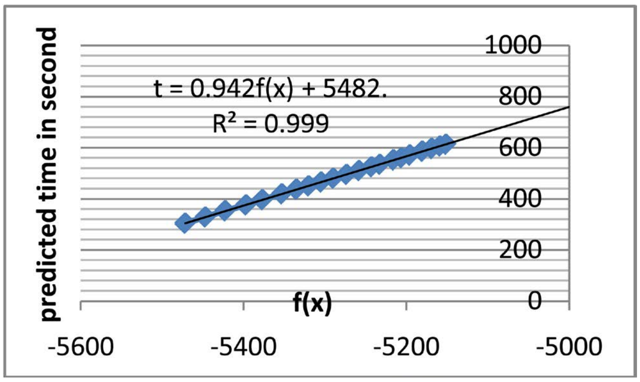

Figure 1. Graph of predicted time versus f(x) at 27.2 KN/m^2 pressure of water treatment plant sludge.

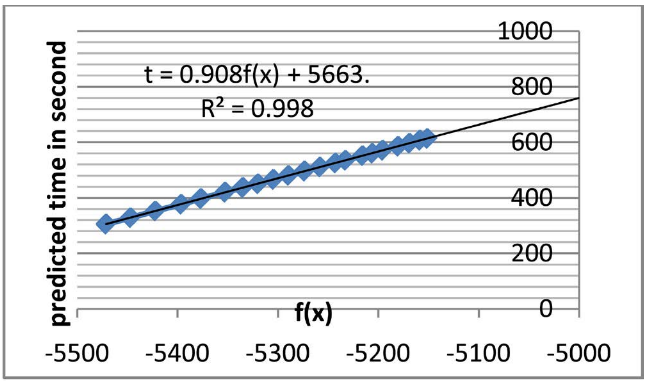

Figure 2. Graph of predicted time versus f(x) at 32.5 KN/m^2 pressure of water treatment plant sludge.

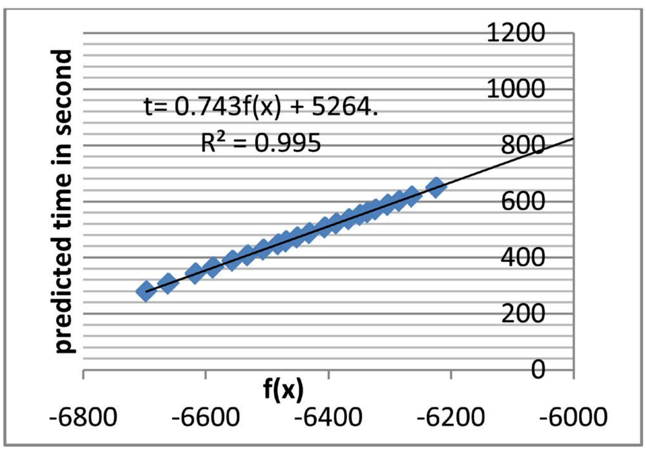

Figure 3. Graph of predicted time versus f(x) at 40.5 KN/m^2 of water treatment plant sludge.

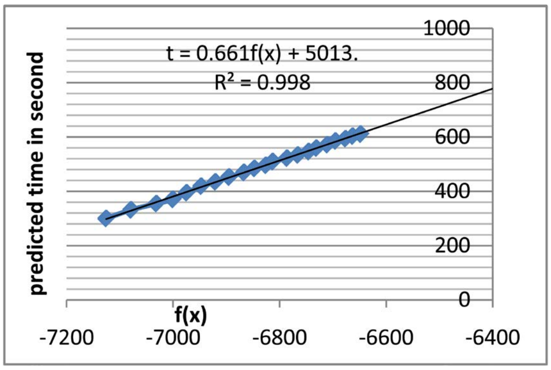

Figure 4. Graph of predicted time versus f(x) at 46.1 KN/m^2 pressure of water treatment plant sludge.

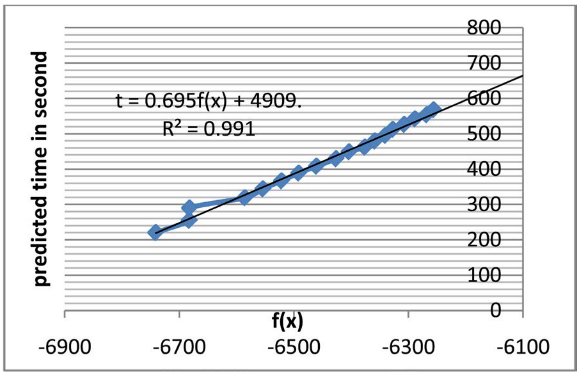

Figure 5. Graph of predicted time versus f(x) at 40.5KN/m^2 pressure of domestic sludge using.

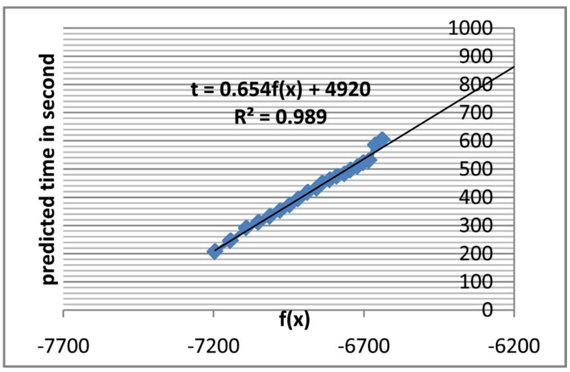

Figure 6. Graph of predicted time versus f(x) at 46.1 KN/m^2 pressure of demostic sludge.

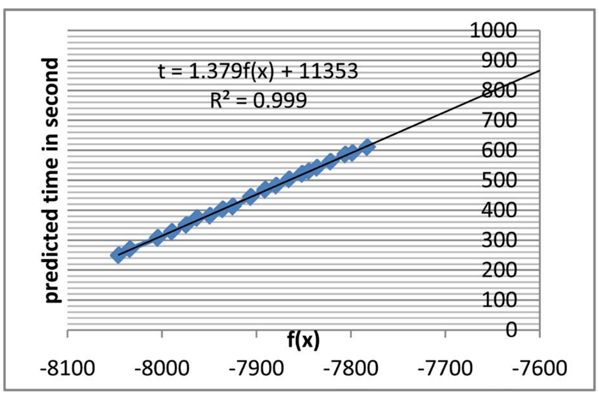

Figure 7. Graph of predicted time versus f(x) at 55.2KN/m^2 pressure of domestic sludge.

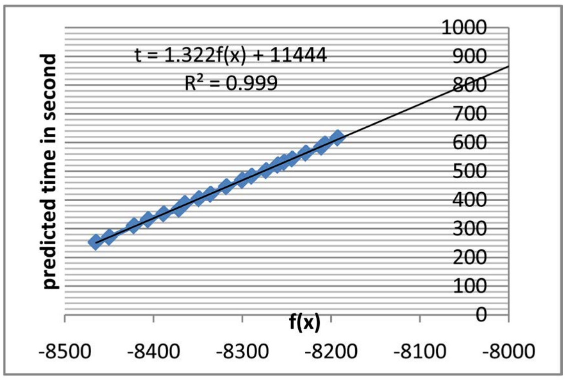

Figure 8. Graph of predicted time versus f(x) at 61.1 KN/m^2 pressure of domestic sludge.

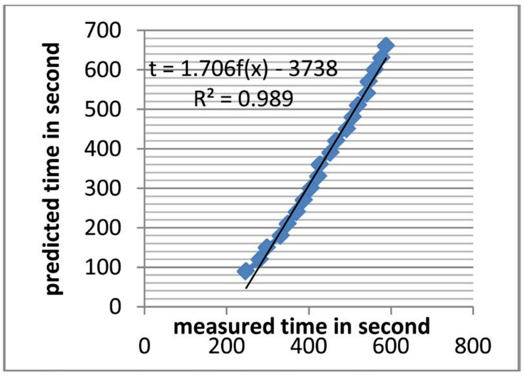

Figure 9. Graph of predicted time versus measured time at 54.3 KN/m^2 pressure of water treatment plant sludge.

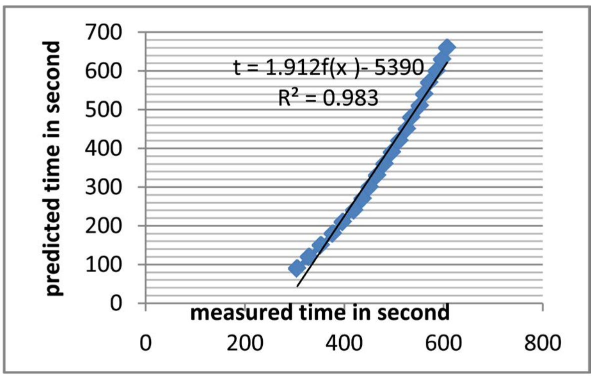

Figure 10. Graph of predicted time versus measured time at 60.4 KN/m^2 pressure of water treatment plant sludge.

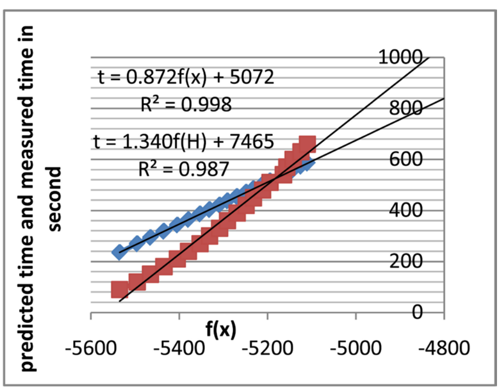

Figure 11. Graph of predicted time and measured time versus f(x) at 27.6 KN/m^2 pressure of domestic sludge.

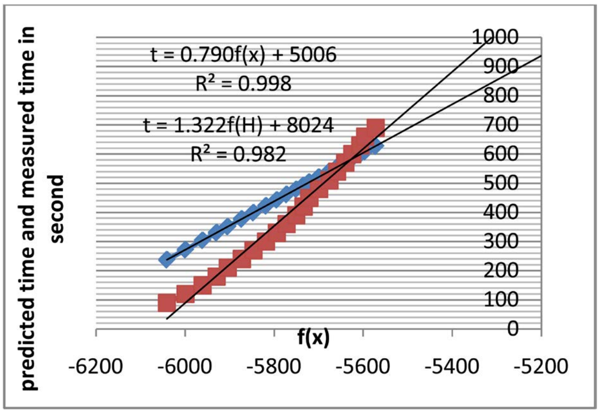

Figure 12. Graph of predicted time and measured time versus f(x) at 32.5 KN/m^2 pressure of domestic sludge.

shows that t increases with decreasing in numerical value of $f(x)$. This is what should actually happen since the sludge surface in the column should decrease as the filtrate increases with time.

The correlation coefficient '*r*' ranging from 0.998 to 1.000 shows a linear relationship between t and $f(x)$. The compressibility coefficient previously suggested by Carman and which has been shown to be in error and Ademiluyi's cake filtration equation which has been derived based on Terzaghi's concept of compressibility coefficient is only applicable for sludges whose Terzaghi's compressibility coefficient is less than one. The new sludge filtration Equation (61) provides for easy estimation of the compressibility coefficient of any sludge. The new equation is applicable to sludges whose Terzaghi's compressibility factor is more than 1.

Generally, the new theory is valid only for cake filtration which is indeed the basis of this investigation. The high correlation coefficients R^2 ranging from 0.996 to 1.000 with the graphs is a true test of valid linear relationship between t and $f(x)$. In Figure 9 and Figure 10, graphs of predicted time versus measured time, the high correlation coefficient R^2 of 0.989 and 0.983 shows that both predicted time and measured time are in agreement. In Figure 11 and Figure 12, graphs of predicted time and measured time versus $f(x)$, the high correlation coefficient R^2 of 0.998 and 0.998 using the new equation and 0.987 and 0.982 using Ademiluyi's equation also confirmed that both predicted time from the new equation and measured time from Ademiluyi's equation are in agreement.

It must be noted that the compressibility attribute of sludge cake previously suggested by Ademiluyi and that of Carman which has been shown to be in error [16] cannot be used in this new Equation (61). The new equation has been derived based on Terzaghi's concept of compressibility factor which has been found to be more than 1 (one) in the two sludges tested.

4. The Concentration Parameter in the New Equation

One of the problems in Carman's equation is the concentration, c which is the mass of dry cake per unit volume of filtrate. This parameter is very difficult to evaluate. To this end, the parameter is taken to be the initial solid content contrary to the theoretical prediction. In the new theory however, the concentration term (So) is the actual initial solid content which can easily be measured in the laboratory.

4.1. Filtration Area

The problem of which areas to be used during sludge dewatering is not found in the new theory unlike Carman's theory in which some previous workers advocate effective area of funnel while another school of thought proposed total area. Since the area needed is the area of the filter column used in calculating the driving head H, the problem of area is solved. To calculate driving head at any filtration time, the area of the column should be used not either the total area or effective area of the filter bed.

4.2. Relationship between the Parameters in the New Sludge Filtration Equation

Effect of dilution on the specific resistance (R^*)

The results of the investigation of the effect of dilution on the specific resistance R^* are displaced in figures. These reveal that the specific resistance R^* are approximately the same in all the piezometric positions tested. Figure 13 shows the graph of the plot of specific resistance R^* versus initial solid content of the sludge using the new equation and Ademiluyi's equation. At any particular time, specific resistance at any height of the sludge cake decreases with increasing solid content. This result is in agreement with that got by [23] and [25] using Carman's equation. Coackley explained that the fall in specific resistance with increasing solid content might be due to variation in the state of peptization of the sludge particles, the reason for obtaining a decreasing specific resistance with increasing corresponding solid.

4.3. Variation of Specific Resistance with Time of Filtration

Figures 14-18 shows the graphs of specific resistance with time, at different fil-

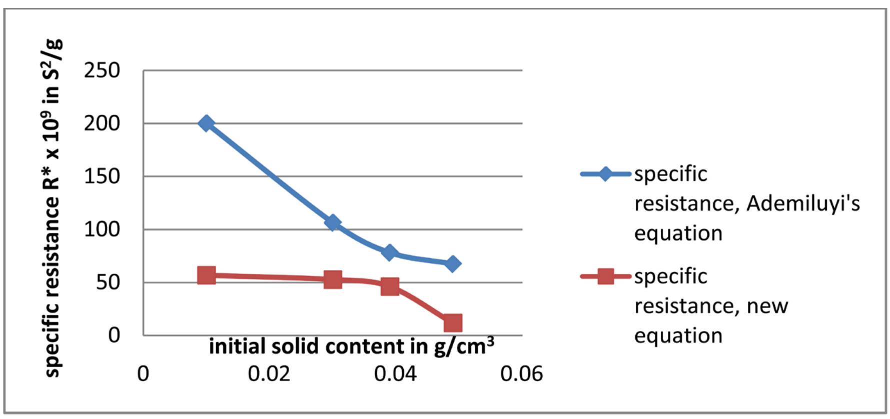

Figure 13. Graph of specific resistance versus initial solid content at filtration time 22 mins for piezometers P_3 of water treatment sludge using Ademiluyi's equation and new equation.

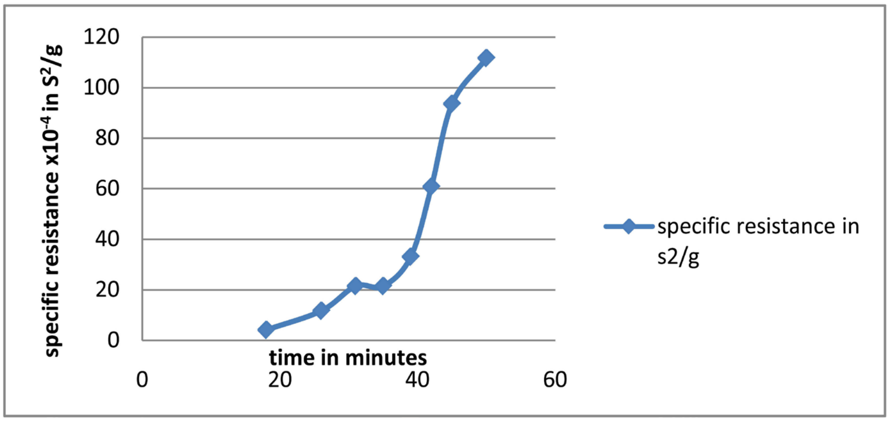

Figure 14. Graph of specific resistance versus time at Piezometric position (P_4) of water treatment sludge for solid content 0.049 g/cm^3.

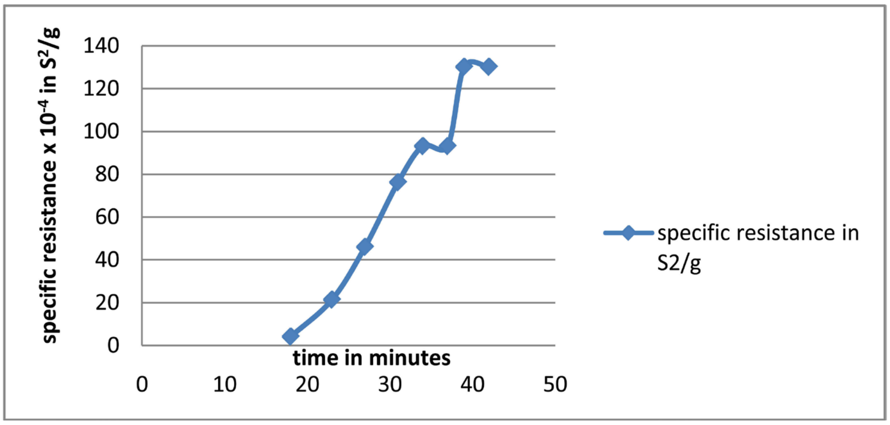

Figure 15. Graph of specific resistance versus time at Piezometric position (P_4) of water treeatment sludge for solid content 0.039 g/cm^3.

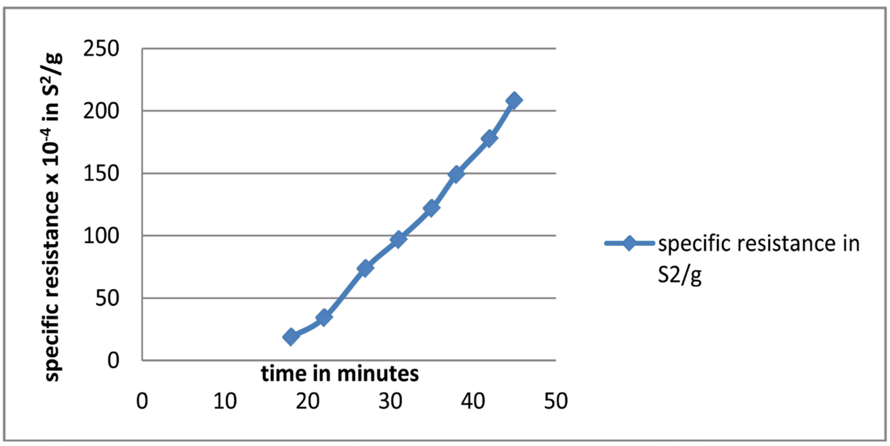

Figure 16. Graph of specific resistance versus time at piezometric position (P_4) of water treatment sludge for solid content 0.03 g/cm^3.

tration tested, specific resistance at any height of the sludge cake increases with increasing filtration time. The increase in specific resistance with time can be explained. During cake filtration, more sludge solids which were once on suspension settle at any piezometric point with time. This increasing sedimentation

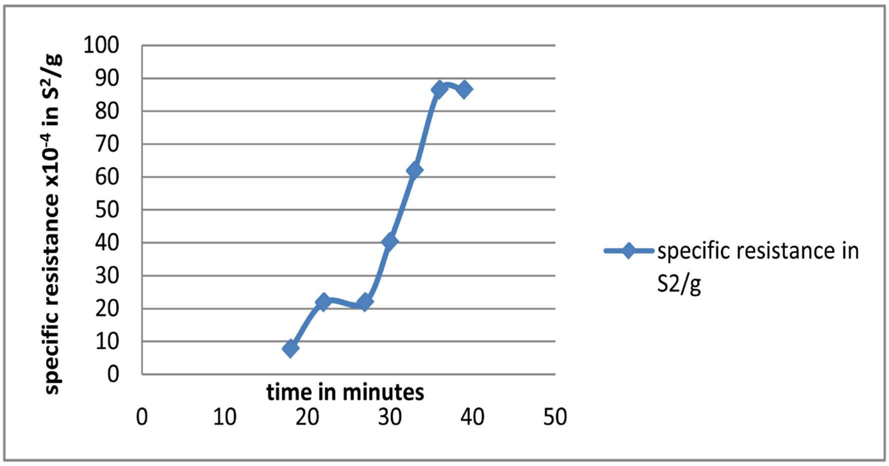

Figure 17. Graph of specific resistance versus time at piezometric position (P_4) of water treatment sludge for solid content 0.020 g/cm^3.

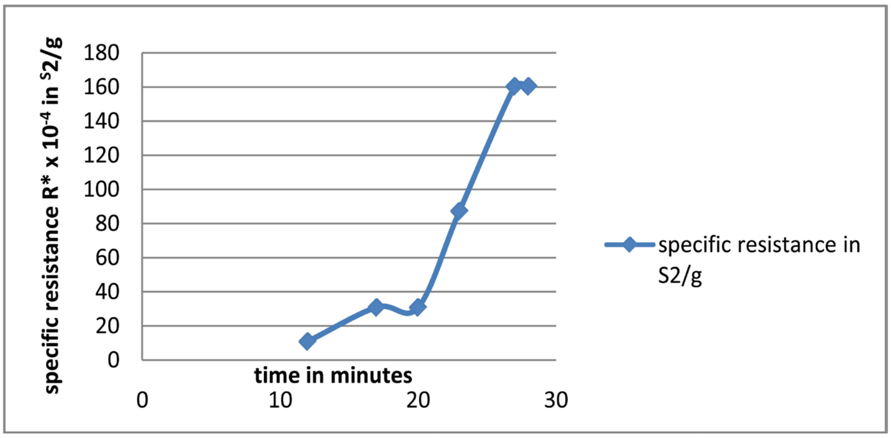

Figure 18. Graph of specific resistance versus time at piezometric positon (P_4) of water treatment sludge for solid content 0.01 g/cm^3.

of initially suspended solids decreases the porosity of cake at the time of sedimentation by blocking the settleable pores. This process will basically increases the specific resistance.

4.4. Effect of Vacuum Pressure on the Specific Resistance

The results of the experimental data collected to investigate the effect of vacuum pressure on the specific resistance is displaced in Figure 19, where a plot of specific resistance with vacuum pressure at 16 minutes filtration time using new equation and Ademiluyi's equation. The general increase in specific resistance with pressure is in agreement with Ademiluyi's and Carman's theories which also stated that specific resistance increases with increasing vacuum pressure. The curve indicates that specific resistance generally increases with increasing vacuum pressure within the range of pressure tested.

4.5. Effect of Piezometric Positions on the Specific Resistanc

The results of the experimental data collected to investigate the effect of piezo-

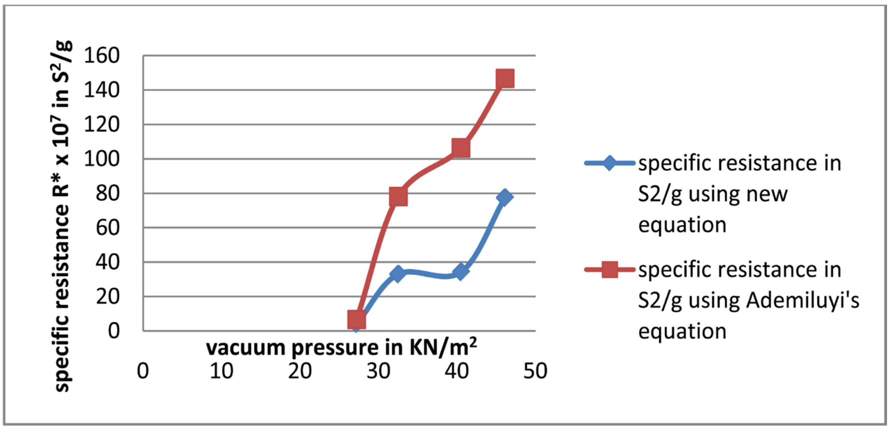

Figure 19. Graph of specific resistance versus vacuum pressure at filtration time of 16 mins for piezometer (P_4) using Ademiluyi's equation and new equation.

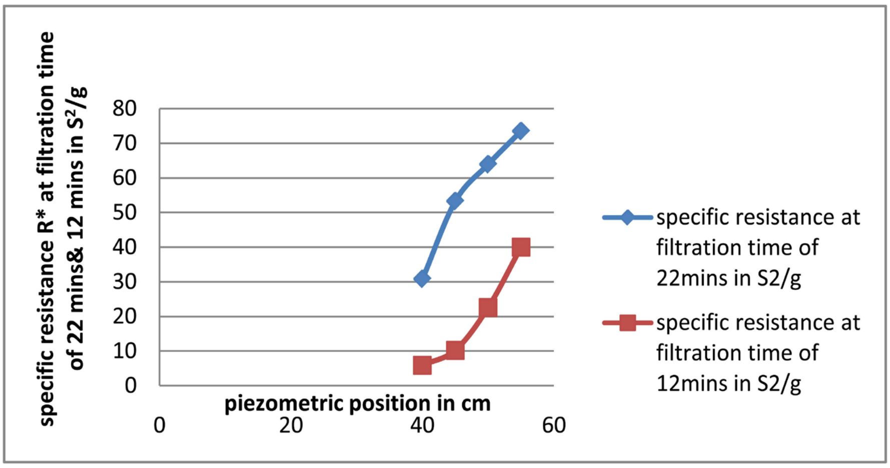

Figure 20. Graph of specific resistance versus piezometric positions at filtration times of 22 mins and 12 mins for piezometer (P_6) of water treatment sludge.

metric positions on the specific resistance is displaced in Figure 20, graph of specific resistance versus piezometric positions at filtration times of 22 mins and 12 mins respectively. The curve indicates that specific resistance increase with increasing piezometric positions. The general increase in specific resistance with increasing piezometric positios is also in agreement with Ademiluyi's theory.

5. Conclusions

The following conclusions are drawn from the study;

1) A new equation to sludge filtration processes has been recommended for use in routine laboratory investigation. This equation is given as:

2)

$$t = \alpha\beta^{s}\gamma^{-s-2}\left(\frac{\beta}{s+1}x^{s+1} - \sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^{i}}{(s-i)}\beta(1+x_0)x^{s-i} - \beta(1+x_0)(-1)^{[s+1/2]}\ln(1+x)\right)$$
$$+\alpha\beta^{s}\gamma^{-s-2}\left(-\frac{\beta}{s+1}x_0^{s+1} + \sum_{i=0}^{\left[s+\frac{1}{2}\right]-1}\frac{(-1)^{i}}{(s-i)}\beta(1+x_0)x_0^{s-i} + \beta(1+x_0)(-1)^{[s+1/2]}\ln(1+x_0)\right)$$

The new theory has been recommended for sludge dewatering studies since the experimental analysis is not rigorous as the traditional theory. Besides, the theory accounts for the compressibility attribute of sludges and the hydrostatic pressure, which are believed to influence sludge dewatering. Also in the derivation of the new basic equation, the compressibility attribute of sludge called compressibility factor is considered rather than compressibility coefficient in Ademiluyi's theory.

Contribution to Knowledge

The new theory is applicable to sludge whose compressibility factor is greater than 1 (one) unlike Ademiluyi's theory which is applicable for sludge whose compressibility coefficient is much less than 1 (one) only.

The controversy among previous writers as regards the area of filter bed to be used (whether effective area or total area of filter bed has been resolved since the new theory makes use of the area of filter column and not that of the filter bed in the evaluation of the driving head (H). The concentration term (So) in the new equation is the actual initial solid content of the sludge as distinct from the Carman's equation in which the concentration term is the mass of dry cakes deposited per unit volume of filtrate. Also in the derivation of the new equation, specific resistance parameter has been treated as a local variable parameter rather than the traditional average value used in sludge filtration studies.

3) Carman's theory requires the linear plot of t/v versus v. Linear regression analysis cannot be used in drawing this straight line since the variables involved are not independent [25]. Hence the straight line of the plot of t/v versus v was drawn by inspection, and the slope of this line was used in calculating the specific resistance but in this new equation, independent variables t and $f(x)$ were used in plotting the straight line graph whose slope is also used in evaluating the specific resistance. The advantage in using this new theory is that the straight line of the plot of t versus $f(x)$ can be drawn by both linear regression analysis and by regression.
4) It has been shown both analytically and experimentally (Data from Ademiluyi's experiment) that
a) Specific resistance increases with time of filtration. This is due to the increasing sedimentation of initially suspended solids which decreases the porosity of cake by blocking the available pores, this finding is in agreement with Ademiluyi's theory.
b) Specific resistance increases toward the filter septum. This may be due to the corresponding decrease of porosity towards the filter septum, in agreement with the finding of Ademiluyi but not in agreement with experimental results obtained from other researchers; Anazodo and Carman, where specific resistance was found to be constant throughout the filtration cycle and along the cake height.
5) It has been found also that specific resistance increase with increasing in vacuum pressure. This finding is in agreement with the finding of both Ademiluyi, and Coackley. The reason for this is that as filtration pressure is increased,

the porosity of the cake decreases thereby increasing the specific resistance.

In view of the variable nature of the specific resistance R^* in the new equation, there's need for a further work. However, differentiating H (driving head) with respect to t (time) and equate to zero may be recommended for evaluating the maximum specific resistance value using the new equation to enhance the usefulness of the expression in specific quantification of sludge filterability. Once this method is established and the maximum specific resistance (MSR) is known, a yield equation should then been derived. The yield of a rotary vacuum filter got from this yield equation should then been compared with the practical yield procured from an industrial vacuum plant.

Limitations of the new equation are,

a) The equation can only be used for sludge whose compressibility factor is more than 1 (one).

b) It is difficult to evaluate its dimensional homogeneity since some of the variables in the new equation have an exponent 's' which is not a dimensionless pure number

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List of Symbols

A = Area of filtration (m^2)

C = Mass of dry cakes deposited per unit volume of filtrate (kg/m^2)

g = Acceleration due to gravity (m/s^2)

H_o = Initial driving head (m)

H = Driving head at any time t, (m)

h_m = mercury rise in manometer (m)

h_f = Head loss (m)

L = Thickness of the cake (m)

L_m = Thickness of the medium (m)

P = applied vacuum pressure (kN/m^2)

P_L = Hydraulic pressure (N/m^2)

P_s = Compressive drag pressure of solids (N/m^2)

q or dv/dt = Flow rate (m^3/s)

R^* = specific resistance at any arbitrary head loss (s^2/g)

R = Average specific resistance (m/kg)

R^1 = Cake resistance (m^{-2})

R_1 = Local flow resistance (m/kg)

R_m = Medium or septum resistance (m^{-2})

s = Compressibility factor (cm^2/g)

S_o = Initial solid content of sludge (kg/m^3)

t = Time taken to obtain filtrate (s)

v = Volume of filtrate (m^3)

W = Mass of dry solids deposited per unit area (kg/m^2)

β = Vacuum pressure (kN/m^2)

p_f = Density of filtrate (kg/m^3)

p_m = Density of mercury (kg/m^3)

γ = Specific weight of filtrate (N/m^3)

μ = Viscosity of filtrate (poise)

$[n/2]$ = greatest integer value of $n/2$

$\tan^n\theta$ = Modified Reduction formula

$\sum_{i=0}^{\left[\frac{n}{2}\right]-1}$ = Summation